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SPECTRUM AND COMPACTNESS OF THE CESÀRO OPERATOR ON WEIGHTED ℓ_p SPACES

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Abstract

An investigation is made of the continuity, the compactness and the spectrum of the Cesàro operator C when acting on the weighted Banach sequence spaces $\ell_p(w)$, $1 < p < \infty$, for a positive decreasing weight w, thereby extending known results for C when acting on the classical spaces ℓ_p. New features arise in the weighted setting (for example, existence of eigenvalues, compactness) which are not present in ℓ_p.

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1. Introduction

The discrete Cesàro operator C is defined on the linear space $\mathbb{C}^{\mathbb{N}}$ (consisting of all scalar sequences) by

$$\mathsf{C}x := \left(x_1, \frac{x_1 + x_2}{2}, \dots, \frac{x_1 + \cdots + x_n}{n}, \dots\right), \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{1.1}$$

The operator C is said to act in a vector subspace $X \subseteq \mathbb{C}^{\mathbb{N}}$ if it maps X into itself. Of particular interest is the situation where X is a Banach space. The fundamental questions in this case are: is $\mathsf{C} : X \to X$ continuous and, if so, what is the spectrum of $\mathsf{C} : X \to X$? Amongst the classical Banach spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ where precise answers are known, we mention ℓ_p $(1 < p < \infty)$ [6, 14], c_0 [14, 18] and both c and ℓ_∞ [1, 14], as well as ces_p, $p \in \{0\} \cup (1, \infty)$ [8], the Bachelis spaces N^p, $2 \le p < \infty$ [9] and the spaces of bounded variation bv_0 [17] and bounded p-variation bv_p, $1 \le p < \infty$ [2]. In all of these cases, the operator norm of $\mathsf{C} : X \to X$ equals its spectral radius and C has at most one eigenvalue, namely, 1. There is no claim that this list of spaces (and references) is complete.

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The aim of this paper is to investigate the two questions mentioned above for C acting on the weighted Banach spaces $\ell_p(w)$. To be precise, let $w = (w(n))_{n=1}^{\infty}$ be a bounded sequence, always assumed to be *strictly* positive. Define the space

$$\ell_p(w) := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|x\|_{p,w} := \left(\sum_{n=1}^{\infty} |x_n|^p w(n) \right)^{1/p} < \infty \right\}$$

for each $1 < p < \infty$, equipped with the norm $\|\cdot\|_{p,w}$. Observe that $\ell_p(w)$ is isometrically isomorphic to ℓ_p via the linear multiplication operator

$$\Phi_w : \ell_p(w) \to \ell_p, \quad x = (x_n)_{n\in\mathbb{N}} \to \Phi_w(x) := (w(n)^{1/p} x_n)_{n\in\mathbb{N}}.$$

Therefore, each $\ell_p(w)$ is a Banach space. The dual space $(\ell_p(w))'$ of $\ell_p(w)$ is the Banach space $\ell_{p'}(v)$, where $1/p + 1/p' = 1$ (that is, p' is the conjugate exponent of p) and $v(n) = w(n)^{-p'/p}$ for $n \in \mathbb{N}$. In particular, $\ell_p(w)$ is reflexive and separable for $1 < p < \infty$. Moreover, the canonical unit vectors $e_k := (\delta_{kn})_{n\in\mathbb{N}}$ for $k \in \mathbb{N}$ form an unconditional basis in $\ell_p(w)$ for $1 < p < \infty$. If $\inf_{n\in\mathbb{N}} w(n) > 0$, then $\ell_p(w) = \ell_p$ with equivalent norms and we are in the standard situation. Accordingly, we are mainly interested in the case where $\inf_{n\in\mathbb{N}} w(n) = 0$.

By Hardy's inequality [13, Theorem 326, page 239], for every $1 < p < \infty$ the restriction of the Cesàro operator $\mathsf{C} : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ as given in (1.1) defines a bounded linear operator from ℓ_p into itself with operator norm equal to p'. Denote these operators by $\mathsf{C}^{(p)}$ so that $\|\mathsf{C}^{(p)}\| = p'$. In Section 2, where the papers [5, 11, 12] are relevant, we discuss various aspects of the continuity of C when it is restricted to $\ell_p(w)$, $1 < p < \infty$; denote this operator by $\mathsf{C}^{(p,w)}$ whenever it is continuous.

For any Banach space X, let I denote the identity operator on X and $\mathcal{L}(X)$ denote the space of all continuous linear operators from X into itself. The *spectrum* and the *resolvent set* of $T \in \mathcal{L}(X)$ are denoted by $\sigma(T)$ and $\rho(T)$, respectively; see [10, Ch. VII], for example. The set of all *eigenvalues* of T, called the *point spectrum* of T, is denoted by $\sigma_{pt}(T)$. The *spectral radius* $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$ of T always satisfies $r(T) \leq \|T\|$, [10, page 567].

Section 3 is devoted to an analysis of the spectrum of C when acting in $\ell_p(w)$. The main result is Theorem 3.3; it is complemented by Example 3.5 which clarifies the scope of this theorem. Unlike for $\mathsf{C}^{(p)}$, it can happen that $\sigma_{pt}(\mathsf{C}^{(p,w)}) \neq \emptyset$. Actually, $\mathsf{C}^{(p,w)}$ can even have infinitely many eigenvalues; see Proposition 3.6. The final section deals with the *compactness* of $\mathsf{C}^{(p,w)}$. Of relevance is how fast w decreases to 0; see Proposition 4.1, Theorem 4.2, Corollary 4.3 and Proposition 4.6. Unlike for C acting in the classical Banach spaces mentioned in the opening paragraph, it may happen in $\ell_p(w)$ that $r(\mathsf{C}^{(p,w)}) < \|\mathsf{C}^{(p,w)}\|$; see Proposition 4.1.

2. Continuity of C in weighted ℓ_p spaces

Some of the concepts and results from [12] that are quoted in this section actually have their origins in the paper [11]. We begin with the following fact.

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Lemma 2.1. *Let $w = (w(n))_{n=1}^{\infty}$ be a positive sequence and $1 < p < \infty$. Then the Cesàro operator* C *maps $\ell_p(w)$ continuously into itself if and only if*

$$\sup_{m\in\mathbb{N}}\Big(\sum_{k=1}^{m} w(k)^{-p'/p}\Big)^{-1}\Big(\sum_{n=1}^{m}\frac{w(n)}{n^p}\Big(\sum_{k=1}^{n} w(k)^{-p'/p}\Big)^{p}\Big) < \infty,$$

that is, if and only if there exists $K > 0$ such that

$$\sum_{n=1}^{m}\frac{w(n)}{n^p}\Big(\sum_{k=1}^{n} w(k)^{-p'/p}\Big)^{p} \le K\Big(\sum_{k=1}^{m} w(k)^{-p'/p}\Big), \quad m \in \mathbb{N}. \tag{2.1}$$

Moreover, if the constant K satisfying (2.1) is chosen as small as possible, then the operator norm of C *is at most $p'K^{1/p}$.*

Proof. Let $T_w : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ denote the linear operator defined by

$$T_w x := \Big(\frac{w(n)^{1/p}}{n}\sum_{k=1}^{n} w(k)^{-1/p} x_k\Big)_{n\in\mathbb{N}} \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{2.2}$$

Then $\Phi_w C = T_w \Phi_w$. Since Φ_w is isometric from $\ell_p(w)$ onto ℓ_p, it follows that C maps $\ell_p(w)$ continuously into itself if and only if T_w maps ℓ_p continuously into itself. But, the matrix of T_w is factorable (cf. [5, Section 4] with $a_n = w(n)^{1/p}/n$ and $b_k = w(k)^{-1/p}$ for $1 \le k \le n$) and so it follows from [5, Theorem 2] that $T_w \in \mathcal{L}(\ell^p)$ if and only if (2.1) holds.

The proof of [5, Theorem 2] yields that the operator norm of C is at most $p'K^{1/p}$. □

Proposition 2.2. *Let $w = (w(n))_{n=1}^{\infty}$ be a positive decreasing sequence and $1 < p < \infty$. Then the Cesàro operator $\mathsf{C}^{(p,w)} \in \mathcal{L}(\ell_p(w))$ and satisfies*

$$1 < \Big(\frac{1}{w(1)}\sum_{n=1}^{\infty}\frac{w(n)}{n^p}\Big)^{1/p} \le \|\mathsf{C}^{(p,w)}\| \le p'. \tag{2.3}$$

Proof. Fix $m \in \mathbb{N}$. Because w is decreasing,

$$\begin{aligned}\sum_{n=1}^{m}\frac{w(n)}{n^p}\Big(\sum_{k=1}^{n} w(k)^{-p'/p}\Big)^{p} &= \sum_{n=1}^{m}\Big(\frac{w(n)^{1/p}}{n}\sum_{k=1}^{n} w(k)^{-p'/p}\Big)^{p}\\ &\le \sum_{n=1}^{m}\Big(\frac{w(n)^{1/p}}{n}\cdot\frac{n}{w(n)^{p'/p}}\Big)^{p} = \sum_{n=1}^{m} w(n)^{-p'/p},\end{aligned}$$

which is precisely (2.1) with $K = 1$. So, Lemma 2.1 implies that C is continuous on $\ell_p(w)$ with $\|\mathsf{C}^{(p,w)}\| \le p'$.

For an alternate proof of the continuity of $\mathsf{C}^{(p,w)}$, based directly on Hardy's inequality in ℓ_p, see [12, Proposition 5.1].

Since $T_w = \Phi_w \mathsf{C}^{(p,w)} \Phi_w^{-1}$, with Φ_w mapping the closed unit ball of $\ell_p(w)$ onto that of ℓ_p and Φ_w^{-1} mapping the closed unit ball of ℓ_p onto that of $\ell_p(w)$, it follows that $\|T_w\| = \|\mathsf{C}^{(p,w)}\|$. Of course,

$$\Phi_w^{-1} x = (w(n)^{-1/p} x_n)_{n\in\mathbb{N}}, \quad x \in \ell_p.$$

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Substituting $x = e_1$ into (2.2) it follows that

$$\|\mathsf{C}^{(p,w)}\| = \|T_w\| \geq \|T_w e_1\|_p = \Big(\frac{1}{w(1)}\sum_{n=1}^{\infty}\frac{w(n)}{n^p}\Big)^{1/p} \geq \Big(1 + \frac{w(2)}{w(1)2^p}\Big)^{1/p} > 1.$$

See also [12, Proposition 5.5]. □

Some comments regarding Proposition 2.2 are in order. As noted above, for each $1 < p < \infty$ we have $\|\mathsf{C}^{(p)}\| = p'$ and, for a positive decreasing weight w, (2.3) holds. These estimates are not the best possible in general. Denote by $\delta_p(w)$ the set of all decreasing, nonnegative sequences in $\ell_p(w)$ and define

$$\Delta_{p,w}(\mathsf{C}^{(p,w)}) := \sup\{\|\mathsf{C}^{(p,w)}x\|_{p,w} : x \in \delta_p(w),\ \|x\|_{p,w} = 1\} \leq \|\mathsf{C}^{(p,w)}\|.$$

The following result follows from [12, Propositions 6.3, 6.5 and 6.6].

Proposition 2.3. *Let $1 < p < \infty$ and $w(n) = 1/n^\alpha$, $n \in \mathbb{N}$, for a fixed $\alpha > 0$. Then*

$$\max\{m_1, m_2\} \leq \Delta_{p,w}(\mathsf{C}^{(p,w)}) \leq \|\mathsf{C}^{(p,w)}\| \leq M_2(r) := [r\zeta(r+\alpha)]^{r/p} \tag{2.4}$$

for $1 \leq r \leq p$, where $m_1 := p/(p+\alpha-1)$ and $m_2 := \zeta(p+\alpha)^{1/p}$, with ζ denoting the Riemann zeta function. Moreover, for $\alpha \leq r < (p+\alpha)$, it is also the case that

$$\|\mathsf{C}^{(p,w)}\| \leq M_3(r) := \Big(\frac{p}{p+\alpha-r}\Big)^{1/p'}\zeta\Big(1 + \frac{r}{p'} + \frac{\alpha}{p}\Big)^{1/p}.$$

We provide some relevant examples.

Example 2.4.

(i) For $w(n) = 1/n^\alpha$, if $\alpha = 0.9$ and $p = 1.1$, then $\max\{m_1, m_2\} \simeq 1.572$ and $M_2(1) = M_3(0.9) \simeq 1.663$ (see [12, pages 15–16]) and so Proposition 2.3 shows that

$$1.572 \leq \|\mathsf{C}^{(p,w)}\| \leq 1.663.$$

On the other hand, $p' = 11$ and so Proposition 2.2 only yields $\|\mathsf{C}^{(p,w)}\| \leq 11$.

(ii) Still for $w(n) = 1/n^\alpha$, but now with $\alpha = 0.5$ and $p = 2$, we have $m_1 = 4/3$ and $M_3(3/4) \simeq 1.593$ (see [12, page 16]) so that Proposition 2.3 reveals that

$$\tfrac{4}{3} \leq \|\mathsf{C}^{(p,w)}\| \leq 1.593.$$

In this case, $p' = 2$, and so Proposition 2.2 only yields $\|\mathsf{C}^{(p,w)}\| \leq 2$.

(iii) Again for $w(n) = 1/n^\alpha$, with $\alpha > 0$, it follows (in the notation of Proposition 2.3) that

$$\Big(\frac{1}{w(1)}\sum_{n=1}^{\infty}\frac{w(n)}{n^p}\Big)^{1/p} = \Big(\sum_{n=1}^{\infty}\frac{1}{n^{p+\alpha}}\Big)^{1/p} = \zeta(p+\alpha)^{1/p} = m_2.$$

Hence, the lower bound in (2.3) reduces to $m_2 \leq \|\mathsf{C}^{(p,w)}\|$ whereas (2.4) yields $\max\{m_1, m_2\} \leq \|\mathsf{C}^{(p,w)}\|$. Of course, (2.3) applies to more general weights w.

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The following example is not a consequence of Proposition 2.3.

EXAMPLE 2.5. Let $p = 2$ and set $w(n) = 2^{-n}$ for $n \in \mathbb{N}$. The proof of Proposition 2.2 yields that $\|\mathsf{C}^{(2,w)}\| = \|T_w\|$. Recall, via (2.2), that

$$T_w x = \Big(\frac{1}{n2^{n/2}} \sum_{k=1}^{n} 2^{k/2} x_k\Big)_{n\in\mathbb{N}}, \qquad x = (x_n)_{n\in\mathbb{N}} \in \ell_2.$$

For every $x \in \ell_2$, it follows via the Cauchy–Schwarz inequality and the identity $\sum_{k=1}^{n} r^k = (r - r^{n+1})/(1 - r)$, for $r \neq 1$, that

$$\begin{aligned}\|T_w x\|_2^2 &= \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}\Big|\sum_{k=1}^{n} 2^{k/2} x_k\Big|^2 \le \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}\Big(\sum_{k=1}^{n} 2^k\Big)\Big(\sum_{k=1}^{n} |x_k|^2\Big)\\ &\le \|x\|_2^2 \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}(2^{n+1} - 2) = \|x\|_2^2 \sum_{n=1}^{\infty} \frac{2(1 - 2^{-n})}{n^2}.\end{aligned}$$

Accordingly, $\|T_w\| \le (\sum_{n=1}^{\infty} (2(1 - 2^{-n})/n^2))^{1/2}$. Observe that

$$\sum_{n=1}^{\infty} \frac{(1 - 2^{-n})}{n^2} = \frac{\pi^2}{6} - \int_0^{1/2} \frac{-\log(1 - t)}{t}\, dt,$$

because of the fact that $\pi^2/6 = \sum_{n=1}^{\infty} (1/n^2)$ and the identity

$$\int_0^{1/2} \frac{-\log(1 - t)}{t}\, dt = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{t^n}{(n + 1)} = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}.$$

The function $f(t) = (-\log(1 - t))/t$ for $t \in (0, 1]$, with $f(0) := 1$, is positive, continuous and increasing on $[0, 1)$ and so

$$1 = f(0) \le f(t) \le f(\tfrac{1}{2}) = 2\log 2 \quad t \in [0, 1/2],$$

which implies that $-\log 2 \le -\int_0^{1/2} (-\log(1 - t))/t\, dt \le -1/2$. Consequently,

$$\sum_{n=1}^{\infty} \frac{2(1 - 2^{-n})}{n^2} \le 2\Big(\frac{\pi^2}{6} - \frac{1}{2}\Big) \simeq 2.2898$$

and so

$$\|\mathsf{C}^{(2,w)}\| = \|T_w\| \le \sqrt{\Big(\frac{\pi^2}{3} - 1\Big)} \simeq 1.513 < p' = 2.$$

Direct calculation yields

$$\|T_w e_1\|_2 = \Big(2\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}\Big)^{1/2} \ge \Big(2\sum_{n=1}^{3} \frac{1}{n^2 2^n}\Big)^{1/2} \simeq 1.073$$

and so

$$1.073 \le \|\mathsf{C}^{(2,w)}\| \le \sqrt{\Big(\frac{\pi^2}{3} - 1\Big)} \simeq 1.513;$$

see also Proposition 2.2.

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3. Spectrum of $\mathsf{C}^{(p,w)}$

The aim of this section is to provide some detailed knowledge of the spectrum of $\mathsf{C}^{(p,w)}$. Unlike for the classical Cesàro operators $\mathsf{C}^{(p)} \in \mathcal{L}(\ell_p)$ for $1 < p < \infty$, it can now happen that eigenvalues appear.

Given a (strictly) positive bounded sequence $w = (w(n))_{n\in\mathbb{N}}$ and $1 < p < \infty$, let $S_w(p) := \{s \in \mathbb{R} : \sum_{n=1}^{\infty} (1/n^s w(n)^{p'/p}) < \infty\}$. For $S_w(p) \neq \emptyset$ we define $s_p := \inf S_w(p)$. Note that $p'/p = 1/(p-1)$ for every $1 < p < \infty$. Moreover, let $R_w := \{t \in \mathbb{R} : \sum_{n=1}^{\infty} n^t w(n) < \infty\}$. For $R_w \neq \mathbb{R}$ we define $t_0 := \sup R_w$.

Fix $1 < p < \infty$ and let $w(n) = 2^{-np/p'}$ for $n \in \mathbb{N}$. Then $S_w(p) = \emptyset$, that is, it can happen that $S_w(p)$ is empty. However, in the event that $S_w(p) \neq \emptyset$, then $s_p \geq 1$. Indeed, for any fixed $s \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} \geq \|w\|_{\infty}^{-p'/p} \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{3.1}$$

So, whenever $s \in S_w(p)$ it follows that $\sum_{n=1}^{\infty} (1/n^s) < \infty$, that is, $s > 1$. Hence, $S_w(p) \subseteq (1, \infty)$, which implies that $s_p \geq 1$. Moreover, for any $r > s \in S_w(p)$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^r w(n)^{p'/p}} < \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}}$$

and so also $r \in S_w(p)$. Accordingly, whenever $S_w(p) \neq \emptyset$, it is an infinite interval, that is, $S_w(p) = [s_p, \infty)$ or $S_w(p) = (s_p, \infty)$ with $s_p \geq 1$. It is a consequence of (3.1) that $1 \notin S_w(p)$ for all $1 < p < \infty$ and all positive bounded sequences w.

In the event that $a_w := \inf_{n\in\mathbb{N}} w(n) > 0$, it follows that necessarily $s_p = 1$. Indeed, in this case, $w(n)^{-p'/p} \leq a_w^{-p'/p}$, $n \in \mathbb{N}$, which implies that $1/n^s w(n)^{p'/p} \leq a_w^{-p'/p}/n^s$ for all $n \in \mathbb{N}$ and $s \in \mathbb{R}$. Hence, $(1, \infty) \subseteq S_w(p)$, and so $s_p \leq 1$. Since we are assuming that $S_w(p) \neq \emptyset$, we already know that $s_p \geq 1$. Accordingly, $s_p = 1$.

Let $1 < p < \infty$ and fix $\alpha > 0$. For $w(n) = 1/n^{\alpha p/p'}$ and any $s \in \mathbb{R}$ it follows that $\sum_{n=1}^{\infty} (1/n^s w(n)^{p'/p}) = \sum_{n=1}^{\infty} (1/n^{s-\alpha}) < \infty$ precisely when $s > (1+\alpha)$ and so $s_p = 1 + \alpha$. Hence, given any $\beta > 1$ and $1 < p < \infty$, there exists a positive decreasing weight $w \downarrow 0$ such that $S_w(p) = (\beta, \infty)$, that is, $s_p = \beta$.

Concerning the set R_w, a similar discussion applies. For $w(n) = 2^{-n}$ it turns out that $R_w = \mathbb{R}$ with $t_0 = \infty$. However, if $R_w \neq \mathbb{R}$, then t_0 is finite with $t_0 \geq -1$ and $R_w = (-\infty, t_0)$ or $R_w = (-\infty, t_0]$. Moreover, $R_w = \emptyset$ is not possible as $\sum_{n=1}^{\infty} n^t w(n) \leq \|w\|_{\infty} \sum_{n=1}^{\infty} n^t < \infty$ whenever $t < -1$. If $a_w > 0$, then necessarily $t_0 = -1$, but $-1 \notin R_w$ as $\sum_{n=1}^{\infty} n^t w(n) \geq a_w \sum_{n=1}^{\infty} n^t$ for all $t \in \mathbb{R}$.

The following result clarifies the connection between s_p and t_0.

Proposition 3.1. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a bounded, strictly positive sequence.*

(i) *For each $1 < p < \infty$ such that $S_w(p) \neq \emptyset$,*

$$t_0 \leq \frac{s_p p}{p'} = (p-1)s_p.$$

In particular, $R_w \neq \mathbb{R}$ whenever there exists $p \in (1, \infty)$ with $S_w(p) \neq \emptyset$.

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(ii) *If $R_w \neq \mathbb{R}$, then $S_w(p) \subseteq [1 + (t_0/(p-1)), \infty)$ for every $1 < p < \infty$.*
(iii) *Suppose that $1 < p < \infty$ satisfies $S_w(p) \neq \emptyset$. Then*

$$S_w(p) \subseteq S_w(q), \quad q \in [p, \infty).$$

In particular, $S_w(q) \neq \emptyset$ and $s_q \leq s_p$ whenever $q \geq p$.
(iv) *If $S_w(p) = \emptyset$ for some $1 < p < \infty$, then $S_w(q) = \emptyset$ for all $1 < q \leq p$.*

Proof. (i) Suppose that $S_w(p) \neq \emptyset$. Fix $s > s_p$. Since $\sum_{n=1}^{\infty} (1/n^s w(n)^{p'/p}) < \infty$, there exists $N \in \mathbb{N}$ such that $1/n^s w(n)^{p'/p} \leq 1$ for $n \geq N$ and hence $n^{sp/p'} w(n) \geq 1$ for $n \geq N$. So, the series $\sum_{n=1}^{\infty} n^{sp/p'} w(n)$ diverges, which yields that $t_0 \leq sp/p'$. Accordingly, $t_0 \leq s_p p/p'$. In particular, $R_w \neq \mathbb{R}$.

(ii) Fix $p \in (1, \infty)$ and any $t < t_0$, in which case $\sum_{n=1}^{\infty} n^t w(n) < \infty$. Hence, there exists $K \in \mathbb{N}$ such that $n^t \leq 1/w(n)$ for $n \geq K$, that is, $n^{tp'/p} \leq 1/w(n)^{p'/p}$ for $n \geq K$. So, for any $s \in \mathbb{R}$ (as $1/n^s > 0$ for $n \in \mathbb{N}$),

$$\frac{1}{n^{s-(tp'/p)}} = \frac{n^{tp'/p}}{n^s} \leq \frac{1}{n^s w(n)^{p'/p}}, \quad n \geq K.$$

Choose now $s \leq 1 + (tp'/p)$. It follows from the previous inequality that $\sum_{n=1}^{\infty} (1/n^s w(n)^{p'/p})$ diverges. Hence, $\sum_{n=1}^{\infty} (1/n^s w(n)^{p'/p})$ diverges whenever $s \leq 1 + (tp'/p)$ for some $t < t_0$, that is, whenever $s \in (-\infty, 1 + (t_0 p'/p))$. So, $S_w(p) \subseteq [1 + (t_0 p'/p), \infty) = [1 + (t_0/(p-1)), \infty)$.

(iii) Fix $s \in S_w(p)$, that is, $\sum_{n=1}^{\infty} (1/n^s w(n)^{p'/p}) < \infty$. For every $1 < q < \infty$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{q'/q}} = \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} \cdot w(n)^{(p'/p)-(q'/q)} \leq \|w\|_{\infty}^{(p'/p)-(q'/q)} \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}},$$

which is finite provided that $p'/p \geq q'/q$. This is equivalent to $(p'-1) \geq (q'-1)$, that is, to $q \geq p$. Hence, whenever $q \geq p$ we have $S_w(p) \subseteq S_w(q)$, which clearly implies $S_w(q) \neq \emptyset$ and $s_q \leq s_p$.

(iv) Follows immediately from part (iii). □

Define $\Sigma := \{1/n : n \in \mathbb{N}\}$ and let $\Sigma_0 := \{0\} \cup \{1/n : n \in \mathbb{N}\}$ be its closure. The following inequalities will be needed later.

Lemma 3.2.

(i) *Let $\lambda \in \mathbb{C} \setminus \Sigma_0$ and set $\alpha := \mathrm{Re}(1/\lambda)$. Then there exist constants $d > 0$ and $D > 0$ (depending on α) such that*

$$\frac{d}{n^{\alpha}} \leq \prod_{k=1}^{n} \left|1 - \frac{1}{k\lambda}\right| \leq \frac{D}{n^{\alpha}}, \quad n \in \mathbb{N}. \tag{3.2}$$

(ii) *For each $m \in \mathbb{N}$,*

$$\frac{(n-1)!}{(n-m)!} \simeq n^{m-1} \quad \textit{for all large } n \in \mathbb{N}. \tag{3.3}$$

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(iii) *Let* $1 < p < \infty$ *and* $w = (w(n))_{n\in\mathbb{N}}$ *be a positive decreasing sequence. Then*

$$(n^m w(n))_{n\in\mathbb{N}} \in \ell_p \quad \forall m \in \mathbb{N} \tag{3.4}$$

if and only if

$$(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p \quad \forall m \in \mathbb{N}. \tag{3.5}$$

Proof. (i) The inequalities in (3.2) follow as in [18, proof of Lemma 7], where the restriction $\alpha < 1$ is assumed. Indeed, with $(1/\lambda) = \alpha + i\beta$ (for $\alpha, \beta \in \mathbb{R}$), and using $1 + x \le e^x$ for $x > 0$,

$$\begin{aligned}\prod_{k=1}^{n}\left|1-\frac{1}{k\lambda}\right| &= \prod_{k=1}^{n}\left(1-\frac{2\alpha}{k}+\frac{\alpha^2+\beta^2}{k^2}\right)^{1/2}\\ &\le \exp\sum_{k=1}^{n}\left(-\frac{\alpha}{k}+\frac{C}{k^2}\right) \le \exp(-\alpha\log(n)+\nu) \le \frac{D}{n^\alpha}.\end{aligned}$$

An application of Taylor's formula to $x \mapsto (1+x)^{-1/2}$ for $x > -1$ yields

$$\begin{aligned}\prod_{k=1}^{n}\left|1-\frac{1}{k\lambda}\right|^{-1} &= \prod_{k=1}^{n}\left(1-\frac{2\alpha}{k}+\frac{\alpha^2+\beta^2}{k^2}\right)^{-1/2} \le \prod_{k=1}^{n}\left(1+\frac{\alpha}{k}+\frac{C'}{k^2}\right)\\ &\le \exp\sum_{k=1}^{n}\left(\frac{\alpha}{k}+\frac{C'}{k^2}\right) \le \exp(\alpha\log(n)+\nu') = d^{-1}n^\alpha.\end{aligned}$$

(ii) Fix $m \in \mathbb{N}$. Then, for all large $n > m$,

$$\frac{(n-1)!}{(n-m)!} = (n-1)\cdots(n-m+1) = n^{m-1}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \simeq n^{m-1}.$$

(iii) Suppose that (3.4) holds. Fix $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ satisfy $k \ge (2 + mp)$. Since $(n^k w(n))_{n\in\mathbb{N}} \in \ell_p$, there exists $N \in \mathbb{N}$ such that

$$w(n) \le \frac{1}{n^k} \le \frac{1}{n^{2+mp}}, \qquad n > N.$$

It follows that

$$\sum_{n=1}^{\infty} n^{mp} w(n) \le \sum_{n=1}^{N} n^{mp} w(n) + \sum_{n=N+1}^{\infty} n^{mp}\left(\frac{1}{n^{2+mp}}\right) < \infty,$$

that is, $(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$. Accordingly, (3.5) is satisfied.

Conversely, suppose that (3.5) holds. Since $(nw(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$, there exists $K \in \mathbb{N}$ such that $w(n) \le 1$ for $n \ge K$ and hence $w(n) \le w(n)^{1/p}$ for $n \ge K$. Fix $m \in \mathbb{N}$. Then $n^m w(n) \le n^m w(n)^{1/p}$ for $n \ge K$. Since $(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$, we can conclude that also $(n^m w(n))_{n\in\mathbb{N}} \in \ell_p$. Hence, (3.4) is satisfied.

This concludes the proof. □

If $S_w(p) \neq \emptyset$, then $s_p \geq 1$ and so $p'/2s_p \leq p'/2$, which is relevant for the following results. Also relevant is that $\|\mathsf{C}^{(p,w}\| < p'$ is possible; see Section 2.

We now come to the main result of this section.

THEOREM 3.3. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a positive decreasing sequence.*

(i) *Suppose that $S_w(p) \neq \emptyset$ for some $1 < p < \infty$. Then for the dual operator $(\mathsf{C}^{(p,w)})' \in \mathcal{L}((\ell_p(w))')$ of $\mathsf{C}^{(p,w)}$,*

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}\right\} \cup \Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})') \tag{3.6}$$

and

$$\sigma_{pt}((\mathsf{C}^{(p,w)})') \setminus \Sigma \subseteq \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\}. \tag{3.7}$$

For the Cesàro operator $\mathsf{C}^{(p,w)}$ itself,

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\} \cup \Sigma \subseteq \sigma(\mathsf{C}^{(p,w)}) \tag{3.8}$$

and

$$\sigma(\mathsf{C}^{(p,w)}) \subseteq \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\} \cap \left\{\lambda \in \mathbb{C} : |\lambda| \leq \|\mathsf{C}^{(p,w)}\|\right\}. \tag{3.9}$$

(ii) *Suppose that $R_w \neq \mathbb{R}$, that is, $t_0 < \infty$. Then, for every $1 < p < \infty$,*

$$\left\{\frac{1}{m} : m \in \mathbb{N},\ 1 \leq m < \frac{t_0}{p} + 1\right\} \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \left\{\frac{1}{m} : m \in \mathbb{N},\ 1 \leq m \leq \frac{t_0}{p} + 1\right\}. \tag{3.10}$$

If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(\mathsf{C}^{(p,w)}) = \Sigma \quad \forall 1 < p < \infty. \tag{3.11}$$

PROOF. The proof is via a series of steps.

(i) By Proposition 2.2, $\mathsf{C}^{(p,w)} \in \mathcal{L}(\ell_p(w))$ with $\|\mathsf{C}^{(p,w)}\| \leq p'$. The dual operator $A := (\mathsf{C}^{p,w})' \in \mathcal{L}(\ell_{p'}(w^{-p'/p}))$ also satisfies $\|A\| \leq p'$ and is given by

$$Ay = \left(\sum_{k=n}^{\infty} \frac{y_k}{k}\right)_{n\in\mathbb{N}}, \quad y = (y_n)_{n\in\mathbb{N}} \in \ell_{p'}(w^{-p'/p}).$$

Step 1. We show $0 \notin \sigma_{pt}(A)$.

Observe that $Ay = 0$ for some $y \in \ell_{p'}(w^{-p'/p})$ implies that $z_n := \sum_{k=n}^{\infty}(y_k/k) = 0$ for all $n \in \mathbb{N}$. Hence, $y_n = n(z_n - z_{n+1}) = 0$ for $n \in \mathbb{N}$, and so A is injective.

Step 2. We show $\Sigma \subseteq \sigma_{pt}(A)$.

Let $\lambda \in \Sigma$, that is, $\lambda = 1/m$ for some $m \in \mathbb{N}$. Via (3.12) below, the nonzero vector $y = (y_n)_{n\in\mathbb{N}}$, defined via $y_1 \in \mathbb{C} \setminus \{0\}$ arbitrary, $y_n := y_1 \prod_{k=1}^{n-1}(1 - (1/\lambda k))$ for $1 < n \leq m$ and $y_n := 0$ for $n > m$, which belongs to $\ell_{p'}(w^{-p'/p})$, satisfies $Ay = \lambda y$.

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Step 3. We show $\{\lambda \in \mathbb{C} : |\lambda - (p'/2s_p)| < p'/2s_p\} \subseteq \sigma_{pt}(A)$.

Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then $Ay = \lambda y$ for some nonzero $y \in \ell_{p'}(w^{-p'/p})$ if and only if $\lambda y_n = \sum_{k=n}^{\infty} (y_k/k)$ for all $n \in \mathbb{N}$. This yields, for every $n \in \mathbb{N}$, that $\lambda(y_n - y_{n+1}) = y_n/n$ and so $y_{n+1} = (1 - (1/\lambda n))y_n$. It follows that

$$y_{n+1} = y_1 \prod_{k=1}^{n} \Big(1 - \frac{1}{\lambda k}\Big), \quad n \in \mathbb{N}, \tag{3.12}$$

with $y_1 \neq 0$. In particular, each eigenvalue of A is simple.

Now let $\lambda \in \mathbb{C} \setminus \Sigma$ satisfy $|\lambda - (p'/2s_p)| < p'/2s_p$ (equivalently, $\alpha := \mathrm{Re}(1/\lambda) > s_p/p'$, that is, $\alpha p' = \mathrm{Re}(p'/\lambda) > s_p$). For such a λ, the vector $y = (y_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ defined by (3.12) actually belongs to $\ell_{p'}(w^{-p'/p})$. Indeed, via Lemma 3.2(i) there exists $c = c(\lambda) > 0$ such that

$$\prod_{k=1}^{n} \Big|1 - \frac{1}{\lambda k}\Big|^{p'} \leq c n^{-\mathrm{Re}(p'/\lambda)}, \quad n \in \mathbb{N}.$$

It then follows from (3.12) that

$$\begin{aligned} \sum_{n=1}^{\infty} |y_n|^{p'} w(n)^{-p'/p} &= |y_1|^{p'} w(1)^{-p'/p} + |y_1|^{p'} \sum_{n=2}^{\infty} \prod_{k=1}^{n} \Big|1 - \frac{1}{\lambda k}\Big|^{p'} w(n)^{-p'/p} \\ &\leq |y_1|^{p'} w(1)^{-p'/p} + c|y_1|^{p'} \sum_{n=2}^{\infty} n^{-\mathrm{Re}(p'/\lambda)} w(n)^{-p'/p}, \end{aligned}$$

where the series $\sum_{n=2}^{\infty} n^{-\mathrm{Re}(p'/\lambda)} w(n)^{-p'/p}$ converges because $\mathrm{Re}(p'/\lambda) \in S_w(p)$, that is, $y \in \ell_{p'}(w^{-p'/p})$. Hence, $\lambda \in \sigma_{pt}(A)$.

Step 4. We show $\sigma_{pt}(A) \setminus \Sigma_0 \subseteq \{\lambda \in \mathbb{C} : |\lambda - (p'/2s_p)| \leq p'/2s_p\}$.

Fix $\lambda \in \sigma_{pt}(A) \setminus \Sigma_0$. According to (3.2), there exists $\beta = \beta(\lambda) > 0$ such that

$$\prod_{k=1}^{n} \Big|1 - \frac{1}{\lambda k}\Big|^{p'} \geq \beta \cdot n^{-\mathrm{Re}(p'/\lambda)}, \quad n \in \mathbb{N}. \tag{3.13}$$

But, as argued in Step 2 (for any $y_1 \in \mathbb{C} \setminus \{0\}$), the eigenvector $y = (y_n)_{n\in\mathbb{N}}$ corresponding to the eigenvalue λ of A, which necessarily belongs to $\ell_{p'}(w^{-p'/p})$, that is, $\sum_{n=1}^{\infty} |y_n|^{p'} w(n)^{-p'/p} < \infty$, is given by (3.12). Then (3.13) implies that also $\sum_{n=1}^{\infty} (1/n^{\mathrm{Re}(p'/\lambda)} w(n)^{p'/p}) < \infty$, that is, $\mathrm{Re}(p'/\lambda) \in S_w(p)$, and so $\mathrm{Re}(p'/\lambda) \geq s_p$. Equivalently, $\mathrm{Re}(1/\lambda) \geq s_p/p'$, that is, $\lambda \in \{\mu \in \mathbb{C} : |\mu - (p'/2s_p)| \leq p'/2s_p\}$.

It is clear that Steps 1–4 establish the two containments in (3.6) and (3.7).

For every $T \in \mathcal{L}(X)$ with X a Banach space, it is known that $\sigma_{pt}(T') \subseteq \sigma(T)$ [10, page 581] with $\sigma(T)$ a closed subset of $\mathbb{C}$. Accordingly, (3.8) follows from (3.6).

Step 5. We show $\sigma(\mathsf{C}^{(p,w)}) \subseteq \{\lambda \in \mathbb{C} : |\lambda - (p'/2)| \leq p'/2\}$.

It suffices to show that every $\lambda \in \mathbb{C}$ with $|\lambda - (p'/2)| > p'/2$ belongs to $\rho(\mathsf{C}^{(p,w)})$. To do this, we argue as in [7]. We recall the formula for $(\mathsf{C} - \lambda I)^{-1} : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ whenever

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$\lambda \notin \Sigma_0$ [18, page 266]: for $n \in \mathbb{N}$, the nth row of the matrix for $(\mathsf{C} - \lambda I)^{-1}$ has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^{n}(1 - \frac{1}{\lambda k})}, \quad 1 \le m < n,$$
$$\frac{n}{1 - n\lambda} = \frac{1}{\frac{1}{n} - \lambda}, \quad m = n,$$

and all the other entries in row n are equal to 0. So, we can write

$$(\mathsf{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda, \tag{3.14}$$

where the diagonal operator $D_\lambda = (d_{nm})_{n,m \in \mathbb{N}}$ is given by $d_{nn} := 1/((1/n) - \lambda)$ and $d_{nm} := 0$ if $n \neq m$. The operator $E_\lambda = (e_{nm})_{n,m \in \mathbb{N}}$ is then the lower triangular matrix with $e_{1m} = 0$ for all $m \in \mathbb{N}$ and every $n \ge 2$, with $e_{nm} := 1/n \prod_{k=m}^{n}(1 - (1/\lambda k))$ if $1 \le m < n$ and $e_{nm} := 0$ if $m \ge n$.

If $\lambda \notin \Sigma_0$, then $d(\lambda) := \mathrm{dist}(\lambda, \Sigma_0) > 0$ and $|d_{nn}| \le 1/d(\lambda)$ for $n \in \mathbb{N}$. Hence, for every $x \in \ell_p(w)$,

$$\begin{aligned}
\|D_\lambda(x)\|_{p,w} &= \Big(\sum_{n=1}^{\infty} |d_{nn} x_n|^p w(n)\Big)^{1/p} \\
&\le \frac{1}{d(\lambda)} \Big(\sum_{n=1}^{\infty} |x_n|^p w(n)\Big)^{1/p} = \frac{1}{d(\lambda)} \|x\|_{p,w}.
\end{aligned}$$

This means that $D_\lambda \in \mathcal{L}(\ell_p(w))$. So, by (3.14), it remains to show that $E_\lambda \in \mathcal{L}(\ell_p(w))$ whenever $\lambda \in \mathbb{C}$ satisfies $|\lambda - (p'/2)| > p'/2$. To this end, we note that if $\lambda \in \mathbb{C} \setminus \Sigma_0$ then with $\alpha := \mathrm{Re}(1/\lambda)$, it follows from (3.2) that

$$|e_{n1}| \le \frac{d^{-1}}{n^{1-\alpha}}, \quad n \ge 2,$$
$$|e_{nm}| \le \frac{d^{-1} D'}{n^{1-\alpha} m^{\alpha}}, \quad 2 \le m < n,$$

for some constants $d > 0$ and $D' > 0$ depending on λ. So, for every $\lambda \in \mathbb{C} \setminus \Sigma_0$, there exists $c = c(\lambda) > 0$ such that

$$|(E_\lambda(x))_n| \le c (G_\lambda(|x|))_n, \quad x \in \mathbb{C}^{\mathbb{N}}, \ n \in \mathbb{N}, \tag{3.15}$$

where $(G_\lambda(x))_n := \sum_{k=1}^{n} (x_k / n^{1-\alpha} k^{\alpha})$ with $\alpha := \mathrm{Re}(1/\lambda)$ and for all $x \in \mathbb{C}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Clearly, (3.15) implies that $E_\lambda \in \mathcal{L}(\ell_p(w))$ whenever $G_\lambda \in \mathcal{L}(\ell_p(w))$.

Claim. The operator $G_\lambda \in \mathcal{L}(\ell_p(w))$ whenever $\lambda \in \mathbb{C}$ satisfies $|\lambda - (p'/2)| > p'/2$.

To establish this claim, fix $\lambda \in \mathbb{C}$ with $|\lambda - (p'/2)| > p'/2$. Then, necessarily, $\lambda \notin \Sigma_0$ with $\alpha := \mathrm{Re}(1/\lambda) < 1/p'$ and so $(1 - \alpha)p > 1$. This implies that $\alpha < 1$. Observe that $G_\lambda \in \mathcal{L}(\ell_p(w))$ if and only if the operator $\tilde{G}_\lambda : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ given by

$$(\tilde{G}_\lambda(x))_n = w(n)^{1/p} \sum_{k=1}^{n} \frac{w(k)^{-1/p}}{n^{1-\alpha} k^{\alpha}} x_k, \quad x \in \mathbb{C}^{\mathbb{N}}, \ n \in \mathbb{N}$$

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defines a continuous linear operator on ℓ_p (the proof of this is along the lines of that of Lemma 2.1). To prove that indeed $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$, we need to distinguish the three cases (a) $\alpha = 0$, (b) $\alpha < 0$ and (c) $0 < \alpha < 1$ and establish relevant inequalities in each case.

Case (a). Since w is decreasing, for every $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} = \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}} \leq \frac{n}{w(n)^{1/(p-1)}}$$

and hence, for every $m \in \mathbb{N}$,

$$\sum_{n=1}^{m} \Big(\frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}} \Big)^p \leq \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)}}. \tag{3.16}$$

Case (b). Observe that for every $n \in \mathbb{N}$,

$$\begin{aligned} &\sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} \\ &\quad \leq \frac{1}{w(n)^{1/(p-1)}} \int_1^{n+1} x^{-\alpha p/(p-1)}\, dx \\ &\quad = \frac{1}{w(n)^{1/(p-1)}} \frac{((n+1)^{-(\alpha p/(p-1))+1} - 1)}{-\frac{\alpha p}{p-1} + 1} \\ &\quad \leq \frac{(p-1)}{(p(1-\alpha)-1)} \frac{(n+1)^{(p(1-\alpha)-1)/(p-1)}}{w(n)^{1/(p-1)}}. \end{aligned}$$

Setting $c := (p-1)/(p(1-\alpha)-1) > 0$ it follows, for every $m \in \mathbb{N}$, that

$$\begin{aligned} &\sum_{n=1}^{m} \Big(\frac{w(n)^{1/p}}{n^{1-\alpha}} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} \Big)^p \\ &\quad \leq c^p \sum_{n=1}^{m} \frac{(n+1)^{p(p(1-\alpha)-1)/(p-1)}}{w(n)^{1/(p-1)} n^{(1-\alpha)p}} \\ &\quad \leq 2^{p(p(1-\alpha)-1)/(p-1)} c^p \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)} n^{\alpha p/(p-1)}}. \end{aligned} \tag{3.17}$$

Case (c). For every $n \in \mathbb{N}$, still with $c = (p-1)/(p(1-\alpha)-1)$,

$$\begin{aligned} \sum_{k=2}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} &\leq \frac{1}{w(n)^{1/(p-1)}} \int_1^n \frac{1}{x^{\alpha p/(p-1)}}\, dx \\ &= \frac{c}{w(n)^{1/(p-1)}} (n^{(p(1-\alpha)-1)/(p-1)} - 1). \end{aligned}$$

Since $(1-\alpha)p > 1$ (that is, $(1-\alpha)p - 1 > 0$) and $\alpha p > 0$ with $1/w(1) \le 1/w(n)$, this implies, for every $n \in \mathbb{N}$, that

$$\begin{aligned}
&\Big(\frac{w(n)^{1/p}}{n^{1-\alpha}} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}}\Big)^p \\
&\quad \le \Big[\frac{w(n)^{1/p}}{n^{1-\alpha} w(1)^{1/(p-1)}} + \frac{w(n)^{1/p} c}{n^{1-\alpha} w(n)^{1/(p-1)}} (n^{(p(1-\alpha)-1)/(p-1)} - 1)\Big]^p \\
&\quad \le \Big[\frac{w(n)^{1/p}}{n^{1-\alpha} w(n)^{1/(p-1)}} + \frac{w(n)^{1/p} c}{n^{1-\alpha} w(n)^{1/(p-1)}} (n^{(p(1-\alpha)-1)/(p-1)} - 1)\Big]^p \\
&\quad = \Big[(1-c)\frac{w(n)^{1/p}}{n^{1-\alpha} w(n)^{1/(p-1)}} + \frac{w(n)^{1/p} c}{n^{1-\alpha} w(n)^{1/(p-1)}} n^{(p(1-\alpha)-1)/(p-1)}\Big]^p \\
&\quad = \Big(\frac{-\alpha p}{p(1-\alpha)-1} \frac{w(n)^{1/p}}{n^{1-\alpha} w(n)^{1/(p-1)}} + \frac{w(n)^{-1/p(p-1)} c}{n^{1-\alpha}} n^{(p(1-\alpha)-1)/(p-1)}\Big)^p \\
&\quad \le \Big(\frac{w(n)^{-1/p(p-1)} c}{n^{1-\alpha}} n^{(p(1-\alpha)-1)/(p-1)}\Big)^p \\
&\quad = c^p w(n)^{-1/(p-1)} n^{-\alpha p/(p-1)}.
\end{aligned}$$

Hence, for every $m \in \mathbb{N}$,

$$\sum_{n=1}^{m} \Big(\frac{w(n)^{1/p}}{n^{1-\alpha}} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}}\Big)^p \le c^p \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)} n^{\alpha p/(p-1)}}. \tag{3.18}$$

The inequalities (3.16)–(3.18) imply that $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$; indeed, in each case, suitable choices of a_n and b_k (with $p = q$) allow us to apply [5, Theorem 2(ii)]. This establishes the claim and hence also Step 5.

Step 6. We note that $\sigma(\mathsf{C}^{(p,w)}) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le \|\mathsf{C}^{(p,w)}\|\}$.

This is well known, [10, Ch. VII Lemma 3.4].

Steps 5 and 6 clearly yield (3.9). The proof of part (i) is thereby complete.

(ii) Suppose first that $R_w \neq \mathbb{R}$. Fix any $1 < p < \infty$.

Step 7. Both of the inclusions in (3.10) are valid.

The Cesàro operator $\mathsf{C}^{(p,w)}$ is clearly injective. So, $0 \notin \sigma_{pt}(\mathsf{C}^{(p,w)})$. Let $\lambda \in \mathbb{C} \setminus \{0\}$. Consider the equation $(\lambda I - \mathsf{C})x = 0$ with $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \setminus \{0\}$. Then $x_1 = \lambda x_1$ and $(2\lambda - 1)x_2 = x_1$ and $(n\lambda - 1)x_n = \lambda(n-1)x_{n-1}$ for all $n \ge 3$. If $m \in \mathbb{N}$ denotes the smallest positive integer such that $x_m \neq 0$, then it follows that $\lambda = 1/m$ and so $x_n = ((n-1)/(n-m))x_{n-1}$ for all $n > m$. Thus,

$$x_n = x_{m+(n-m)} = \frac{(n-1)!}{(m-1)!(n-m)!} x_m, \quad n \ge m. \tag{3.19}$$

According to (3.3), we have $((n-1)!/((m-1)!(n-m)!)) \simeq (1/(m-1)!)n^{m-1}$ for each $m \in \mathbb{N}$. So, $x \in \ell_p(w)$ if and only if the series $\sum_{n=m+1}^{\infty} n^{(m-1)p} w(n)$ converges. But the

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series $\sum_{n=m+1}^{\infty} n^{(m-1)p} w(n)$ converges precisely when $(m-1)p \in R_w$. In this case, $(m-1)p \leq t_0$, that is, $m \leq (t_0/p) + 1$. So, $\sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \{1/m : m \in \mathbb{N},\ 1 \leq m \leq (t_0/p) + 1\}$.

Conversely, if $m < (t_0/p) + 1$ for some $m \in \mathbb{N}$, that is, $(m-1)p < t_0$, then $(m-1)p \in R_w$ as $t_0 = \sup R_w$. Then the vector $x \in \mathbb{C}^{\mathbb{N}}$ defined according to (3.19), with $x_1 = \cdots = x_{m-1} = 0$ and for any arbitrary $x_m \neq 0$, belongs to $\ell_p(w)$. Therefore, $1/m \in \sigma_{pt}(\mathsf{C}^{(p,w)})$.

Step 8. Assume now that $R_w = \mathbb{R}$. Then (3.11) is valid.

Fix $1 < p < \infty$. As argued in Step 7, the point $1/m \in \sigma_{pt}(\mathsf{C}^{(p,w)})$ if and only if $(m-1)p \in R_w$. But, for $R_w = \mathbb{R}$, this is satisfied for every $m \in \mathbb{N}$ and so $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$. On the other hand, it is also shown in the proof of Step 7 that *every* eigenvalue λ of $\mathsf{C} : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ must have the form $\lambda = 1/m$ for some $m \in \mathbb{N}$. Since every eigenvalue of $\mathsf{C}^{(p,w)}$ is also an eigenvalue of C (as $\ell_p(w) \subseteq \mathbb{C}^{\mathbb{N}}$), it follows that $\sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \Sigma$. □

Remark 3.4.

(i) If $s_p \notin S_w(p)$ for some $1 < p < \infty$, then the argument of Step 4 in the proof of Theorem 3.3 implies that (3.6) reduces to the equality

$$\sigma_{pt}((\mathsf{C}^{(p,w)})') = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}\right\} \cup \Sigma.$$

Also, if $t_0 \notin R_w$, then (3.10) reduces to the equality

$$\sigma_{pt}(\mathsf{C}^{(p,w)}) = \left\{\frac{1}{m} : m \in \mathbb{N},\ 1 \leq m < \frac{t_0}{p} + 1\right\}, \quad 1 < p < \infty.$$

(ii) For $w(n) = 1$, for all $n \in \mathbb{N}$, in which case $\ell_p(w) = \ell_p$ and $s_p = 1$, we have that $\mathsf{C}^{(p,w)} = \mathsf{C}^{(p)}$ for all $1 < p < \infty$ with $\|\mathsf{C}^{(p,w)}\| = \|\mathsf{C}^{(p)}\| = p'$. Then (3.8) and (3.9) imply the known fact

$$\sigma(\mathsf{C}^{(p)}) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\}. \tag{3.20}$$

Since $t_0 = -1$, we also recover from (3.10) the known fact $\sigma_{pt}(\mathsf{C}^{(p)}) = \emptyset$.

(iii) According to (3.8), for w positive, decreasing and with $S_w(p) \neq \emptyset$,

$$\frac{p'}{s_p} \leq \max\left\{1, \frac{p'}{s_p}\right\} \leq \|\mathsf{C}^{(p,w)}\| \leq p'. \tag{3.21}$$

In particular, whenever $s_p = 1$ (see, for example, Example 3.5(i) below), the inequalities in (3.21) imply that necessarily $\|\mathsf{C}^{(p,w)}\| = p'$ is as large as possible. For the special case where $w(n) = 1/n^{\alpha}$, $n \in \mathbb{N}$, for some $\alpha > 0$, direct calculation yields $s_p = 1 + (\alpha p'/p)$ and so $S_w(p) \neq \emptyset$ for all $1 < p < \infty$. It follows that

$$\frac{p'}{s_p} = \frac{p}{\alpha + p - 1} = m_1,$$

where m_1 occurs in the lower bound for $\|\mathsf{C}^{(p,w)}\|$ as given in (2.4); see Proposition 2.3. Hence, (3.21) yields $m_1 \leq \|\mathsf{C}^{(p,w)}\|$. Combined with Example 2.4(iii) we can conclude that

$$\max\{m_1, m_2\} \leq \|\mathsf{C}^{(p,w)}\|.$$

This provides an alternate proof to that in [12] of the same estimate in (2.4).

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(iv) An examination of the argument for Step 2 in the proof of Theorem 3.3(i) shows that the assumption $S_w(p) \neq \emptyset$ is not used there, that is, we always have

$$\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})')$$

for every $1 < p < \infty$ and every positive decreasing weight w.

We now present some relevant examples.

Example 3.5.

(i) Suppose that $w(n) = 1/(\log(n+1))^\gamma$ for $n \in \mathbb{N}$ with $\gamma \geq 0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} < \infty$$

if and only if $s > 1$ and hence $s_p = 1$ for every $1 < p < \infty$. In view of Remark 3.4(iii) we have that $\|\mathsf{C}^{(p,w)}\| = p'$. Moreover, $\sum_{n=1}^{\infty} n^t w(n) < \infty$ if and only if $t < -1$ or $t \leq -1$ for $\gamma > 1$. Hence, $t_0 = -1$. According to Theorem 3.3, for each $1 < p < \infty$,

$$\sigma(\mathsf{C}^{(p,w)}) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\}, \quad \sigma_{pt}(\mathsf{C}^{(p,w)}) = \emptyset.$$

In particular, equality may occur in (3.9). For the case where $\gamma = 0$ (so that $w(n) = 1$ for $n \in \mathbb{N}$), we recover the known result about the spectrum of $\mathsf{C}^{(p)} \in \mathcal{L}(\ell_p)$ for $1 < p < \infty$ [6, 14].

(ii) More generally, suppose that $w(n) = 1/n^\beta(\log(n+1))^\gamma$ for $n \in \mathbb{N}$ with $\beta \geq 0$ and $\gamma \geq 0$. Then $\sum_{n=1}^{\infty} (1/n^s w(n)^{p'/p}) < \infty$ if and only if $s > (\beta p'/p) + 1$ and so $s_p = (\beta p'/p) + 1$ for every $1 < p < \infty$. Moreover, $\sum_{n=1}^{\infty} n^t w(n) < \infty$ if and only if $t < (\beta - 1)$ or $t \leq (\beta - 1)$ for $\gamma > 1$. Hence, $t_0 = \beta - 1$. According to Theorem 3.3, for each $1 < p < \infty$,

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2((\beta p'/p)+1)}\right| \leq \frac{p'}{2((\beta p'/p)+1)}\right\} \cup \Sigma \subseteq \sigma(\mathsf{C}^{(p,w)})$$

and

$$\sigma_{pt}(\mathsf{C}^{(p,w)}) = \left\{\frac{1}{m} : m \in \mathbb{N},\ 1 \leq m < \frac{\beta - 1}{p} + 1\right\}.$$

In particular, $\sigma_{pt}(\mathsf{C}^{(p,w)}) = \emptyset$ whenever $\beta \in [0, 1]$. We claim that actually

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2((\beta p'/p)+1)}\right| \leq \frac{p'}{2((\beta p'/p)+1)}\right\} \cup \Sigma = \sigma(\mathsf{C}^{(p,w)}), \tag{3.22}$$

which shows that equality may occur in (3.8).

Keeping in mind the argument for Step 5 in the proof of Theorem 3.3, to verify (3.22) it suffices to prove that every $\lambda \in \mathbb{C} \setminus \{0\}$ satisfying

$$|\lambda - (p'/2((\beta p'/p) + 1))| > p'/2((\beta p'/p) + 1)$$

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belongs to $\rho(\mathsf{C}^{(p,w)})$, that is, the operator $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$. So, fix such a λ and note that

$$\alpha := \operatorname{Re}\left(\frac{1}{\lambda}\right) < \left(\beta\frac{p'}{p} + 1\right)/p' = \frac{\beta}{p} + \frac{1}{p'}.$$

We also observe, for our particular w, that the operator $\tilde{G}_\lambda$ is given by

$$(\tilde{G}_\lambda(x))_n = \frac{1}{n^{1-\alpha+(\beta/p)}\log^{\gamma/p}(n+1)}\sum_{k=1}^{n}\frac{x_k}{k^{\alpha-(\beta/p)}\log^{-\gamma/p}(k+1)}$$

for $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. So, $\tilde{G}_\lambda$ is given by the factorable matrix with $a_n := n^{-(1-\alpha+(\beta/p))}\log^{-\gamma/p}(n+1)$ and $b_k := k^{-(\alpha-(\beta/p))}\log^{\gamma/p}(k+1)$, where $\alpha < (\beta/p) + (1/p') = (\beta/p) + 1 - (1/p)$ implies that $1 - \alpha + (\beta/p) > 1/p$ and

$$\left(1 - \alpha + \frac{\beta}{p}\right) + \left(\alpha - \frac{\beta}{p}\right) = 1 = \frac{1}{p} + \frac{1}{p'}$$

and also that $(\gamma/p) + (-\gamma/p) = 0$. According to [5, Corollary 9(ii)], it follows that $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$ and the claim is proved.

Finally, since $s_p = (\beta + p - 1)/(p - 1)$, it follows from (3.21) that

$$p' \cdot \frac{p-1}{\beta + p - 1} \le \|\mathsf{C}^{(p,w)}\| \le p', \quad 1 < p < \infty,$$

with $(p-1)/(\beta+p-1) \uparrow 1$ for $\beta \downarrow 0$. This example also shows that the inequality $t_0 \le s_p p'/p$ (cf. Proposition 3.1(i)) can be strict. For $\beta \downarrow 0$ it follows from (3.8) and (3.9) that

$$\sigma(\mathsf{C}^{(p,w)}) \uparrow \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2}\right| < \frac{p'}{2}\right\},$$

whose closure equals $\sigma(\mathsf{C}^{(p)}) = \sigma(\mathsf{C}^{(p,w)})$ for w as in (i).

It is clear from (3.10) that $\mathsf{C}^{(p,w)}$ has at most finitely many eigenvalues whenever $t_0 \in \mathbb{R}$. The following result characterizes the case where $\sigma_{pt}(\mathsf{C}^{(p,w)})$ is an *infinite set*; see also Remark 3.8(i) below. Recall that a sequence $u = (u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is *rapidly decreasing* if $(n^m u_n)_{n\in\mathbb{N}} \in \ell_1$ for every $m \in \mathbb{N}$. The space of all rapidly decreasing, $\mathbb{C}$-valued sequences is usually denoted by s.

Proposition 3.6. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a positive decreasing sequence.*

(i) *The following assertions are equivalent.*

(1) $R_w = \mathbb{R}$.
(2) $(n^m w(n))_{n\in\mathbb{N}} \in \ell_1$ *for all* $m \in \mathbb{N}$.
(3) $(n^m w(n))_{n\in\mathbb{N}} \in c_0$ *for all* $m \in \mathbb{N}$.
(4) $w \in s$.

(ii) *For each $1 < p < \infty$, the following assertions are equivalent.*

(5) $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$.
(6) $(n^m w(n))_{n\in\mathbb{N}} \in \ell_p$ *for all* $m \in \mathbb{N}$.

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(iii) *Any one of the equivalent assertions* (1)–(4) *implies that both* (5) *and* (6) *are valid for every* $1 < p < \infty$.
(iv) *If* (6) *holds for some* $1 < p < \infty$*, then each assertion* (1)–(4) *is satisfied.*

Proof. (i) That (1) if and only if (2) follows from the definition of R_w. That (2) implies (3) is immediate from $\ell_1 \subseteq c_0$. Assume (3). Fix $t \in \mathbb{N}$ and set $m = t + 2$. Then $(n^m w(n))_{n\in\mathbb{N}} \in c_0$ implies that $\sup_{n\in\mathbb{N}} n^m w(n) < \infty$. Accordingly,

$$\sum_{n=1}^{\infty} n^t w(n) = \sum_{n=1}^{\infty} \frac{1}{n^2} n^m w(n) \leq \frac{\pi^2}{6} \sup_{n\in\mathbb{N}} n^m w(n) < \infty.$$

Since t is arbitrary, we can conclude that (2) holds. That (2) if and only if (4) is clear from the definition of the space s.

(ii) Since $\mathsf{C}^{(p,w)}$ is injective, $0 \notin \sigma_{pt}(\mathsf{C}^{(p,w)})$. By (3.3) and (3.19), $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $\mathsf{C}^{(p,w)}$ if and only if $\lambda = 1/m$ for some $m \in \mathbb{N}$ with the corresponding one-dimensional eigenspace generated by a vector $x^{[m]} = (x_n^{[m]})_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ satisfying $x_n^{[m]} \simeq n^{m-1}$. So, $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$ if and only if $(n^{m-1})_{n\in\mathbb{N}} \in \ell_p(w)$ for all $m \in \mathbb{N}$, that is, if and only if $(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$ for all $m \in \mathbb{N}$, which is equivalent to (6) via Lemma 3.2(iii).

(iii) Follows immediately from parts (i) and (ii) and the fact that (2) implies (6), since $\ell_1 \subseteq \ell_p$ for every $1 < p < \infty$.

(iv) Immediate from $\ell_p \subseteq c_0$ for every $1 < p < \infty$. □

Given a decreasing sequence $w = (w(n))_{n\in\mathbb{N}}$ of positive real numbers, set $\alpha_n := -\log w(n)$ for $n \in \mathbb{N}$. Then $w(n) = e^{-\alpha_n}$ for $n \in \mathbb{N}$. Moreover, $\alpha_n \to \infty$ for $n \to \infty$ if and only if $w(n) \to 0$ for $n \to \infty$.

Corollary 3.7. *Let* $w = (w(n))_{n\in\mathbb{N}}$ *be a positive decreasing sequence.*

(i) *If* $w \in s$*, then* $\lim_{n\to\infty} (\log n)/\alpha_n = 0$.
(ii) *If* $\lim_{n\to\infty} (\log n)/\alpha_n = 0$ *and* $w(N) < 1$ *for some* N*, then* $w \in s$.

Proof. (i) Since $w \in s$, condition (3) in Proposition 3.6 implies that

$$\forall m \in \mathbb{N} \quad \exists n_m \in \mathbb{N} \quad \forall n \geq n_m : \ n^m w(n) = \frac{n^m}{e^{\alpha_n}} < 1,$$

that is, that

$$\forall m \in \mathbb{N} \quad \exists n_m \in \mathbb{N} \quad \forall n \geq n_m : \ n^m < e^{\alpha_n}.$$

It follows that

$$\forall m \in \mathbb{N} \quad \exists n_m \in \mathbb{N} \quad \forall n \geq n_m : \ m \log n < \alpha_n.$$

This implies that necessarily $\alpha_n > 0$ for all $n \geq n_m$, and so

$$\forall m \in \mathbb{N} \quad \exists n_m \in \mathbb{N} \quad \forall n \geq n_m : \ \frac{\log n}{\alpha_n} < \frac{1}{m}.$$

This means precisely that $\lim_{n\to\infty} (\log n)/\alpha_n = 0$.

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(ii) Fix $m \in \mathbb{N}$. Then there is $n_0 \in \mathbb{N}$ with $n_0 \geq N$ such that $(\log n)/\alpha_n < 1/(m+1)$ for all $n \geq n_0$. Since $w(N) < 1$ implies that $\alpha_n = -\log w(n) > 0$ for all $n \geq n_0$, we can conclude that $(m+1)\log n < \alpha_n$, that is, $n^{m+1}w(n) < 1$ for all $n \geq n_0$. So, $\sup_{n\in\mathbb{N}} n^{m+1}w(n) < \infty$. It follows that

$$n^m w(n) \leq \frac{1}{n} \sup_{r\in\mathbb{N}} r^{m+1} w(r), \quad n \in \mathbb{N},$$

with $(1/n)\sup_{r\in\mathbb{N}} r^{m+1}w(r) \to 0$ as $n \to \infty$. By the equivalence of (3) and (4) in Proposition 3.6(i), it follows that $w \in s$. □

Remark 3.8.

(i) Concerning condition (5) in Proposition 3.6 (for any given $1 < p < \infty$), we claim that the *entire* set $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$ whenever $\sigma_{pt}(C^{(p,w)})$ is an infinite set. To see this, suppose that $1/m \in \sigma_{pt}(C^{(p,w)})$ for some $m \in \mathbb{N}$. According to the argument in Step 7 of the proof of Theorem 3.3, we can conclude that $(n^{m-1})_{n\in\mathbb{N}} \in \ell_p(w)$. So, for all $1 \leq k < m$, it follows that

$$\sum_{n=1}^{\infty} (n^k)^p w(n) \leq \sum_{n=1}^{\infty} (n^{m-1})^p w(n) < \infty$$

and hence, via (3.3), that the vector $(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ given by (3.19), with k in place of m, also belongs to $\ell_p(w)$, that is, it is an eigenvector of $\mathsf{C}^{(p,w)}$ corresponding to $\lambda = 1/k$. This shows that $\{1/k\}_{k=1}^m \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$ whenever $1/m \in \sigma_{pt}(\mathsf{C}^{(p,w)})$, which clearly implies the stated claim.

(ii) Let $1 < p_0 < \infty$. The constant vector $\mathbf{1} := (1, 1, \ldots) \in \mathbb{C}^{\mathbb{N}}$ satisfies $\mathsf{C}\mathbf{1} = \mathbf{1}$ and so $1 \in \sigma_{pt}(C^{(p_0,w)})$ if and only if $\mathbf{1} \in \ell_{p_0}(w)$, that is, if and only if $w \in \ell_1$. In this case, $1 \in \sigma_{pt}(C^{(p,w)})$ for every $1 < p < \infty$. Then Theorem 3.3(ii) implies that necessarily $t_0 \in (0, \infty]$.

(iii) Let $w(n) = 1/n^\alpha$, for all $n \in \mathbb{N}$ and some $\alpha > 0$. Then $\sum_{n=1}^\infty n^t w(n) < \infty$ if and only if $t < (\alpha - 1)$, and so $t_0 = (\alpha - 1)$. In particular, $R_w \neq \mathbb{R}$. Moreover, for any $1 < p < \infty$,

$$\left\{\frac{1}{m} : m \in \mathbb{N},\ 1 \leq m < \frac{t_0}{p} + 1\right\} = \left\{\frac{1}{m} : m \in \mathbb{N},\ 1 \leq m < \frac{(\alpha-1)}{p} + 1\right\}.$$

So, given any $1 < p < \infty$, it is possible to choose an appropriate $\alpha > 0$ such that $\sigma_{pt}(\mathsf{C}^{(p,w)})$ is a *finite* set with any preassigned cardinality; see (3.10).

(iv) Condition (1) of Proposition 3.6, that is, $R_w = \mathbb{R}$, implies that necessarily $S_w(p) = \emptyset$ for every $1 < p < \infty$; see Proposition 3.1(i).

Let $w = (w(n))_{n\in\mathbb{N}}$ be any (strictly) positive decreasing sequence and let $1 < p < \infty$. The Cesàro operator $\mathsf{C}^{(p,w)}$ is similar (via an isometry) to an operator $T_w \in \mathcal{L}(\ell_p)$ which is defined by the factorable matrix $A(w) = (a_{nk})_{n,k\in\mathbb{N}}$ with entries $a_{nk} = a_n b_k = (w(n))^{1/p}/n) \cdot w(k)^{-1/p}$ for $1 \leq k \leq n$ and $a_{nk} = 0$ for $k > n$ (see the proof of Lemma 2.1). In particular, $\sigma(\mathsf{C}^{(p,w)}) = \sigma(T_w)$. Moreover, the matrix $A(w)$ satisfies the following two conditions:

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(i) $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| = \sup_{n\in\mathbb{N}} (w(n)^{1/p}/n) \sum_{k=1}^{n} w(k)^{-1/p} \leq 1$, because w decreasing implies that $\sum_{k=1}^{n} w(k)^{-1/p} \leq n w(n)^{-1/p}$, $n \in \mathbb{N}$; and

(ii) $f_k := \lim_{n\to\infty} a_{nk} = w(k)^{-1/p} \lim_{n\to\infty} (w(n)^{1/p}/n) = 0$, $k \in \mathbb{N}$, because $w \in \ell_\infty$.

If, in addition, the matrix $A(w)$ also satisfies the condition:

(iii) $\alpha := \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \lim_{n\to\infty} (w(n)^{1/p}/n) \sum_{k=1}^{n} w(k)^{-1/p}$ exists;

then the linear operator corresponding to $A(w)$ is a selfmap of c, the space of all convergent sequences, that is, $A(w)$ is conservative, [19, page 112].

Suppose now that the matrix $A(w)$ satisfies condition (iii) with $\alpha = 1$. Then $A(w)$ is *regular* and the linear operator corresponding to $A(w)$ is limit preserving over c, [19, page 114]. Define $\eta := \limsup_{n\to\infty} a_n b_n$. For the operator T_w (which is similar to the Cesàro operator $\mathsf{C}^{(p,w)}$) it turns out that $\eta = 0$ and so a result of Rhoades and Yildirim [19, Theorem 3] yields

$$\{\lambda \in \mathbb{C} : \left|\lambda - \tfrac{1}{2}\right| \leq \tfrac{1}{2}\} \subseteq \sigma(\mathsf{C}^{(p,w)}) \tag{3.23}$$

after noting that $S := \overline{\{a_n b_n : n \in \mathbb{N}\}} = \Sigma_0 \subseteq \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \leq \tfrac{1}{2}\}$.

It is worthwhile to compare (3.8) with (3.23). So, let $1 < p < \infty$ and let w be a positive decreasing sequence such that $S_w(p) \neq \emptyset$. Then

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\} \subseteq \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\} \subseteq \sigma(\mathsf{C}^{(p,w)})$$

with the first inclusion holding if and only if $s_p \leq p'$. Observe that if $(w(n)^{-1/p}/n)_{n\in\mathbb{N}} \in \ell_{p'}$, then $s_p \leq p'$ is valid and, conversely, if $s_p < p'$, then $(w(n)^{-1/p}/n)_{n\in\mathbb{N}} \in \ell_{p'}$. In this case, (3.8) is a better inclusion than (3.23). For instance, if $w(n) := 1/n^r$ for all $n \in \mathbb{N}$ and some $r > 0$, then $(w(n)^{-1/p}/n)_{n\in\mathbb{N}} \in \ell_{p'}$ if and only if $r < 1$. On the other hand, the reverse inclusion

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\} \subseteq \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

holds if and only if $p' \leq s_p$. Observe that if $(w(n)^{-1/p}/n)_{n\in\mathbb{N}} \notin \ell_{p'}$, then $p' \leq s_p$ is valid and, conversely, if $p' < s_p$, then $(w(n)^{-1/p}/n)_{n\in\mathbb{N}} \notin \ell_{p'}$. In this case, modulo the additional requirement that $\alpha = 1$ (see condition (iii)), in which case (3.23) is actually valid, we see that (3.23) is a better inclusion than (3.8).

The following example shows that condition (iii) above and the property $S_w(p) \neq \emptyset$ can be compatible.

EXAMPLE 3.9. Fix $1 < p < \infty$. For each $n \in \mathbb{N}$ set $w(n) = 1/(\log(n+1))^p$, in which case $w(n) \downarrow 0$. Then $S_w(p) = (1, \infty)$ and

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\} = \sigma(C^{(p,w)}) \quad \text{with } \sigma_{pt}(C^{(p,w)}) = \emptyset;$$

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see Example 3.5(i) with $\gamma = p$. Moreover, concerning condition (iii), observe that

$$\frac{w(n)^{1/p}}{n}\sum_{k=1}^{n} w(k)^{-1/p} = \frac{1}{n\log(n+1)}\sum_{k=1}^{n}\log(k+1), \quad n \in \mathbb{N}.$$

The inequalities

$$((n+1)\log(n+1) - n) \le \sum_{k=1}^{n}\log(k+1) \le ((n+2)\log(n+2) - n - 2\log 2), \quad n \in \mathbb{N}$$

then imply that

$$\alpha = \lim_{n\to\infty}\frac{w(n)^{1/p}}{n}\sum_{k=1}^{n} w(k)^{-1/p} = 1.$$

Note also that $(w(n)^{-1/p}/n)_{n\in\mathbb{N}} = ((\log(n+1))/n)_{n\in\mathbb{N}} \in \ell_{p'}$.

We conclude this section with some comments about the mean ergodicity and the linear dynamics of $\mathsf{C}^{(p,w)}$. For X a Banach space, recall that $T \in \mathcal{L}(X)$ is *mean ergodic* if its sequence of Cesàro averages $T_{[n]} := (1/n)\sum_{m=1}^{n} T^m$ for $n \in \mathbb{N}$ converges to some operator $P \in \mathcal{L}(X)$ for the strong operator topology, that is, $\lim_{n\to\infty} T_{[n]}x = Px$ for each $x \in X$, [10, Ch. VIII]. Since $(1/n)T^n = T_{[n]} - ((n-1)/n)T_{[n-1]}$ for $n \in \mathbb{N}$ (with $T_{[0]} := I$), a necessary condition for T to be mean ergodic is that $\lim_{n\to\infty}(1/n)T^n = 0$ (in the strong operator topology).

Let w be a positive decreasing sequence and let $1 < p < \infty$ with $S_p(w) \neq \emptyset$. If $s_p < p'$, then it follows from (3.6) that $\mu := \frac{1}{2}(1 + (p'/s_p)) \in \sigma_{pt}((\mathsf{C}^{(p,w)})')$ and so there exists a nonzero vector $x' \in \ell_{p'}(w^{-p'/p})$ such that $(\mathsf{C}^{(p,w)})'x' = \mu x'$. Choose any $x \in \ell_p(w) \setminus \{0\}$ satisfying $\langle x, x'\rangle \neq 0$. Then

$$\left\langle \frac{1}{n}(\mathsf{C}^{(p,w)})^n x, x'\right\rangle = \frac{1}{n}\langle x, ((\mathsf{C}^{(p,w)})')^n x'\rangle = \frac{\mu^n}{n}\langle x, x'\rangle, \quad n \in \mathbb{N}$$

with $\mu > 1$ and so the set $\{(1/n)(\mathsf{C}^{(p,w)})^n x : n \in \mathbb{N}\}$ is unbounded in $\ell_p(w)$. In particular, the sequence $\{(1/n)(\mathsf{C}^{(p,w)})^n\}_{n\in\mathbb{N}}$ cannot converge to 0 for the strong operator topology in $\mathcal{L}(\ell_p(w))$. Accordingly, $\mathsf{C}^{(p,w)}$ fails to be mean ergodic whenever $s_p < p'$. This is the case when $w(n) = 1$ for all $n \in \mathbb{N}$, in which case $s_p = 1$, and we recover the known fact that the classical Cesàro operator $\mathsf{C}^{(p)}$ fails to be mean ergodic for every $1 < p < \infty$; see [3, Section 4], where it is also shown that the Cesàro operator fails to be mean ergodic in the classical Banach sequence spaces c_0, c, ℓ_p $(1 < p \le \infty)$, bv_0 and bv, but that it is mean ergodic in bv_p $(1 < p < \infty)$. For w as in Example 3.5(i), we recall that, also, $s_p = 1$ for every $1 < p < \infty$, and so $\mathsf{C}^{(p,w)}$ is not mean ergodic.

Concerning the dynamics of a continuous linear operator T defined on a separable Banach space X, recall that T is *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X. If, for some $x \in X$, the projective orbit $\{\lambda T^n x : \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then X is said to be *supercyclic*. Clearly, hypercyclicity always implies supercyclicity.

Let now w be a positive decreasing sequence and let $1 < p < \infty$. According to Remark 3.4(iv), the *infinite* set $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})')$. Then, by a result of Ansari and Bourdon [4, Theorem 3.2], $\mathsf{C}^{(p,w)}$ is not supercyclic and hence also not hypercyclic.

4. Compactness of $\mathsf{C}^{(p,w)}$

According to (3.20), for each $1 < p < \infty$ the classical Cesàro operator $\mathsf{C}^{(p)} \in \mathcal{L}(\ell_p)$ is surely not compact. However, in the presence of a positive weight $w \downarrow 0$, this may no longer be the case for $\mathsf{C}^{(p,w)}$ acting on $\ell_p(w)$. We begin with the following fact.

PROPOSITION 4.1. *Let w be a positive decreasing weight.*

(i) *For every $1 < p < \infty$ we have $\Sigma \subseteq \sigma(\mathsf{C}^{(p,w)})$.*
(ii) *Suppose that $\mathsf{C}^{(p,w)}$ is a compact operator for some $1 < p < \infty$. Then*

$$\sigma(\mathsf{C}^{(p,w)}) = \Sigma_0 \quad \textit{and} \quad \sigma_{pt}(\mathsf{C}^{(p,w)}) = \Sigma. \tag{4.1}$$

Moreover, $w \in s$ and $r(\mathsf{C}^{(p,w)}) < \|\mathsf{C}^{(p,w)}\|$.

PROOF. (i) According to Remark 3.4(iv), we have $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})')$. But it is always the case that $\sigma_{pt}((\mathsf{C}^{(p,w)})') \subseteq \sigma(\mathsf{C}^{(p,w)})$ [10, page 581], and so $\Sigma \subseteq \sigma(\mathsf{C}^{(p,w)})$.

(ii) Since $\mathsf{C}^{(p,w)}$ is injective, $0 \notin \sigma_{pt}(\mathsf{C}^{(p,w)})$. The compactness of $\mathsf{C}^{(p,w)}$ then implies $\sigma_{pt}(\mathsf{C}^{(p,w)}) = \sigma(\mathsf{C}^{(p,w)}) \setminus \{0\}$ [15, Theorem 3.4.23]. According to the proof of Step 8 for Theorem 3.3, we also have that $\sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \Sigma$. In view of part (i), the equalities in (4.1) follow.

By Theorem 3.3(ii) we must have $R_w = \mathbb{R}$ (if not, then t_0 is finite and so (3.10) would imply that $\sigma_{pt}(\mathsf{C}^{(p,w)})$ is finite, which is a contradiction to (4.1)). Then, via Proposition 3.6(i), we can conclude that $w \in s$.

It follows from (2.3) and the equality $r(\mathsf{C}^{(p,w)}) = 1$ (see (4.1)) that $r(\mathsf{C}^{(p,w)}) < \|\mathsf{C}^{(p,w)}\|$. □

To decide when $\mathsf{C}^{(p,w)}$ is compact, first observe that $\mathsf{C}^{(p,w)} = \Phi_w^{-1} T_w \Phi_w$ (see Lemma 2.1 and its proof), where $T_w \in \mathcal{L}(\ell_p)$ is given by (2.2). Given any $x \in B_p := \{x \in \ell_p : \|x\| \leq 1\}$ and $i \in \mathbb{N}$, it follows from Hölder's inequality that

$$\begin{aligned}\sum_{n=i}^{\infty} |(T_w x)_n|^p &= \sum_{n=i}^{\infty} \frac{w(n)}{n^p} \Big| \sum_{k=1}^{n} \frac{1}{w(k)^{1/p}} \cdot x_k \Big|^p \\ &\leq \sum_{n=i}^{\infty} \frac{w(n)}{n^p} \Big(\sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \Big)^{p/p'}.\end{aligned}$$

So, T_w (and hence also $\mathsf{C}^{(p,w)}$) will be compact whenever w satisfies the following compactness criterion:

$$\sum_{n=1}^{\infty} \frac{w(n)}{n^p} \Big(\sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \Big)^{p/p'} < \infty. \tag{4.2}$$

Indeed, (4.2) implies that $\lim_{i\to\infty} \sum_{n=i}^{\infty} |(T_w x)_n|^p = 0$ *uniformly* with respect to $x \in B_p$, from which the relative compactness in ℓ_p of the bounded set $T_w(B_p) \subseteq \ell_p$ follows, [10, pages 338–339].

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We introduce some notation. Let w be a positive decreasing sequence. Define

$$A_n(p,w) := w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}}, \quad n \in \mathbb{N},\ 1 < p < \infty.$$

The compactness criterion (4.2) then states that $\mathsf{C}^{(p,w)}$ is a compact operator if $\sum_{n=1}^{\infty}((A_n(p,w))^{p/p'}/n^p) < \infty$.

THEOREM 4.2. *Suppose, for some $1 < p < \infty$, that there exist constants $M > 0$ and $0 \le \alpha < 1$ such that*

$$A_n(p,w) \le Mn^{\alpha}, \quad n \in \mathbb{N}.$$

Then $\mathsf{C}^{(q,w)}$ is a compact operator for every $1 < q \le p$. In particular, $w \in s$.

PROOF. Observe, for fixed $1 < q \le p$, that

$$\gamma := \frac{q'}{q} - \frac{p'}{p} = \frac{1}{q-1} - \frac{1}{p-1} = \frac{p-q}{(q-1)(p-1)} \ge 0.$$

For each $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{1}{w(k)^{q'/q}} = \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot w(k)^{-\gamma}.$$

Accordingly, for each $n \in \mathbb{N}$,

$$\begin{aligned} A_n(q,w) &= \frac{w(n)^{q'/q}}{w(n)^{p'/p}} \cdot w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot w(k)^{-\gamma} \\ &= w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot \left(\frac{w(n)}{w(k)}\right)^{\gamma}. \end{aligned}$$

Since w is decreasing, $w(n)/w(k) \le 1$ for all $1 \le k \le n$ and so

$$A_n(q,w) \le w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} = A_n(p,w) \le Mn^{\alpha}.$$

Accordingly,

$$\sum_{n=1}^{\infty} \frac{(A_n(q,w))^{q/q'}}{n^q} \le M^{q/q'} \sum_{n=1}^{\infty} \frac{n^{\alpha q/q'}}{n^q} = M^{q/q'} \sum_{n=1}^{\infty} \frac{1}{n^{q-(\alpha q/q')}}.$$

But $q - \alpha q/q' = q - \alpha(q-1) = q(1-\alpha) + \alpha > (1-\alpha) + \alpha = 1$ and so

$$\sum_{n=1}^{\infty} \frac{(A_n(q,w))^{q/q'}}{n^q} < \infty.$$

Then the compactness criterion yields that $\mathsf{C}^{(q,w)}$ is a compact operator.

That $w \in s$ is a consequence of Proposition 4.1(ii). $\square$

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The following consequence of Theorem 4.2 leads to a rich supply of weights w for which $\mathsf{C}^{(p,w)}$ is compact.

Corollary 4.3. *Let w be a positive weight with $w \downarrow 0$. If the limit*

$$l := \lim_{n\to\infty} \frac{w(n)}{w(n-1)} \tag{4.3}$$

exists in $\mathbb{R} \setminus \{1\}$, then $\mathsf{C}^{(p,w)}$ is compact for every $1 < p < \infty$.

Proof. Fix $1 < p < \infty$. According to Theorem 4.2 (with $\alpha = 0$), it suffices to prove that $\sup_{n\in\mathbb{N}} A_n(p, w) < \infty$. Set $a_n := \sum_{k=1}^{n} w(k)^{-p'/p}$ and $b_n := w(n)^{-p'/p}$ for $n \in \mathbb{N}$. Since $w \downarrow 0$, we have $b_n \uparrow \infty$. Moreover, the limit

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} &= \lim_{n\to\infty} \frac{w(n)^{-p'/p}}{w(n)^{-p'/p} - w(n-1)^{-p'/p}} \\ &= \lim_{n\to\infty} \frac{1}{1 - (w(n)/w(n-1))^{p'/p}} = \frac{1}{1 - l^{p'/p}}\end{aligned}$$

exists in $\mathbb{R}$ as $l \neq 1$. According to the Stolz–Cesàro criterion [16, Theorem 1.22], it follows that $\lim_{n\to\infty} (a_n/b_n) = 1/(1 - l^{p'/p}) \in \mathbb{R}$, that is, $\lim_{n\to\infty} A_n(p, w) = 1/(1 - l^{p'/p}) \in \mathbb{R}$. In particular, $\sup_{n\in\mathbb{N}} A_n(p, w) < \infty$ is indeed satisfied. □

Remark 4.4.

(i) Let w be a positive decreasing weight.

  (a) According to (3.8), if $\mathsf{C}^{(p,w)}$ is a compact operator for some $1 < p < \infty$, then $S_w(p) = \emptyset$.

  (b) The condition $w \downarrow 0$ by itself need not imply that $S_w(p) = \emptyset$ (see, for instance, Example 3.5).

(ii) Suppose $S_w(p) \neq \emptyset$ for some $1 < p < \infty$. Then $\mathsf{C}^{(q,w)}$ fails to be compact for every $q \in [p, \infty)$. This follows from part (i)(a) and Proposition 3.1(iii).

(iii) The following examples (a)–(c) all fall within the scope of Corollary 4.3. So, in each case, $w \in s$ and the identities in (4.1) hold; see Proposition 4.1.

  (a) For any fixed $a > 1$ and $r \geq 0$ set $w(n) := n^r/a^n$ for $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{w(n)}{w(n-1)} = a^{-1} \neq 1.$$

  (b) For any fixed $a \geq 1$, the weight $w(n) := a^n/n!$ for $n \in \mathbb{N}$ satisfies

$$\lim_{n\to\infty} \frac{w(n)}{w(n-1)} = 0 \neq 1. \tag{4.4}$$

  (c) The weight $w(n) := 1/n^n$ for $n \in \mathbb{N}$ also satisfies (4.4).

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We point out, since w is decreasing, that $w(n)/w(n-1) \in (0,1]$ for all $n \in \mathbb{N}$. Hence, whenever the limit (4.3) exists, then necessarily $l \in [0,1]$.

As an application, suppose that the positive decreasing weight w has the property that $l := \lim_{n\to\infty}(w(n)/w(n-1))$ exists in $[0,1)$. Then, for each $r > 0$, the positive decreasing weight $w^r : n \mapsto w(n)^r$ for $n \in \mathbb{N}$ satisfies $\lim_{n\to\infty}(w(n)^r/w(n-1)^r) = l^r \in [0,1)$. Hence, $\mathsf{C}^{(p,w^r)}$ is a compact operator in $l_p(w^r)$ for every $1 < p < \infty$.

(iv) The following criterion is sufficient to ensure that the limit (4.3) exists in $\mathbb{R} \setminus \{1\}$. Hence, both Proposition 4.1 and Corollary 4.3 are applicable to such a weight w. In particular, $w \in s$.

Let $\beta = (\beta_n)_{n\in\mathbb{N}}$ be a positive increasing sequence with $\beta \uparrow \infty$ such that $\lim_{n\to\infty}(\beta_n - \beta_{n-1}) = \infty$. Then the weight $w(n) := e^{-\beta_n}$ for $n \in \mathbb{N}$ satisfies $l := \lim_{n\to\infty}(w(n)/w(n-1)) = 0 \neq 1$.

It is routine to verify that $\lim_{n\to\infty}(w(n)/w(n-1)) = 0$.

For the weight $w(n) := a^{-n}$ for $n \in \mathbb{N}$ (with $a > 1$), we have that $\beta_n := -\log w(n) = n\log(a) \uparrow \infty$, but $(\beta_n - \beta_{n-1})\log(a) \not\to \infty$ for $n \to \infty$. So, the above criterion is *not* applicable to this weight. However, according to part (iii)(a) of this remark (with $r = 0$), the weight w is admissible for Corollary 4.3.

The following examples illustrate that Theorem 4.2 is more general than Corollary 4.3.

Example 4.5.

(i) Fix $0 < \beta < 1$ and set $w_\beta(n) := e^{-n^\beta}$ for $n \in \mathbb{N}$, in which case $w_\beta \downarrow 0$, but

$$\lim_{n\to\infty} \frac{w_\beta(n)}{w_\beta(n-1)} = \lim_{n\to\infty} e^{(n-1)^\beta - n^\beta} = \lim_{n\to\infty} e^{-\beta/n^{(1-\beta)}} = 1,$$

as $(n-1)^\beta - n^\beta = n^\beta[(1-1/n)^\beta - 1] = n^\beta[1 - \beta/n + o(1/n) - 1] \simeq -\beta/n^{1-\beta}$ for $n \to \infty$, so Corollary 4.3 is not applicable. We show that Theorem 4.2 does apply.

Fix $1 < p < \infty$ and set $\gamma := p'/p$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned} A_n(p, w_\beta) &= e^{-\gamma n^\beta} \sum_{k=1}^{n} e^{\gamma k^\beta} \le e^{-\gamma n^\beta} \int_1^{n+1} e^{\gamma x^\beta}\,dx \\ &= \frac{e^{-\gamma n^\beta}}{\beta} \int_1^{(n+1)^\beta} e^{\gamma t} t^{(1/\beta)-1}\,dt \le \frac{e^{-\gamma n^\beta}}{\beta} \int_1^{(n+1)^\beta} e^{\gamma t} t^m\,dt, \end{aligned}$$

where $m \in \mathbb{N}$ is chosen minimally such that $(m-1) < (1/\beta) - 1 \le m$. Integration by parts $(m+1)$ times yields

$$\begin{aligned} \int_1^{(n+1)^\beta} e^{\gamma t} t^m\,dt \le a_0 &+ a_1(n+1)^\beta e^{\gamma(n+1)^\beta} + a_2(n+1)^{2\beta} e^{\gamma(n+1)^\beta} \\ &+ \cdots + a_m(n+1)^{m\beta} e^{\gamma(n+1)^\beta} \end{aligned}$$

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for positive constants $a_0, a_1, \dots, a_m$. It follows that

$$\int_1^{(n+1)^\beta} e^{\gamma t} t^m \, dt \le M(1+n)^{m\beta} e^{\gamma(1+n)^\beta}, \quad n \in \mathbb{N},$$

for some constant $M > 0$. Accordingly,

$$A_n(p, w_\beta) \le \frac{M}{\beta}(1+n)^{m\beta} e^{\gamma((1+n)^\beta - n^\beta)}, \quad n \in \mathbb{N}.$$

Since $(n+1)^\beta - n^\beta \simeq \beta/n^{1-\beta}$ and $(1+n)^{m\beta} \simeq n^{m\beta}$ for $n \to \infty$, there exists $K > 0$ (independent of n) such that

$$A_n(p, w_\beta) \le K n^{m\beta}, \quad n \in \mathbb{N}.$$

Since $(m-1) < (1/\beta) - 1$ implies that $\alpha := m\beta \in (0,1)$, Theorem 4.2 yields that $\mathsf{C}^{(p,w_\beta)}$ is compact.

For $\beta \ge 1$, the compactness of $\mathsf{C}^{(p,w_\beta)}$ follows from Corollary 4.3. Indeed, if $\beta = 1$, then $w_\beta(n) = e^{-n}$ for $n \in \mathbb{N}$ and so Remark 4.4(iii)(a) implies the compactness of $\mathsf{C}^{(p,w_\beta)}$. For $\beta > 1$, observe from above that

$$\lim_{n\to\infty} \frac{w_\beta(n)}{w_\beta(n-1)} = \lim_{n\to\infty} e^{(n-1)^\beta - n^\beta} = \lim_{n\to\infty} e^{-\beta n^{\beta-1}} = 0$$

and so the compactness of $\mathsf{C}^{(p,w_\beta)}$ follows again from Corollary 4.3.

(ii) There also exist positive decreasing weights $w \in s$ such that the sequence $\{w(n)/w(n-1)\}_{n\in\mathbb{N}}$ fails to converge at all, yet $\mathsf{C}^{(p,w)}$ is a compact operator for every $1 < p < \infty$.

Define $w(n) := 1/j^j$, $n = 2j-1$, and $w(n) := 1/2j^j$, $n = 2j$, for each $j \in \mathbb{N}$. Then w is (strictly) decreasing to 0. For $n_j := 2j$, $j \in \mathbb{N}$, we have $w(n_j)/w(n_j - 1) = \frac{1}{2}$ for all $j \in \mathbb{N}$ and so $\lim_{j\to\infty}(w(n_j)/w(n_j-1)) = \frac{1}{2}$, whereas for $n_r := 2r+1$, $r \in \mathbb{N}$, the subsequence $\{w(n_r)/w(n_r-1)\}_{r\in\mathbb{N}}$ of $\{w(n)/w(n-1)\}_{n\in\mathbb{N}}$ converges to 0. Accordingly, the sequence $\{w(n)/w(n-1)\}_{n\in\mathbb{N}}$ is not convergent and so Corollary 4.3 is not applicable.

Fix $1 < p < \infty$ and set $\gamma := p'/p > 0$. To establish the compactness of $\mathsf{C}^{(p,w)}$, observe, for every $j \in \mathbb{N}$, that

$$A_{2j}(p, w) = \frac{1}{(2j^j)^\gamma}\Big(\sum_{k=1}^{j}(k^k)^\gamma + \sum_{k=1}^{j}(2k^k)^\gamma\Big) = \frac{1+2^\gamma}{2^\gamma}\frac{1}{(j^j)^\gamma}\sum_{k=1}^{j}(k^k)^\gamma \tag{4.5}$$

and that

$$A_{2j-1}(p, w) = 1 + \frac{1}{(j^j)^\gamma}\sum_{k=1}^{2(j-1)} w(k)^{-\gamma} = 1 + \frac{(j-1)^{(j-1)\gamma}}{(j^j)^\gamma} A_{2(j-1)}(p, w), \tag{4.6}$$

with $\lim_{j\to\infty}((j-1)^{(j-1)\gamma}/(j^j)^\gamma) = 0$. Set $a_j := \sum_{k=1}^{j}(k^k)^\gamma$ and $b_j := (j^j)^\gamma$ for $j \in \mathbb{N}$. Then $b_j \uparrow \infty$. Moreover,

$$\lim_{j\to\infty}\frac{a_j - a_{j-1}}{b_j - b_{j-1}} = \lim_{j\to\infty}\frac{(j^j)^\gamma}{(j^j)^\gamma - ((j-1)^{j-1})^\gamma} = \lim_{j\to\infty}\frac{1}{1 - \frac{(j-1)^{(j-1)\gamma}}{(j^j)^\gamma}} = 1.$$

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According to the Stolz–Cesàro criterion [16, Theorem 1.22], it follows that, also, $\lim_{j\to\infty}(a_j/b_j) = 1$. So, via (4.5) and (4.6), we obtain $\lim_{j\to\infty} A_{2j}(p,w) = (1+2^\gamma)/2^\gamma$ and $\lim_{j\to\infty} A_{2j-1}(p,w) = 1$. In particular, $\sup_{i\in\mathbb{N}} A_i(p,w) < \infty$ and so Theorem 4.2 applies (with $\alpha = 0$). Hence, $\mathsf{C}^{(p,w)}$ is compact and $w \in s$.

The following result is a comparison-type criterion for compactness. One knows something about the compactness of $\mathsf{C}^{(p,w)}$ in $\ell_p(w)$ for a certain weight w and $1 < p < \infty$ and one has a second weight v whose growth relative to w is *controlled*. Then, also, $\mathsf{C}^{(p,v)} \in \mathcal{L}(\ell_p(v))$ is compact.

Proposition 4.6. *Let w be a positive decreasing sequence. Suppose, for some $1 < p < \infty$, that there exists $0 \le \alpha < 1$ such that*

$$A_n(p,w) \le Mn^\alpha, \quad n \in \mathbb{N} \tag{4.7}$$

for some constant $M > 0$.

Let v be any positive decreasing sequence such that $\{v(n)/w(n)\}_{n\in\mathbb{N}} \in \ell_\infty$ and satisfying

$$w(n) \le Kn^\beta v(n), \quad n \in \mathbb{N} \tag{4.8}$$

for some $0 \le \beta < (p-1)(1-\alpha)$ and some constant $K > 0$. Then $\mathsf{C}^{(q,v)} \in \mathcal{L}(\ell_q(v))$ is a compact operator for every $1 < q \le p$.

Proof. Let $L := \sup_{n\in\mathbb{N}}(v(n)/w(n))$. Then, for each $n \in \mathbb{N}$, we have via (4.7) and (4.8) that

$$\begin{aligned}
A_n(p,v) &= v(n)^{p'/p}\sum_{k=1}^n \frac{1}{v(k)^{p'/p}} = \left(\frac{v(n)}{w(n)}\right)^{p'/p} w(n)^{p'/p}\sum_{k=1}^n \frac{1}{w(k)^{p'/p}}\cdot\left(\frac{w(k)}{v(k)}\right)^{p'/p}\\
&\le L^{p'/p} w(n)^{p'/p}\sum_{k=1}^n \frac{1}{w(k)^{p'/p}}(Kk^\beta)^{p'/p}\\
&\le (LK)^{p'/p} w(n)^{p'/p}\sum_{k=1}^n \frac{1}{w(k)^{p'/p}} n^{\beta p'/p}\\
&= (LK)^{p'/p} n^{\beta p'/p} A_n(p,w) \le M(LK)^{p'/p} n^{\alpha+(\beta p'/p)}.
\end{aligned}$$

Moreover, $\alpha + (\beta p'/p) = \alpha + (\beta/(p-1)) < 1$ because $0 \le \beta < (p-1)(1-\alpha)$ implies $\beta/(p-1) < (1-\alpha)$ which implies $\alpha + (\beta/(p-1)) < 1$. So, Theorem 4.2 applied to v (with $\alpha + (\beta/(p-1))$ in place of α) implies that $\mathsf{C}^{(q,v)} \in \mathcal{L}(\ell_q(v))$ is compact for all $1 < q \le p$. □

Example 4.7. Let $v(n) := 1/e^{n^\beta}\log^\gamma(n+1)$ for $n \in \mathbb{N}$, where $0 < \beta < 1$ and $\gamma > 0$. Then $\mathsf{C}^{(p,v)} \in \mathcal{L}(\ell_p(v))$ is compact for every $1 < p < \infty$. Observe that $\lim_{n\to\infty}(v(n)/v(n-1)) = 1$ and so Corollary 4.3 is not applicable.

So, fix $1 < p < \infty$. Define $w(n) := e^{-n^\beta}$ for $n \in \mathbb{N}$. According to Example 4.5(i), there exist constants $M > 0$ and $0 < \alpha < 1$ such that

$$A_n(p,w) \le Mn^\alpha, \quad n \in \mathbb{N}.$$

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Since $v(n) \le w(n)$ for $n \in \mathbb{N}$, it is clear that $\{v(n)/w(n)\}_{n\in\mathbb{N}} \in \ell_\infty$. Choose any $r \in (0, (p-1)(1-\alpha))$. Then

$$\frac{w(n)}{v(n)} = \log^\gamma(n+1) = \frac{\log^\gamma(n+1)}{n^r} \cdot n^r \le Kn^r, \quad n \in \mathbb{N}$$

for some $K > 0$ (as $\lim_{n\to\infty} (\log^\gamma(n+1))/n^r = 0$). According to Proposition 4.6, we can conclude that $\mathsf{C}^{(p,v)}$ is compact in $\ell_p(v)$.

References

[1] A. M. Akhmedov and F. Başar, 'On the fine spectrum of the Cesàro operator in c_0', *Math. J. Ibaraki Univ.* **36** (2004), 25–32.
[2] A. M. Akhmedov and F. Başar, 'The fine spectrum of the Cesàro operator C_1 over the sequence space bv_p, $(1 \le p < \infty)$', *Math. J. Okayama Univ.* **50** (2008), 135–147.
[3] A. A. Albanese, J. Bonet and W. J. Ricker, 'Convergence of arithmetic means of operators in Fréchet spaces', *J. Math. Anal. Appl.* **401** (2013), 160–173.
[4] S. I. Ansari and P. S. Bourdon, 'Some properties of cyclic operators', *Acta Sci. Math. (Szeged)* **63** (1997), 195–207.
[5] G. Bennett, 'Some elementary inequalities', *Q. J. Math.* **38** (1987), 401–425.
[6] A. Brown, P. R. Halmos and A. L. Shields, 'Cesàro operators', *Acta Sci. Math. (Szeged)* **26** (1965), 125–137.
[7] G. P. Curbera and W. J. Ricker, 'Spectrum of the Cesàro operator in ℓ^p', *Arch. Math.* **100** (2013), 267–271.
[8] G. P. Curbera and W. J. Ricker, 'Solid extensions of the Cesàro operator on ℓ^p and c_0', *Integral Equations Operator Theory* **80** (2014), 61–77.
[9] G. P. Curbera and W. J. Ricker, 'The Cesàro operator and unconditional Taylor series in Hardy spaces', *Integral Equations Operator Theory*, to appear, doi:10.1007/s00020-015-2230-9.
[10] N. Dunford and J. T. Schwartz, *Linear Operators I: General Theory*, 2nd printing (Wiley Interscience Publications, New York, 1964).
[11] G. J. O. Jameson, 'Norms and lower bounds of operators on the Lorentz sequence space $d(w, 1)$', *Illinois J. Math.* **43** (1999), 79–99.
[12] G. J. O. Jameson and R. Lashkaripour, 'Norms of certain operators on weighted ℓ^p spaces and Lorentz sequences spaces', *J. Inequal. Pure Appl. Math.* **3**(Article 6) (2002), 17 pp.
[13] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities* (Cambridge University Press, Cambridge, 1964).
[14] G. Leibowitz, 'Spectra of discrete Cesàro operators', *Tamkang J. Math.* **3** (1972), 123–132.
[15] R. E. Megginson, *An Introduction to Banach Space Theory* (Springer, New York, 1998).
[16] M. Mureşan, *A Concrete Approach to Classical Analysis* (Springer, Berlin, 2008).
[17] J. I. Okutoyi, 'On the spectrum of C_1 as an operator on bv_0', *J. Aust. Math. Soc.* **48** (1990), 79–86.
[18] J. B. Reade, 'On the spectrum of the Cesàro operator', *Bull. Lond. Math. Soc.* **17** (1985), 263–267.
[19] B. E. Rhoades and M. Yildirim, 'Spectra for factorable matrices on ℓ^p', *Integral Equations Operator Theory* **55** (2006), 111–126.

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