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Severity reduction and private market distortion effects of voluntary and mandatory public annuity plans[†]

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Abstract

Many economies have recently adopted the defined-contribution retirement financing system, but one disadvantage of this system is that retirees have to bear longevity risk. As a result, several economies have also introduced the public annuity plans. We analyze the similarities and differences between voluntary public annuity with ceiling (VPAC) plan and mandatory public annuity with flexibility (MPAf) plan that are empirically observed. Introducing either plan reduces the severity of adverse selection in public annuities, but further distorts the private annuity market. These two plans have systematically different effects on retirees' utility levels: the good health group is adversely affected and the average health group benefits. On the other hand, the poor health group benefits from the VPAC plan but may be adversely affected under the MPAf plan.

Keywords: Voluntary and mandatory public annuity plans; adverse selection; severity reduction effect; private market distortion effect

1. Introduction

Many developed countries in Europe and North America have adopted the unfunded pay-as-you-go (PAYGO) pension system. Due to unexpected and substantial fertility and mortality changes, a lot of them have suffered budgetary difficulties in recent decades. Learning from the experience of these countries, more and more emerging economies opt for the defined-contribution system. At the same time, some developed countries have also reformed their pension system by adding a new (and usually smaller) defined-contribution component [OECD (2021), pp. 49–51]. While the governments adopting this kind of arrangement are less likely to have budgetary problems when facing demographic changes, one disadvantage of this system is that the retirees have to bear more risks, including the longevity risk (the risk of outliving their resources when they live longer than anticipated).

In this context, an important policy debate is whether appropriate financial instruments are available for retirees to insure against longevity risk during the wealth decumulation phase [Mitchell and Piggott (2011)]. In principle, they can rely on the private financial market to hedge this risk. However, it is well known that few individuals purchase annuity products, and, for those

[†] A previous version of this paper was circulated under the title “Voluntary versus mandatory public annuity plans: A common framework to understand their pros and cons.” We are grateful to an Associate Editor, two referees, Jimmy Chan, Stephen Chiu, Yujing Xu, and the participants of the Econometric Society World Congress (2020) for helpful comments and suggestions, and to Tingyu Pang and Yinan Ying for excellent research assistance. We thank the Research Grants Council of Hong Kong (Project No. 17503219) for generous financial support.

who purchase, they only spend a small portion of their wealth on annuities [Modigliani (1986), Brown (2001), Benartzi et al. (2011)].

Perhaps because of the retirees' lack of enthusiasm in participating in the private annuity market, several economies including Denmark, Hong Kong, India, Lithuania, Singapore, and Sweden have introduced the public annuity (PA) plans in recent years. Lau and Zhang (2023) summarize the similarities and differences of public annuitization policies in these economies. In particular, there is a major difference among these plans regarding how the retirees are expected to participate: retirees in Hong Kong and India can choose whether to purchase the PA or not, while the participation in the other four economies is mandatory.¹ Moreover, a simple two-way classification of voluntary versus mandatory plans does not describe very accurately the observed practices that the PA plans in Singapore and Sweden are mandatory but with some flexibility in annuitization choice, while the PA plan in Hong Kong is voluntary but with a restriction on the maximum purchase amount. It is not entirely clear what are the roles of *flexibility in a mandatory PA plan* and *restrictiveness in a voluntary PA plan*?

Given these different practices, our first objective is to understand the economic reasons of introducing two observed PA plans: the mandatory public annuity with flexibility (MPAf) and voluntary public annuity with ceiling (VPAc) plans. Another objective is to understand how different retirees are affected if the government decides to introduce a particular PA plan.

We analyze these two PA plans in a model with asymmetric information on survival probability. The model structure is similar to those in existing annuity studies such as Abel (1986) and Brugiavini (1993), but our focus is on the roles of flexibility versus restrictiveness in a mandatory or voluntary PA plan. We obtain two major sets of results. First, when compared with the private annuity payout before the PA plan is introduced, the equilibrium PA payout is higher, but that of the private annuity is lower. Second, we find that the MPAf and VPAC plans have systematically different effects on the welfare of retirees with different survival probabilities. The average health group benefits and the good health group is adversely affected in either the MPAf or VPAC plan. The poor health group benefits from the VPAC plan, but is adversely affected in the MPAf plan.

The underlying reasons of both sets of results are that introducing either PA plan leads to two effects. Because of the quantity restrictions of the PA plan, the severity of adverse selection in public annuities is reduced, leading to a higher equilibrium PA payout. We label it the *severity reduction effect*. However, this advantage is accompanied by a negative effect of more distortion in the private annuity market, leading to a lower equilibrium private annuity payout. We label it the *private market distortion effect*. Moreover, when compared with a MPAf plan with the same restriction on the maximum amount of PA purchase, the strength of the severity reduction effect and that of the private market distortion effect in the VPAC plan are both smaller. The interaction of these two effects also explains the systematically different effects on the welfare of distinct groups of retirees. Whether the MPAf or VPAC plan is adopted, the average health group benefits because the severity reduction effect dominates, and the good health group is adversely affected because the private market distortion effect dominates. On the other hand, the impact on the poor health group depends on which PA plan is adopted. In particular, they are adversely affected in the MPAf plan because of the distortion caused by the requirement to buy the minimum mandated level, but benefit from the VPAC plan because the restriction of the maximum purchase does not affect their preferred level of PA purchase.

The rest of this paper is organized as follows. Section 2 provides literature review. Section 3 introduces the model. In Section 4, we examine separately two observed PA plans: the VPAC plan and the MPAf plan. In particular, we study the relative importance of the severity reduction and private market distortion effects in these plans. Section 5 examines the effects of the PA plans on the utility levels of different retirees. Section 6 concludes.

2. Related literature

Our study is related to two strands of the literature. First, it is related to the literature of retirement financing [such as McGrattan and Prescott (2017), Hosseini and Shourideh (2019)]. In particular, our study is related closely to the idea of government-provided annuity during the wealth decumulation phase of retirement financing. In the context of PAYGO system, Diamond (2004) suggests the idea of government-provided annuity because of the potential efficiency due to economies of scale and lower administrative cost. Fong et al. (2011) evaluate the money's worth of the mandatory public annuities offered by the Singaporean government and find that the annuity purchasers benefit from the lower cost of PA provision. They also examine the possible crowding out effect of the government-provided annuities on private annuity providers.

Motivated by recent PA practices, Lau and Zhang (2023) examine retirement financing policy questions related to the guarantee and non-escalating payments of the voluntary PA plan. However, the mandatory PA plan and the private annuity market have not been examined in that paper. This paper studies the more fundamental question of voluntary versus mandatory PA plans when the private annuity market is present.

Second, our study is related to the annuity demand literature in the presence of adverse selection. The seminal study of Yaari (1965) starts the research on annuity demand. Davidoff et al. (2005) extend Yaari's (1965) analysis by considering weaker conditions. Rothschild and Stiglitz (1976) and Wilson (1977) examine information asymmetry in the insurance market generally, and Abel (1986) and Eichenbaum and Peled (1987) study adverse selection in the annuity market. Subsequently, simulation studies [such as Friedman and Warshawsky (1990)] and empirical studies [such as Finkelstein and Poterba (2002, 2004), Einav et al. (2010)] support the idea that information asymmetry causes adverse selection in the annuity market.

The major forms of annuity contract discussed in the literature are either non-exclusive contract with linear pricing [Abel (1986), Brugiavini (1993), Hosseini (2015)] or exclusive contract specifying both prices and quantities [Eckstein et al. (1985), Eichenbaum and Peled (1987)]. Motivated by the form of PA contracts offered in various economies, we differ from these studies in one key aspect: we examine the annuities specifying *linear price* for all buyers but having possible *quantity restrictions* on the minimum and maximum amounts of purchase. This extension allows us to examine the roles of mandatory and voluntary elements in a PA plan and apply the annuity demand theory to study various empirically relevant PA plans.

3. A model of private and public annuities

In this section, we introduce a simple two-period model of annuity purchase when public and private annuities coexist. In reality, we seldom observe a complete crowding out of the private annuity market by the provision of public annuities.

In the literature, several important factors have been emphasized in understanding annuitization behavior, especially the low level of purchase of private annuities. The factors include high annuity price (which may arise from adverse selection, high administrative cost, etc.), illiquidity concern, intra-family risk sharing, and crowding out effect of government pension. Among them, adverse selection due to asymmetric information on survival probability is a major factor, based on previous theoretical work [such as Eckstein et al. (1985); Abel (1986); Hosseini (2015)] and empirical studies [such as Finkelstein and Poterba (2004); Einav et al. (2010)].

Incorporating the above idea, our model contains the feature of information asymmetry on survival probability.² In the following analysis, we first establish the model environment before the PA plan is introduced and then analyze the outcomes after the introduction of various PA plans.

3.1 Asymmetric information on survival probability

We consider a continuum of retirees who live for two periods at most: Period 1 with certainty and Period 2 with some probability (θ). The two periods correspond to, respectively, the early and advanced stages of retirement. Retirees have different probabilities of surviving to Period 2, represented by a cumulative distribution $F(\theta)$ where $\theta \in [\underline{\theta}, \bar{\theta}]$ and $0 \leq \underline{\theta} < \bar{\theta} \leq 1$.³ Generally, an individual knows her behavior and preference in health-related activities and has a better sense about her health and survival probability. In this model, we assume that θ is private information and is known by the individual in Period 1.

We assume that non-exclusive annuity contracts with linear pricing (as in Abel, 1986; Hosseini, 2015), rather than exclusive contracts with price convexity [as in Eckstein et al. (1985); Eichenbaum and Peled (1987)], are offered, because all observed PA contracts in various economies are similar to this type. Moreover, as pointed out by Abel (1986), it is hard to determine whether a buyer also holds annuities from other providers. This difficulty affects the effectiveness of exclusive contracts in the annuity market.⁴ Motivated by observed practices, we also assume that quantity restrictions are imposed on the public annuities. As will be elaborated in Section 4.3, a distinguishing feature between public and private annuity provision in the presence of adverse selection is the effectiveness of imposing the quantity restrictions.

In the model, the lifetime utility of a retiree with survival probability θ is given by

$$U(c_{1\theta}, c_{2\theta}; \theta) = u(c_{1\theta}) + \frac{\theta}{1 + \rho} u(c_{2\theta}), \tag{1}$$

where $c_{i\theta}$ is the level of consumption expenditure in Period i ($i = 1, 2$) and ρ is the subjective discount rate. We assume that the utility function $U(c_{1\theta}, c_{2\theta}; \theta)$ in (1) is homothetic, with the property that the marginal rate of substitution is a homogeneous function of degree 0:

$$\frac{\partial U(tc_{1\theta}, tc_{2\theta}; \theta) / \partial c_{1\theta}}{\partial U(tc_{1\theta}, tc_{2\theta}; \theta) / \partial c_{2\theta}} = \frac{\partial U(c_{1\theta}, c_{2\theta}; \theta) / \partial c_{1\theta}}{\partial U(c_{1\theta}, c_{2\theta}; \theta) / \partial c_{2\theta}} \tag{2}$$

for $t > 0$.⁵ Moreover, the standard assumptions that $u(c)$ is strictly concave and $\lim_{c \rightarrow 0} u'(c) = \infty$ hold.

For simplicity, we do not consider bequest motive in this paper, because it is not likely to be a major factor in understanding the similarities and differences of voluntary versus mandatory PA plans. Assuming away bequest motive means that the annuity always dominates the risk-free bond as the financial tool to hedge longevity risk [e.g. Yaari (1965), Davidoff et al. (2005)]. As a result, retirees do not consider the purchase of risk-free bond in our model.

3.2 Before the introduction of the PA plan

We first consider the outcome before the introduction of the PA plan. A typical retiree makes the consumption and annuitization choices to maximize her expected lifetime utility, given by (1). The outcome under this environment will be useful in subsequent analysis, particularly the analysis (in Section 5) of the impact on different retirees when the government considers whether to adopt a particular PA plan or not.

In this paper, we consider an annuity contract with survival-contingent payments only. The private annuity contract operates as follows. If a retiree buys one dollar of annuity in Period 1, she will receive \widehat{V} dollars in Period 2 if she is alive. With this annuity product, her budget constraints are given by:

$$\widehat{c}_{1\theta} = w - \widehat{v}_\theta, \tag{3}$$

and

$$\widehat{c}_{2\theta} = \widehat{V}\widehat{v}_\theta, \tag{4}$$

where \widehat{v}_θ is the annuitization amount of a retiree with parameter θ and w is the retiree's wealth. Since borrowing against annuity contracts is not allowed in most societies, the restriction of

non-negative amounts of annuity purchase ($\widehat{v}_\theta \geq 0$) is imposed. Note that we use the symbol $\widehat{\cdot}$ above a variable to denote the corresponding variable in the economy before the introduction of the PA (i.e., with private annuity market only).

The annuity buyer's optimization problem, after knowing her private information θ , is to choose \widehat{v}_θ to maximize

$$U(\widehat{c}_{1\theta}, \widehat{c}_{2\theta}; \theta) = u(\widehat{c}_{1\theta}) + \frac{\theta}{1 + \rho} u(\widehat{c}_{2\theta}) = u(w - \widehat{v}_\theta) + \frac{\theta}{1 + \rho} u(\widehat{V}\widehat{v}_\theta). \tag{5}$$

It is easy to show that the optimal choice of \widehat{v}_θ is an interior solution, which is characterized by:⁶

$$u'(w - \widehat{v}_\theta^*) = \frac{\theta}{1 + \rho} \widehat{V} u'(\widehat{V}\widehat{v}_\theta^*), \tag{6}$$

where the symbol * associated with a variable denotes the optimal choice of that variable.

The left-hand side (LHS) in the first-order condition (6) is the marginal cost of annuitizing one more dollar, because the buyer has to reduce one unit of consumption in Period 1. In terms of utility, the marginal cost is $u'(\widehat{c}_{1\theta}^*) = u'(w - \widehat{v}_\theta^*)$. If the buyer is alive in Period 2, she receives \widehat{V} extra units of annuity income. Thus, the increase in her utility level is given by $\widehat{V} u'(\widehat{c}_{2\theta}^*) = \widehat{V} u'(\widehat{V}\widehat{v}_\theta^*)$. After discounting the expected value of this future utility benefit back to Period 1, we have the right-hand side (RHS) term of (6). At the optimal choice, the marginal benefit and marginal cost are equal according to (6).

Differentiating (6) totally, we obtain

$$\frac{\partial \widehat{v}_\theta^*}{\partial \theta} = \frac{-\widehat{V} u'(\widehat{c}_{2\theta}^*)}{(1 + \rho) u''(\widehat{c}_{1\theta}^*) + \theta \widehat{V}^2 u''(\widehat{c}_{2\theta}^*)} > 0, \tag{7}$$

which suggests that the optimal annuity choice \widehat{v}_θ^* is increasing in the survival probability θ . The relationship is shown in Panel A of Figure 1.

Based on the buyers' choices described above, annuity providers' revenue in Period 1 is given by $\int_{\underline{\theta}}^{\bar{\theta}} \widehat{v}_\theta^* dF(\theta)$, and their expected payment in Period 2 is $\int_{\underline{\theta}}^{\bar{\theta}} \theta \widehat{V}^* \widehat{v}_\theta^* dF(\theta)$. Following many researchers [such as Abel (1986), Hosseini (2015)], we assume the zero-profit condition for the provision of annuities. This assumption offers the advantage that the correlation between survival probability and annuity purchase, which is the source of adverse selection, is reflected in the equilibrium annuity payout level. Under the zero-profit condition for private annuity market, the equilibrium value of the payout term \widehat{V} , denoted by \widehat{V}^* , is determined according to

$$\widehat{V}^* = \frac{(1 + r) \int_{\underline{\theta}}^{\bar{\theta}} \widehat{v}_\theta^* dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta \widehat{v}_\theta^* dF(\theta)}, \tag{8}$$

after using the discount factor $1/(1 + r)$ to adjust for the revenue and payment terms in the two different periods, where r is the interest rate.⁷ According to Lau et al. (2022), the equilibrium payout \widehat{V}^* is unique for all time-separable utility functions (1) if the distribution of retirees' survival probabilities satisfies a rather weak condition. For example, a uniform distribution or a truncated normal distribution ($\underline{\theta} \leq \theta \leq \bar{\theta}$) with variance larger than some threshold level satisfies this sufficient condition.

Comparing the payout \widehat{V}^* with the actuarially fair payout $\frac{1+r}{E(\theta)}$ based on average survival probability of the population, it is straightforward to conclude that

$$\widehat{V}^* < \frac{1 + r}{E(\theta)}, \tag{9}$$

because \widehat{v}_θ^* and θ are positively correlated according to (7).

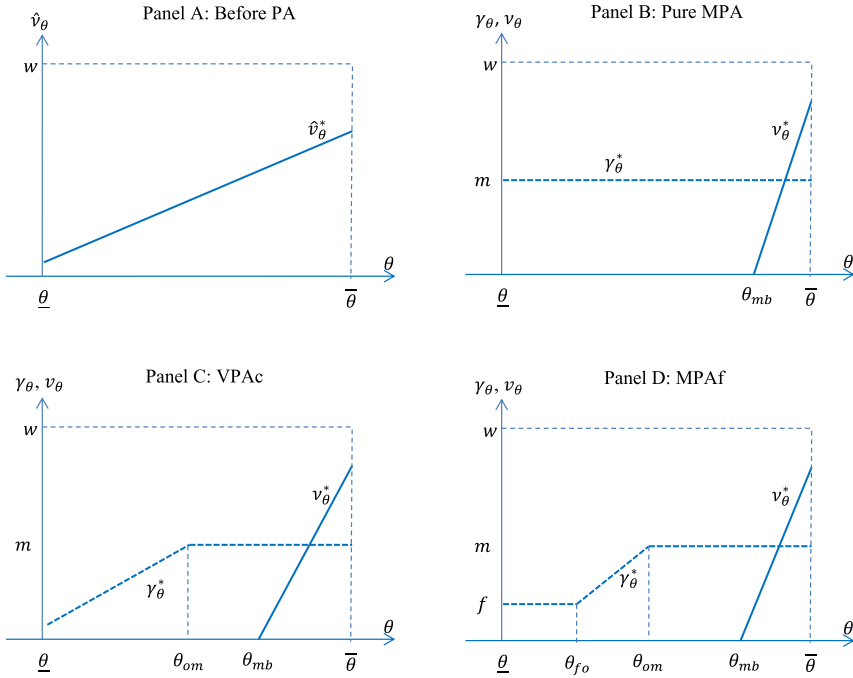


Figure 1. Annuity choices under various plans.

The value of payout \widehat{V}^* is below the actuarially fair level, indicating some efficiency loss due to adverse selection, which is often cited as an important reason calling for government intervention.

3.3 Introducing the PA plan

In the presence of market imperfection such as adverse selection, the government may choose to intervene in the annuity market in different ways, such as taxation, regulation, or annuity provision. Drawing on the PA practices in various economies [see, e.g. Section 2 and Table 1 of Lau and Zhang (2023)], we consider the case that the government (directly or indirectly through a statutory body) steps in the market as the PA provider.

We assume that the PA contract provides survival-contingent payments only. If a retiree buys one dollar of the PA in Period 1, she will receive G dollars in Period 2 if she is alive. Therefore, the retiree’s budget constraints, when both private and public annuities are available, are given by

$$c_{1\theta} = w - \gamma_\theta - v_\theta, \tag{10}$$

and

$$c_{2\theta} = G\gamma_\theta + Vv_\theta, \tag{11}$$

where γ_θ ($\gamma_\theta \geq 0$) is her purchase amount of PA and v_θ ($v_\theta \geq 0$) is her purchase amount of private annuity. The other variables are the same as before, except that the symbol $\widehat{}$ is absent after the PA plan is introduced.

Combining (1), (10), and (11), the problem becomes

$$\max_{\gamma_\theta, v_\theta} U(c_{1\theta}, c_{2\theta}; \theta) = u(w - \gamma_\theta - v_\theta) + \frac{\theta}{1 + \rho} u(G\gamma_\theta + Vv_\theta).$$

Solving the above problem leads to the following two conditions:

$$\frac{\partial U}{\partial \gamma_\theta} = -u'(w - \gamma_\theta - v_\theta) + \frac{\theta}{1 + \rho} Gu'(G\gamma_\theta + Vv_\theta), \tag{12}$$

and

$$\frac{\partial U}{\partial v_\theta} = -u'(w - \gamma_\theta - v_\theta) + \frac{\theta}{1 + \rho} Vu'(G\gamma_\theta + Vv_\theta), \tag{13}$$

where $u'(w - \gamma_\theta - v_\theta)$ in either (12) or (13) is the marginal cost of buying one more unit of a public or private annuity. On the other hand, the marginal benefit of buying a private annuity is different from that of buying the PA, with $\frac{\theta}{1+\rho} Gu'(G\gamma_\theta + Vv_\theta)$ in (12) being the marginal benefit of buying the PA, and $\frac{\theta}{1+\rho} Vu'(G\gamma_\theta + Vv_\theta)$ in (13) being the marginal benefit of buying the private annuity.

3.4 Pure mandatory PA plan

Before analyzing two PA plans which are empirically relevant, we first consider a benchmark case of a pure mandatory public annuity plan (pure MPA plan) in which all retirees are required to purchase the same level of PA. The results of this case, together with those in the opposite extreme case of a pure VPA plan,⁸ are helpful to understand the similarities and differences of the two PA plans to be considered in the next section.

The pure MPA plan is represented by

$$\gamma_\theta = m. \tag{14}$$

For this plan, the behavior of PA purchase is straightforward, as all retirees are required to buy the same amount (m) of public annuities. On the other hand, the behavior in the private annuity market is more interesting, particularly when m is set at a not-too-high level such that retirees with high value of θ still have residual annuitization demand after purchasing the mandated amount of public annuities. In this case, retirees with good health satisfy their residual demands by purchasing annuities from the private sector. The optimal choice of private annuity (v_θ^*), which is obtained by combining (14) and $\frac{\partial U}{\partial v_\theta} \Big|_{v_\theta=v_\theta^*} = 0$ in (13), is given by

$$u'(w - m - v_\theta^*) = \frac{\theta}{1 + \rho} Vu'(Gm + Vv_\theta^*). \tag{15}$$

Using similar procedure as in (7), we can show that

$$\frac{\partial v_\theta^*}{\partial \theta} > 0. \tag{16}$$

Since v_θ^* is always increasing for those who buy private annuity, we can determine the threshold level of survival probability θ_{mb} by substituting $v_\theta^* = 0$ in (15) to obtain

$$\theta_{mb} = \frac{(1 + \rho) u'(w - m)}{V^* u'(G^* m)}, \tag{17}$$

where V^* is the equilibrium value of payout of the private annuity, which is determined according to the zero-profit condition as

$$V^* = \frac{(1 + r) \int_{\underline{\theta}}^{\bar{\theta}} v_\theta^* dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta v_\theta^* dF(\theta)} = \frac{(1 + r) \int_{\theta_{mb}}^{\bar{\theta}} v_\theta^* dF(\theta)}{\int_{\theta_{mb}}^{\bar{\theta}} \theta v_\theta^* dF(\theta)}. \tag{18}$$

It is straightforward to see that the equilibrium value of PA payout (G^*) for the pure MPA plan under the zero-profit condition is given by⁹

$$G^* = \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} \gamma_{\theta}^* dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta \gamma_{\theta}^* dF(\theta)} = \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} m dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta m dF(\theta)} = \frac{1+r}{E(\theta)}, \tag{19}$$

the actuarially fair level.

Buyers' choices after the introduction of the pure MPA plan are summarized as follows. Buyers with $\theta \leq \theta_{mb}$ purchase the PA ($\gamma_{\theta}^* = m$) only, where θ_{mb} is defined in (17). On the other hand, buyers with $\theta > \theta_{mb}$ purchase both the PA (at the mandated amount, $\gamma_{\theta}^* = m$) and the private annuity (v_{θ}^*), where v_{θ}^* is determined according to (15) and is increasing in θ .

Buyers' annuitization choices under the pure MPA plan are shown in Panel B of Figure 1.

4. Two observed PA plans

While analyzing the pure mandatory PA and pure voluntary PA plans allows us to obtain some interesting results, these two plans are less relevant because they are not commonly observed, as we have briefly described in Section 1. In existing voluntary PA plans, there is a restriction on the maximum amount of purchase. Presumably, the PA provider wants to exercise financial prudence to ensure that some buyers, particularly those who are likely to live very long, would not be able to purchase a substantial amount of the PA. On the other hand, there are some elements of flexibility in existing mandatory PA plans, perhaps because the presence of some factors (such as the concern about whether the annuity buyers can afford to purchase the mandated level) makes a uniform mandated level of annuity purchase undesirable for the retirees.

We now analyze buyers' behavior under two empirically relevant PA plans: a voluntary plan with a restriction on the maximum amount of purchase and a mandatory plan with flexibility.

4.1 VPA plan with ceiling

The Hong Kong Mortgage Corporation (HKMC) Annuity Plan was first launched in July 2018. The plan is a voluntary one. Individuals over a certain age (60 currently) are allowed to purchase the public annuities, with the amount of purchase ranging from HKD50,000 (roughly USD6,400) to a level set by the PA provider from time to time. Currently, the maximum amount of purchase is HKD5 million (roughly USD641,000), which was set at June 2022.¹⁰

Motivated by the observed practices in Hong Kong, we consider the VPA plan with ceiling, which is specified as:¹¹

$$0 \leq \gamma_{\theta} \leq m, \tag{20}$$

where

$$m \leq \bar{m}, \tag{21}$$

with \bar{m} being determined according to¹²

$$u' \left(w - \frac{G^* - V^*}{\hat{V}^* - V^*} \bar{m} \right) = \frac{\bar{\theta}}{1 + \rho} V^* u' \left(\hat{V}^* \frac{G^* - V^*}{\hat{V}^* - V^*} \bar{m} \right), \tag{22}$$

and V^* is determined according to (18) and G^* is the equilibrium payout of the VPAC plan, given by (26) below. Since setting a very high level of m may not be financially prudent and may lead to a complete or substantial crowding out of the private annuity market, it is not likely that many governments want to do so. As will be shown in subsequent analysis, imposing the not-too-high ceiling condition (21) ensures that the private annuity market is not substantially crowded out.

When the payout of the PA is higher than that of the private annuity (to be shown in Proposition 1) and when the optimal amount of PA purchased by individual θ , denoted by γ_θ^* , is less than the ceiling level m , the optimal choice γ_θ^* is defined by

$$u'(w - \gamma_\theta^*) = \frac{\theta}{1 + \rho} Gu'(G\gamma_\theta^*), \tag{23}$$

which is obtained by combining $v_\theta = 0$ and $\frac{\partial U_\theta}{\partial \gamma_\theta} \Big|_{\gamma_\theta = \gamma_\theta^*} = 0$ in (12). Following similar procedure as in (7), it is straightforward to show that

$$\frac{\partial \gamma_\theta^*}{\partial \theta} = \frac{-Gu'(c_{2\theta}^*)}{(1 + \rho) u''(c_{1\theta}^*) + \theta G^2 u''(c_{2\theta}^*)} > 0. \tag{24}$$

Buyers' choices under the VPAC plan are summarized as follows. When this plan is introduced, (a) buyers with $\theta < \theta_{om}$ purchase the PA only, with the PA purchase (γ_θ^*) being determined according to (23) and increasing in θ , where

$$\theta_{om} = \frac{(1 + \rho) u'(w - m)}{G^* u'(G^* m)}; \tag{25}$$

(b) buyers with $\theta_{om} \leq \theta \leq \theta_{mb}$ purchase the PA only, and purchase the maximum amount of PA ($\gamma_\theta^* = m$), where θ_{mb} is defined in (17); and (c) buyers with $\theta > \theta_{mb}$ purchase both the PA (at the maximum amount, $\gamma_\theta^* = m$) and the private annuity (v_θ^*), with v_θ^* being determined according to (15) and increasing in θ . (The proof of the above results, which is straightforward but tedious, is given in the Online Appendix.)

Consistent with the buyers' behavior, the equilibrium value of PA payout of the VPAC plan is given by¹³

$$G^* = \frac{(1 + r) \int_{\underline{\theta}}^{\bar{\theta}} \gamma_\theta^* dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta \gamma_\theta^* dF(\theta)} = \frac{(1 + r) \left[\int_{\underline{\theta}}^{\theta_{om}} \gamma_\theta^* dF(\theta) + \int_{\theta_{om}}^{\bar{\theta}} m dF(\theta) \right]}{\int_{\underline{\theta}}^{\theta_{om}} \theta \gamma_\theta^* dF(\theta) + \int_{\theta_{om}}^{\bar{\theta}} \theta m dF(\theta)}. \tag{26}$$

Buyers' annuitization choices under the VPAC plan are shown in Panel C of Figure 1.

4.2 MPA plan with flexibility

The Central Provident Fund (CPF) Lifelong Income for the Elderly (LIFE) program in Singapore was introduced in 2009 and has become mandatory since 2013. Participants are required to set aside the Full Retirement Sum (FRS) to buy the lifelong annuity provided by the CPF Board. The FRS level is adjusted periodically and the current level is, for example, 192,000 Singaporean dollars (about USD142,000) for those aged 55 in 2022. In 2016, the CPF Board introduced the Retirement Sum Topping-Up Scheme. Participants can use this scheme to top-up their Retirement Sum up to the Enhanced Retirement Sum (ERS), which is 1.5 times of the FRS.¹⁴ We observe that a floor (i.e. the FRS) and a ceiling (i.e. the ERS) are imposed on the amount of PA purchase in the CPF LIFE plan. Some flexibility between these two levels is given to the participants in this MPA program.

Motivated by the observed practices in Singapore,¹⁵ the MPA plan with flexibility is specified as:

$$f \leq \gamma_\theta \leq m, \tag{27}$$

where parameters m and f satisfy (21) and¹⁶

$$\widehat{v}_\theta^* \leq f < m. \tag{28}$$

If condition (28) does not hold, the lowest mandated level (f) of the MPAf plan is even lower than the purchase level of private annuity of the least healthy retiree before the PA plan is introduced. We impose condition (28) to eliminate this uninteresting case.¹⁷

Following similar procedure as above, buyer’s behavior under this PA plan is given as follows. After the PA plan is introduced, (a) buyers with $\theta \leq \theta_{f_0}$ only purchase the minimum required level ($\gamma_{\theta}^* = f$) of PA, where

$$\theta_{f_0} = \frac{(1 + \rho) u'(w - f)}{G^* u'(G^* f)}; \tag{29}$$

(b) buyers with $\theta_{f_0} < \theta < \theta_{om}$ purchase the PA only, with the PA purchase (γ_{θ}^*) being determined according to (23) and increasing in θ , where θ_{om} is defined in (25); (c) buyers with $\theta_{om} \leq \theta \leq \theta_{mb}$ purchase the maximum amount of PA ($\gamma_{\theta}^* = m$) only, where θ_{mb} is defined in (17); and (d) buyers with $\theta > \theta_{mb}$ purchase both the PA (at the maximum amount, $\gamma_{\theta}^* = m$) and the private annuity (v_{θ}^*), with v_{θ}^* being determined according to (15) and increasing in θ . Moreover, the equilibrium PA payout of the MPAf plan is given by

$$G^* = \frac{(1 + r) \int_{\underline{\theta}}^{\bar{\theta}} \gamma_{\theta}^* dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta \gamma_{\theta}^* dF(\theta)} = \frac{(1 + r) \left[\int_{\underline{\theta}}^{\theta_{f_0}} f dF(\theta) + \int_{\theta_{f_0}}^{\theta_{om}} \gamma_{\theta}^* dF(\theta) + \int_{\theta_{om}}^{\bar{\theta}} m dF(\theta) \right]}{\int_{\underline{\theta}}^{\theta_{f_0}} \theta f dF(\theta) + \int_{\theta_{f_0}}^{\theta_{om}} \theta \gamma_{\theta}^* dF(\theta) + \int_{\theta_{om}}^{\bar{\theta}} \theta m dF(\theta)}. \tag{30}$$

Buyers’ annuitization choices under the MPAf plan are shown in Panel D of Figure 1.

4.3 Severity reduction and private market distortion effects in a two-tier annuity market

In Sections 4.1 and 4.2, we obtained the results about annuity buyers’ choices conditional on a higher payout for the PA ($G^* > V^*$). The following proposition, which compares each of the equilibrium payouts of the public and private annuities with the payout level of the private annuity market before the PA plan is introduced, confirms that $G^* > V^*$ is an endogenous outcome under either the VPac or MPAf plan.¹⁸

Proposition 1. Consider the introduction of either the VPac plan or MPAf plan. Compared with the equilibrium payout of the private annuity (\widehat{V}^*) before the introduction of the PA plan,

(a) the equilibrium payout of the PA is higher:

$$G^* > \widehat{V}^*; \tag{31}$$

and (b) the equilibrium payout of the private annuity is lower:

$$V^* < \widehat{V}^*. \tag{32}$$

Proof. See Appendix A. □

Combining parts (a) and (b) of Proposition 1, together with the positive correlation of the annuity purchase amount and θ according to (24) for those buyers whose purchase amounts are not constrained by the quantity restrictions of the PA plan, we obtain¹⁹

$$V^* < G^* < \frac{1 + r}{E(\theta)}. \tag{33}$$

An implication of (33) is the outcome of a two-tier annuity market. Less healthy people only purchase the public annuities, but healthier people purchase both public and private annuities. Moreover, we observe from the above analysis that there are two key effects when the PA plan is introduced.

First, the design of the PA plan matters. The imposed quantity restrictions (for both the ceiling and floor) on the PA purchase lead to a reduction in the severity of adverse selection, which

originally arises from the positive correlation of the risk type (based on survival probability) and the PA purchase amount. Specifically, the ceiling restriction of both PA plans reduces the positive correlation of these two variables at the upper end by restricting the purchase amount of healthier retirees to the maximum level (m), and the floor restriction of the MPAf plan reduces the positive correlation at the lower end by requiring the less healthy retirees to purchase the minimum mandated level (f). We call this effect the severity reduction effect.

The restrictions imposed by the PA plans may not be implemented in the private annuity market, and even if some of them are implemented, the effects are different. Obviously, private annuity companies cannot make retirees' purchase of private annuity products mandatory. On the other hand, imposing the maximum purchase level is possible for private annuity companies, but this policy is not effective in reducing the severity of adverse selection of the competitive private market. In a competitive market where different companies offer similar annuity products, if a company sets a purchase ceiling, a retiree with good health can buy the maximum amount from this company and then go to another one to satisfy her unfulfilled demand. It is not likely (and also not legal in many countries) that different companies share their customer lists and restrict the customers from going to other companies to buy annuities.²⁰ Since the overall severity of adverse selection is determined by the total amount of private annuity purchased by all retirees, the effect of imposing purchase restriction on the severity of adverse selection in the PA sector cannot be replicated in the private market.

Second, the PA plan affects indirectly the private annuity market. Since the public and private annuity contracts provide the same financial function to the retirees, and the payout of the PA is higher, the retirees satisfy their annuity demand by first purchasing the more attractive PA. Because of the restrictions imposed by either VPac or MPAf plan, the demands of some retirees may not be completely satisfied. As shown in (18), the presence of the PA plan causes less healthy retirees to drop out completely from the private annuity market. The equilibrium price of private annuities will be higher (and annuity payout lower) since the low-risk buyers do not participate in the private annuity market. At the same time, healthier retirees' purchases of the private annuity are also distorted by the PA plan, since their budget constraints are changed after purchasing m units of PA at a cheaper price.²¹ When the utility function is homothetic, combining these two factors leads to (32), as shown in Appendix A. We label this effect the private market distortion effect.

4.4 Comparing PA plans with different degrees of choice restriction

Based on Proposition 1, we trace the effects of the PA plans on the two equilibrium annuity payouts (G^* and V^*) to the severity reduction and private market distortion effects. The above analysis focuses on the similarities of the VPac and MPAf plans and shows that both effects are present in each plan. We now examine their differences by analyzing the relative importance of the two effects in the two plans.

In general, different economies adopting different PA plans are also likely to select different ceiling (m) and floor (f) levels. It is usually difficult to obtain unambiguous results when comparing different PA plans with changes in both m and f parameters. Instead, we conduct a comparative static exercise based on the following distinction when comparing parameters m and f of various PA plans: there is no floor ($f = 0$) for the VPac plan based on (20), there is a floor ($0 \leq \hat{v}_\theta^* < f < m$) for the MPAf plan based on (28), and the floor is the same as the ceiling ($f = m$) for the pure MPA plan based on (14).

We now compare the equilibrium payouts of various PA plans with the same level of m but different values of f . The results are summarized in the following proposition.

Proposition 2. *Comparing a VPac plan, a MPAf plan, and a pure MPA plan such that each plan has the same level of m , the equilibrium payouts of the private and public annuities in the three plans are ranked as follows:*

$$V_{MPA}^* < V_{MPAf}^* < V_{VPAc}^* < \widehat{V}^* < G_{VPAc}^* < G_{MPAf}^* < G_{MPA}^*. \tag{34}$$

The proof of Proposition 2 is very similar to that of Proposition 1. It is available in the Online Appendix.

Among different PA plans with the same ceiling level (m), the floor parameter (f) of a particular PA plan can be interpreted as representing its degree of choice restriction. Comparing with the pure MPA plan with the same ceiling level, the MPAf plan has less choice restriction ($f < m$). As a result, the severity reduction effect is not so strong and the private market distortion effect is also weaker, leading to the results that G_{MPAf}^* is lower (than G_{MPA}^*) in the PA sector but V_{MPAf}^* is higher (than V_{MPA}^*) in the private annuity market. On the other hand, we observe that while both the VPAC and MPAf plans with the same ceiling level have elements of flexibility and restrictiveness, the relative emphasis is different. Comparing with the VPAC plan, the MPAf plan has a higher degree of choice restriction ($f > 0$). As a result, G_{MPAf}^* is higher (than G_{VPAc}^*) in the PA sector but V_{MPAf}^* is lower (than V_{VPAc}^*) in the private annuity market.

More generally, the intuition of Proposition 2 can be understood as follows. When the degree of choice restriction of the PA plan is increased by raising the floor parameter f from 0 (VPAC plan) to an intermediate value $0 \leq \widehat{\theta} < f < m$ (MPAf plan) and then to m (pure MPA plan), poor health buyers are required to purchase more units of the PA. As a result, the severity reduction effect in the PA sector becomes stronger (i.e. G_{MPA}^* is the highest). On the other hand, fewer buyers (only those who are very healthy) participate in the private annuity market, resulting in a stronger private market distortion effect (i.e. V_{MPA}^* is lowest).

To summarize, the tradeoff between the severity reduction and private market distortion effects is present in each of the three PA plans: VPAC, MPAf, and pure MPA plans. The PA plan with a lower degree of choice restriction (such as the VPAC plan) has a smaller gain in the severity reduction effect but also a smaller loss in the private market distortion effect. On the other hand, the PA plan with a higher degree of choice restriction has a larger gain in the severity reduction effect but also a larger loss in the private market distortion effect.

5. Effects on retirees’ welfare

In the previous sections, we analyze the diverse practices of voluntary versus mandatory PA plans, and the focus is on the equilibrium annuity payout values. We now analyze a related issue regarding the annuity buyers’ utility levels: what will be the effects on different retirees when a government introduces a VPAC or MPAf plan? This question is particularly important when the government is contemplating whether they want to adopt a PA plan, and which PA plan to adopt if they decide to go.²²

Define $\widehat{U}_\theta^* = U(\widehat{c}_{1\theta}^*, \widehat{c}_{2\theta}^*; \theta)$ as the maximized value of $U(\widehat{c}_{1\theta}, \widehat{c}_{2\theta}; \theta)$ in (5) before the PA plan is introduced, and $U_\theta^* = U(c_{1\theta}^*, c_{2\theta}^*; \theta)$ as the maximized value of $U(c_{1\theta}, c_{2\theta}; \theta)$ in (1) after the PA plan is introduced. We are interested in the difference of these two maximized values for buyers with various survival probabilities, before and after a particular PA plan is introduced. The following lemma is useful for subsequent analysis.

Lemma 1. *When the government introduces a PA plan (either a VPAC or MPAf plan), the derivative of the change in the maximized utility of an annuity buyer with respect to her survival probability θ is given by*

$$\frac{\partial (U_\theta^* - \widehat{U}_\theta^*)}{\partial \theta} = \frac{1}{1 + \rho} [u(c_{2\theta}^*) - u(\widehat{c}_{2\theta}^*)]. \tag{35}$$

Proof. See Appendix B. □

Lemma 1 has an interesting interpretation. The derivative

$$\frac{\partial \widehat{U}_\theta^*}{\partial \theta} = \frac{1}{1 + \rho} u(\widehat{c}_{2\theta}^*), \tag{36}$$

can be regarded as the marginal benefit of longevity (MBL), measured in utility level, before the PA plan is introduced. When θ increases a little, a buyer has a higher chance of surviving to Period 2. Thus, the buyer has a higher chance of consuming $\widehat{c}_{2\theta}^*$ and thereby obtains the benefit of $u(\widehat{c}_{2\theta}^*)$. The RHS term in (36) represents this benefit, which is discounted back to Period 1. Similarly,

$$\frac{\partial U_\theta^*}{\partial \theta} = \frac{1}{1 + \rho} u(c_{2\theta}^*) \tag{37}$$

captures the MBL after the PA plan is introduced. Combining them leads to (35), which is the difference of the MBL before and after the PA plan is introduced.

Since $u'(\cdot) > 0$, whether $U_\theta^* - \widehat{U}_\theta^*$ is increasing or decreasing in θ depends on the difference between $c_{2\theta}^*$ and $\widehat{c}_{2\theta}^*$. In subsequent analysis, we will use the gap between $c_{2\theta}^*$ and $\widehat{c}_{2\theta}^*$ to examine whether $U_\theta^* - \widehat{U}_\theta^*$ is increasing or decreasing in survival probability (θ) for different intervals of θ .

5.1 Effects under a VPA with ceiling plan

We consider the VPAC plan with the not-too-high ceiling condition (21). We first obtain a useful result in Lemma 2. The proof is given in the Online Appendix.

Lemma 2. *If (21) holds, then*

$$U_\theta^* - \widehat{U}_\theta^* < 0. \tag{38}$$

The intuition of Lemma 2 is as follows. When the VPAC plan is introduced, the retirees benefit from the higher level of PA payout G^* (the severity reduction effect), but they can at most enjoy this benefit up to m units of PA purchase. On the other hand, retirees who are very healthy and have residual demand for the private annuity suffer from the lower level of private annuity payout V^* (the private market distortion effect). Lemma 2 shows that when m is set according to condition (21), the loss from the private market distortion effect for the healthiest retirees (with $\theta = \bar{\theta}$) dominates the benefit from the severity reduction effect. As a result, (38) holds and the healthiest annuity buyers are adversely affected by the introduction of the VPAC plan.

With the help of Lemmas 1 and 2, we obtain the systematically different effects of introducing the VPAC plan on two categories of buyers, based on their survival probabilities (θ).²³ The result is summarized in the following proposition.

Proposition 3. *When the government introduces a VPAC plan such that parameter m satisfies (21), the VPAC plan systematically separates annuity buyers in two categories according to the change in utility level: (a) an increase in a buyer’s utility level for the poor health group; and (b) a decrease in a buyer’s utility level for the good health group.*

Proof. See Appendix B. □

The policy implications of Proposition 3, together with those of Proposition 4 regarding the MPAF plan in Section 5.2, will be discussed in Section 5.3.

5.2 Effects under a MPA with flexibility plan

We now consider the MPAf plan. The following lemma is useful for analyzing the utility effects under this plan. The proof is given in the Online [Appendix](#).

Lemma 3. *If*

$$f > \underline{f}, \tag{39}$$

where \underline{f} is the larger root to

$$U_{\underline{\theta}}(f) = u(w - f) + \frac{\theta}{1 + \rho} u(G^* f) = \widehat{U}_{\underline{\theta}}^*, \tag{40}$$

with G^* given by (30) and $\widehat{U}_{\underline{\theta}}^* = u(w - \widehat{v}_{\underline{\theta}}^*) + \frac{\theta}{1 + \rho} u(\widehat{V}^* \widehat{v}_{\underline{\theta}}^*)$, then

$$U_{\underline{\theta}}^* - \widehat{U}_{\underline{\theta}}^* < 0. \tag{41}$$

If

$$\widehat{v}_{\underline{\theta}}^* \leq f \leq \underline{f}, \tag{42}$$

then (41) does not hold.

Under the MPAf plan, retirees with poor health (such as those with $\theta < \theta_{fo}$) benefit from the higher PA payout (G^*) but are adversely affected by the restriction to buy the mandated level (f) of PA purchase. Lemma 3 shows that if the floor is set at a not-too-low level (i.e. (39) holds), then for retirees with the lowest survival probability ($\underline{\theta}$), the mandated floor level deviates quite substantially from their annuity demand (as measured by $\widehat{v}_{\underline{\theta}}^*$ before the PA plan is introduced). As a result, (41) holds and annuity buyers with the lowest survival probability are adversely affected by the introduction of the MPAf plan, because the loss from the restriction to buy the mandated PA level dominates the benefit from the higher PA payout.

We now focus on the more interesting case that the not-too-low floor condition (39) holds.²⁴ The following proposition summarizes the results that introducing the MPAf plan has systematically different effects on three categories of buyers, based on their survival probabilities (θ).

Proposition 4. *When the government introduces a MPAf plan such that parameter m satisfies (21) and parameter f satisfies (39), the MPAf plan systematically separates annuity buyers in three categories according to the change in utility level: (a) a decrease in a buyer’s utility level for the poor health group; (b) an increase in a buyer’s utility level for the group with average health; and (c) a decrease in a buyer’s utility level for the good health group.*

The proof of Proposition 4 is given in the Online [Appendix](#). Most parts of the proof are similar to those of Proposition 3.

Propositions 3 and 4 appear to be quite different, with one threshold for the VPac plan in Proposition 3 but two thresholds for the MPAf plan in Proposition 4 when (39) holds.²⁵ There is actually a similarity between Propositions 3 and 4 in that the utility difference ($U_{\theta}^* - \widehat{U}_{\theta}^*$) for either the VPac or MPAf plan is first increasing in θ and then decreasing in θ after reaching a maximum. As a result, there is a single peak in the $U_{\theta}^* - \widehat{U}_{\theta}^*$ function for either PA plan. The above framework also helps understand the main factor leading to Proposition 4. When f is set according to condition (39), the floor has a strong effect in restricting the PA purchases by retirees of poor health. Thus, the distortion caused by the mandated floor level dominates the severity reduction effect, leading to (41) for the least healthy buyers. Together with the single peak property of the $U_{\theta}^* - \widehat{U}_{\theta}^*$ function, it can be shown (in the Online [Appendix](#)) that there are three categories of buyers.²⁶

5.3 Which PA plan to adopt: Voluntary or mandatory?

Our analysis provides some guidance regarding which PA plan the government wants to adopt. Whatever the MPAf or VPAC plan is adopted, the average health group benefits but the good health group is adversely affected.²⁷ The intuition is that the annuity buyers with average health purchase some, but not excessive, amount of annuity before the PA plan is introduced. Thus, when the PA plan is introduced, they benefit from the higher PA payout but their optimal choices of annuity quantity are not severely distorted by the restriction of either plan. On the other hand, the negative private market distortion effect is more important to the good health group, leading to a drop in utility level. For the poor health group, the change in utility level depends on which PA plan is adopted. In particular, retirees with poor health are adversely affected in the MPAf plan with condition (39) because the minimum mandated level of annuity purchase differs substantially from their desired levels in the absence of the plan, but benefit from the VPAC plan because there is no floor restriction that would otherwise distort the amount of their PA purchase.

While the government's ultimate decision to adopt a particular PA plan depends on many economic and political factors, our analysis based on a simple two-period model brings out some important factors (the severity reduction and private market distortion effects) clearly and shows how the two existing PA plans have systematically different utility effects on the retirees with different health characteristics.²⁸

6. Conclusion

There are two major types of observed PA plans: voluntary plans with ceiling restriction and mandatory plans with some flexibility. We study the similarities and differences of these PA plans. In both plans, the restrictive element due to the quantity restriction leads to a positive effect that mitigates the severity of adverse selection in the PA sector, but it also causes a negative effect that amplifies the distortion in the private market. On the other hand, the flexibility component performs the moderating function that reduces the magnitude of each of these two effects. Our results suggest that the consequence of offering the PA plan is a compromise of its positive severity reduction effect on public annuities and its negative private market distortion effect on private annuities.

Our study provides guidance to the economies planning to introduce the PA plan. First, the introduction of the PA plan, either voluntary or mandatory, leads to a two-tier annuity market. Retirees first buy the PA because it offers a higher payout. The private annuity market still has a role to play, as some buyers have excessive demands that are unfulfilled by the restriction of the PA plan and purchase annuities from the private market. Second, the introduction of a PA plan, either mandatory or voluntary, divides the population into different groups based on whether they benefit or are adversely affected in the newly introduced PA plan. Our analysis suggests that the retirees with average survival probability benefit from the PA plan, whereas the retirees with a high level of survival probability are adversely affected. The effects on the retirees with a low level of survival probability depend on which PA plan is offered; in particular, they are adversely affected in the MPAf plan when the floor parameter (f) is set at a not-too-low level. Before deciding to introduce a particular PA plan to the retirees, the government should consider seriously the effects on different groups of retirees who benefit from or are adversely affected by the plan.

Supplementary material. To view supplementary material for this article, please visit <https://doi.org/10.1017/S1365100523000093>.

Notes

1 Public annuities, offered only by the government or its delegated agents, can be either mandatory (as in Denmark, Lithuania, Singapore, or Sweden) or voluntary (as in Hong Kong or India). On the other hand, while the decision in mandating the citizens to buy annuities can only be made by the government, the annuities can be either supplied by a public authority (as in Singapore) or by the private sector (as in the UK before the past decade). In this paper, we only consider

mandatory PA plans in which the annuities are provided by the government, but do not consider issues related to mandatory annuities that are provided by the private sector.

2 We assume asymmetric, instead of symmetric, information between annuity buyers and providers because of three reasons. First, it is consistent with a lot of empirical evidence, even though there are skepticisms such as whether some evidence based on self-assessed survival probability information is reliable or not. Second, the results under the assumption of symmetric information are straightforward and well known. Third and most important, it can be shown that offering public annuities does not deliver beneficial effect to the retirees under the assumption of symmetric information. (An example is presented in the Online Appendix.) On the other hand, the PA plans affect the annuity payouts and buyers' welfare when survival probability information is asymmetric, as shown in Sections 4 and 5.

3 In the literature of annuity market under asymmetric information, another commonly used specification is one with two health states. For example, Brugiavini (1993) assumes that survival probability takes two possible states in Section 3.1 of her paper but assumes that it is a continuous random variable in Section 3.2. Some of the phenomena (such as the positive correlation of annuity purchase and risk type) can be shown using either specification, and the analysis in a two-state model is sometimes preferred because of its simplicity. We adopt the more general specification of continuous survival probability distribution in this paper, because some of our results (such as the welfare effect on three distinct groups of annuity buyers in Proposition 4) are clearly revealed when this specification is used, but not with a two-state model.

4 This well-known difficulty regarding exclusive contracts with convex pricing in the insurance markets (such as the annuity market) has also been mentioned by Cawley and Philipson (1999, p. 831): "several small contracts would be cheaper than a large one under convex pricing" and by Chiappori (2000, p. 369): "agents can always 'linearize' the schedule by buying a large number of small contracts from different insurers."

5 The homotheticity property is satisfied, for example, when the utility function is additively separable over time, as in (1), with CRRA specification for $u(c_{it})$: $u(c_{it}) = \frac{c_{it}^{1-\phi}-1}{1-\phi}$, where ϕ is the coefficient of relative risk aversion. This specification has been commonly used in the literature [such as Abel (1986), Einav et al. (2010), and Hosseini (2015)].

6 Since the first-order condition holds at an arbitrary value of \widehat{V} and not only at the equilibrium value \widehat{V}^* , defined in (8), we use \widehat{V} in (6). The same idea is behind (15) and (23). The more general specification is particularly useful in Appendix A.

7 We use a life-cycle model with constant interest rate in this paper, instead of an overlapping-generations model with endogenous interest rate [such as Heijdra et al. (2014)]. It is generally more difficult to obtain analytical results in overlapping-generations models.

8 Note that under the assumption of no cost differential in private or public provision of the annuities, the outcome under a pure VPA plan is the same as that in Section 3.2.

9 Some researchers [such as Diamond (2004)] argue that it is less costly for the government to provide the annuity product, perhaps because of the lower cost due to economies of scale or the government's willingness to bear various risks associated with issuing long-term financial products. Since this paper focuses on the flexibility versus restrictiveness issue of PA policy design, whether there is any cost differential in the private versus PA provision is not a crucial factor. We aim to deliver clean results in an environment without this cost differential and assume the zero-profit condition for both private and PA provision.

10 https://www.hkma.gov.hk/eng/our_business/hkmc_annuity_plan.html

11 There is also a minimum purchase requirement in the HKMC Annuity Plan in Hong Kong. However, this level is very low (at HKD50,000, which is 1% of the maximum purchase level). This restriction is imposed to avoid dealing with very low levels of PA purchase. We do not model this aspect in our paper.

12 Note that in the definition of \bar{m} according to (22), G^* and V^* are functions of m but not \bar{m} . It is easy to show from (22) that \bar{m} is unique for a VPAC plan with ceiling m .

13 Similar to the uniqueness of \widehat{V}^* in Section 3.2, we can use the results in Lau et al. (2022) to show that G^* is unique when the retirees' survival probabilities are, for example, uniformly distributed.

14 <https://www.cpf.gov.sg/member/faq/retirement-income/general-information-on-retirement/what-are-the-retirement-sums-applicable-to-me>

15 Similar to Singapore, the PA plan in Sweden is mandatory in participation but has some flexibility. Members of the Premium Pension Authority in Sweden are allowed to annuitize 25%, 50%, 75%, or 100% of their pension entitlements. Since it is less convenient to conduct the analysis when the annuitization choice is a discrete variable, we model the MPAF plan based on the experience in Singapore.

16 Compared with condition (21) on the ceiling parameter (m), condition (28) on the floor parameter (f) is less restrictive. This is because unlike parameter m of the PA plan which directly affects the extent of crowding out of the private annuity market, parameter f does not.

17 We have tried different parameter values in numerical analysis for the VPAC and MPAF plans and confirmed that conditions (21) and (28) hold in many cases.

18 Note that for the VPAC plan, there is another possible equilibrium in which the retirees proportionally divide their annuity purchase between the public and private annuities according to, for example, $\gamma_{\theta}^* = (m/\widehat{v}_{\theta}^*)\widehat{v}_{\theta}^*$ and $v_{\theta}^* = (1 - m/\widehat{v}_{\theta}^*)\widehat{v}_{\theta}^*$, where the scaling factor $(m/\widehat{v}_{\theta}^*)$ ensures that the ceiling of the VPAC plan is not violated for all retirees. We do not consider this

equal-payout outcome ($G^* = V^*$) because of two reasons. First, this outcome cannot be an equilibrium for MPAf plan with (28) because in the presence of the minimum mandated level of PA purchase (f), least healthy buyers still have to purchase f units of PA ($\gamma_\theta^* = f$), which is larger than $(m/\widehat{v}_\theta^*)\widehat{v}_\theta^*$. More importantly, this outcome is not an equilibrium even for the VPAC plan if there is a slight cost disadvantage in the provision of private annuities, due to taxation requirement or smaller scale of operation.

19 A direct way to prove $V^* < G^*$ is presented in the Online Appendix.

20 On the other hand, the government, being the sole PA provider, can easily verify a retiree’s identity and enforce the maximum purchase restriction.

21 The ratio $\frac{v_\theta(\widehat{V}^*, w, m)}{\widehat{v}_\theta^*}$ in (A14), which is zero for $\theta \leq \frac{(1+\rho)u'(w-m)}{V^*u'(G^*m)}$, represents the truncation effect that less healthy retirees will not participate in the private annuity market after the PA plan is introduced, according to (18). On the other hand, $\frac{v_\theta(\widehat{V}^*, w, m)}{\widehat{v}_\theta^*}$ is increasing in θ when $\theta > \frac{(1+\rho)u'(w-m)}{V^*u'(G^*m)}$, which captures the effect of the PA plan on the purchase level of private annuity by healthier retirees.

22 Paul Keating, former prime minister of Australia, observed that when Australia’s superannuation system was designed (by his government) three decades ago, people retired at about 65 and died at 81, but they are now living nine years longer. To fill the gap that there is no policy in Australia for the 80 to 100 category except for relying on the aged pension, he suggested a national insurance scheme in which “fund members who died early would pay for those who lived longer, covering the extended cost of health and housing provision for the elderly.” [The Guardian (2018)]. The analysis of this section is useful if his idea is implemented and the government chooses between the VPAC and MPAf plans. Detailed discussion about retirement income protection in Australia, especially the issues related to longevity insurance products, can be found in Bateman and Piggott (2011).

23 The threshold (θ^{VPAC}) separating the annuity buyers into two categories is defined such that (a) $U_\theta^* - \widehat{U}_\theta^* > 0$ for $\theta \in (\underline{\theta}, \theta^{VPAC})$; and (b) $U_\theta^* - \widehat{U}_\theta^* < 0$ for $\theta \in (\theta^{VPAC}, \bar{\theta})$. See Appendix B and Figure A2.

24 If the government chooses a mandatory PA plan, it is probably because they want the retirees to purchase at least a non-trivial amount of public annuities to insure against longevity risk. Condition (39) is more consistent with this idea.

25 It is straightforward to show that: *If m satisfies (21) and f satisfies (42), the MPAf plan systematically separates annuity buyers in two categories according to the change in utility level: an increase in a buyer’s utility level for the group with poor or average health and a decrease in a buyer’s utility level for the good health group.* The intuition of this result is that the floor parameter f has only a mild effect in restricting the PA purchases of poor health retirees when f is relatively low such that (42) holds. Thus, the severity reduction effect dominates the distortion caused by the mandated floor level for the least healthy annuity buyers (with $\theta = \underline{\theta}$), and there are two categories of buyers in this case. Since this outcome is similar to that under the VPAC plan, we do not focus on this case in Proposition 4.

26 When (39) holds, there are two thresholds (θ_L^{MPAf} and θ_H^{MPAf}) separating the annuity buyers into three categories. They are defined such that (a) $U_\theta^* - \widehat{U}_\theta^* < 0$ for $\theta \in (\underline{\theta}, \theta_L^{MPAf})$; (b) $U_\theta^* - \widehat{U}_\theta^* > 0$ for $\theta \in (\theta_L^{MPAf}, \theta_H^{MPAf})$; and (c) $U_\theta^* - \widehat{U}_\theta^* < 0$ for $\theta \in (\theta_H^{MPAf}, \bar{\theta})$. (These points are illustrated in Figure A3 of the Online Appendix.)

27 The terms “good health,” “average health,” and “poor health” can be precisely defined based on the threshold(s) of Propositions 3 and 4. We use these terms in a general and non-technical sense here (without referring to the precise definition based on these thresholds), when we discuss the underlying economic reasons.

28 This paper analyzes voluntary and mandatory PA plans in the presence of adverse selection. In this context, the crucial element is health heterogeneity, not wealth heterogeneity. For modeling purpose, we use the simplifying assumption of wealth homogeneity, which helps deliver sharp and useful results. A natural question is whether some of the above results will disappear when there is wealth heterogeneity, which is observed in many economies. In the Online Appendix, we present computational results with wealth heterogeneity. The main results of this paper continue to hold in the more general environment.

29 The underlying idea of the proof is as follows. It can be seen from (23) and (A1) that the function $\gamma_\theta(G, w, m, f)$ represents the amount of PA purchase by a buyer with survival probability θ when the payout is G (and other parameters of the plan are m and f). Based on buyers’ choices in (A1), we define $J(G, w, m, f)$ in (A2) such that the PA is in surplus if $J(G, w, m, f) > G$ but is in deficit if $J(G, w, m, f) < G$. Thus, under the zero-profit condition, the equilibrium payout of the PA, denoted by G^* , is defined by (A3).

30 Expanding (A4) and using (8) and (A2), we obtain

$$2 \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} x\gamma_x(\widehat{V}^*, w, m, f)\widehat{v}_y^*dF(x)dF(y) - 2 \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} x\gamma_y(\widehat{V}^*, w, m, f)\widehat{v}_x^*dF(x)dF(y) < 0,$$

and thus,

$$\left[\int_{\underline{\theta}}^{\bar{\theta}} x\gamma_x(\widehat{V}^*, w, m, f) dF(x) \right] \left[\int_{\underline{\theta}}^{\bar{\theta}} \widehat{v}_y^* dF(y) \right] < \left[\int_{\underline{\theta}}^{\bar{\theta}} x\widehat{v}_x^* dF(x) \right] \left[\int_{\underline{\theta}}^{\bar{\theta}} \gamma_y(\widehat{V}^*, w, m, f) dF(y) \right].$$

After replacing the arbitrary indexes x and y by θ , we obtain (A5).

31 The steps leading to (A16) are similar (but opposite in sign) to those leading to (A5).

32 These two inequalities, together with (31) and the two budget constraints, lead to $w = \widehat{c}_{1\theta}^* + \frac{\widehat{c}_{2\theta}^*}{\widehat{V}^*} > c_{1\theta}^* + \frac{c_{2\theta}^*}{G^*} = w$. Thus, it is a contradiction.

33 These two inequalities, together with (6), (23), and (31), lead to $u'(c_{1\theta}^*) < u'(\widehat{c}_{1\theta}^*) = \frac{\theta \widehat{V}^*}{1+\rho} u'(\widehat{c}_{2\theta}^*) < \frac{\theta G^*}{1+\rho} u'(c_{2\theta}^*) = u'(c_{1\theta}^*)$. Thus, it is a contradiction.

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Appendix

We prove Propositions 1 and 3 in Appendices A and B, respectively.

Appendix A: Proof of Proposition 1

We focus on the MPAf plan in this Appendix. The proof for the simpler VPac plan is similar and is given in the Online Appendix.

(A) Under the MPAf plan with (27) and (28), define the function

$$\gamma_\theta (G, w, m, f) = \begin{cases} f & \text{if } \gamma_\theta^* \leq f \\ \gamma_\theta^* & \text{if } f < \gamma_\theta^* < m, \\ m & \text{if } \gamma_\theta^* \geq m \end{cases} \tag{A1}$$

where the optimal PA purchase γ_θ^* is defined in (23). Based on (A1), define the function

$$J(G, w, m, f) = \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} \gamma_\theta (G, w, m, f) dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta \gamma_\theta (G, w, m, f) dF(\theta)} \tag{A2}$$

Based on (30) and (A2), the equilibrium payout (G^*) is defined by²⁹

$$J(G^*, w, m, f) = \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} \gamma_\theta (G^*, w, m, f) dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta \gamma_\theta (G^*, w, m, f) dF(\theta)} = G^* \tag{A3}$$

We study $J(G, w, m, f)$ as a function of G when $G \in \left[\widehat{V}^*, \frac{1+r}{E(\theta)} \right]$ and look for the intersection of this function and the 45-degree line, as shown in Panel A of Figure A1. First, consider $J(G, w, m, f)$ at the beginning point $G = \widehat{V}^*$. It can be shown from (6), (23), and (A1) that $\frac{\gamma_\theta(\widehat{V}^*, w, m, f)}{\widehat{v}_\theta^*}$ (i) is larger than 1 and strictly decreasing for low value of θ such that $\widehat{v}_\theta^* < f$, (ii) is less than 1 and strictly decreasing for high value of θ such that $\widehat{v}_\theta^* > m$, or (iii) equals to 1 for the intermediate values of θ . As a result, the function $\frac{\gamma_\theta(\widehat{V}^*, w, m, f)}{\widehat{v}_\theta^*}$ is weakly decreasing for all $\theta \in [\underline{\theta}, \bar{\theta}]$, with a strictly decreasing part over some interval. Thus, the Chebyshev’s Sum

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} (x - y) \left[\frac{\gamma_x(\widehat{V}^*, w, m, f)}{\widehat{v}_x^*} - \frac{\gamma_y(\widehat{V}^*, w, m, f)}{\widehat{v}_y^*} \right] \widehat{v}_x^* \widehat{v}_y^* dF(x) dF(y) \tag{A4}$$

is negative, where x and y are two arbitrary indexes. This leads to³⁰

$$J(\widehat{V}^*, w, m, f) = \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} \gamma_\theta(\widehat{V}^*, w, m, f) dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta \gamma_\theta(\widehat{V}^*, w, m, f) dF(\theta)} > \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} \widehat{v}_\theta^* dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta \widehat{v}_\theta^* dF(\theta)} = \widehat{V}^* \tag{A5}$$

Second, consider $J(G, w, m, f)$ at the end point $G = \frac{1+r}{E(\theta)}$. It is easy to show that

$$J\left(\frac{1+r}{E(\theta)}, w, m, f\right) < \frac{1+r}{E(\theta)}, \tag{A6}$$

because when adverse selection is present, the PA payout based on the retirees’ behavior (the LHS term) is always lower than the (hypothetical) payout based on the average survival probability of the retirees (the RHS term).

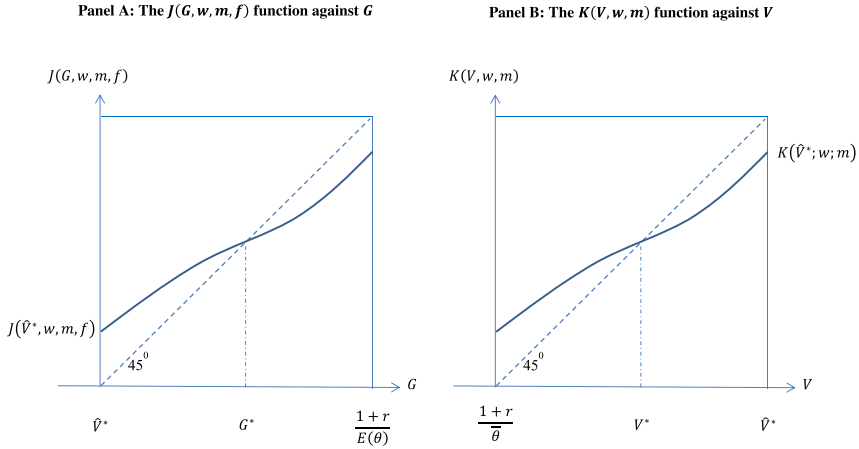


Figure A1. The determination of equilibrium annuity payouts.

The $J(G, w, m, f)$ function is above the 45-degree line at $G = \widehat{V}^*$, and below it at $G = \frac{1+r}{E(\theta)}$. Together with the continuity of the $J(G, w, m, f)$ function, we conclude that there exists an equilibrium value of G^* which is larger than \widehat{V}^* . (See Panel A of Figure A1.) This proves part (a) of Proposition 1.

(B) Based on the optimal private annuity purchase (v_θ^*) in (15) with $G = G^*$, define the function

$$v_\theta(V, w, m) = \max \{0, v_\theta^*\}, \tag{A7}$$

where other parameter values (including G^*) are unchanged. Based on (A7), we define the function

$$K(V, w, m) = \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} v_\theta(V, w, m) dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta v_\theta(V, w, m) dF(\theta)}. \tag{A8}$$

Based on (18) and (A8), the equilibrium private annuity payout (V^*) in the presence of the PA plan is defined by

$$K(V^*, w, m) = \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} v_\theta(V^*, w, m) dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta v_\theta(V^*, w, m) dF(\theta)} = V^*. \tag{A9}$$

We study $K(V, w, m)$ as a function of V when $V \in \left[\frac{1+r}{\theta}, \widehat{V}^*\right]$ and look for the intersection of this function and the 45-degree line, as shown in Panel B of Figure A1. First, consider $K(V, w, m)$ when $V = \frac{1+r}{\theta}$. It is easy to show that

$$K\left(\frac{1+r}{\theta}, w, m\right) > \frac{1+r}{\theta}, \tag{A10}$$

because there are some buyers who have residual demand (after buying the PA) even when the annuity payout is as low as $\frac{1+r}{\theta}$. As long as some buyers demand the private annuity, (A10) holds.

Second, consider $K(V, w, m)$ when $V = \widehat{V}^*$. Our objective is to compare $v_\theta(\widehat{V}^*, w, m)$ with $\widehat{v}_\theta^* \equiv v_\theta(\widehat{V}^*, w, 0)$. A direct comparison turns out to be difficult, and we rely on an intermediate term: $v_\theta(\widehat{V}^*, w^h, 0)$, as follows.

For a retiree whose desired annuity choice is not constrained by the PA plan (i.e. it is an interior solution), we consider the hypothetical environment in which she is given extra resources up to

the new wealth level of w^h but is only allowed to buy the private annuity with payout \widehat{V}^* . We want to find the level of w^h such that the individual's optimal choices of $c_{1\theta}$ and $c_{2\theta}$ will be the same (and thus the first-order condition is the same) as those chosen at her original level of wealth (w) when both PA (with the equilibrium payout G^*) and private annuity (with the hypothetical payout \widehat{V}^*) are available. Since $\gamma_\theta^* = m$ for any retiree in this group, we can use (3), (4), (10), and (11) to show that the optimal annuity choices are related by

$$v_\theta \left(\widehat{V}^*, w^h, 0 \right) = v_\theta \left(\widehat{V}^*, w, m \right) + \frac{G^*}{\widehat{V}^*} m, \tag{A11}$$

and the actual and hypothetical levels of wealth are related by

$$w^h = w + \left(\frac{G^*}{\widehat{V}^*} - 1 \right) m, \tag{A12}$$

where $v_\theta(\widehat{V}^*, w, m)$ and $v_\theta(\widehat{V}^*, w^h, 0)$ are determined according to (A7).

We now examine $\frac{v_\theta(\widehat{V}^*, w^h, 0)}{v_\theta(\widehat{V}^*, w, 0)}$, where $v_\theta(\widehat{V}^*, w^h, 0)$ and $v_\theta(\widehat{V}^*, w, 0)$ are the optimal choices in two similar environments which have the same annuity payout (\widehat{V}^*) but different wealth levels. When the utility function $U(c_{1\theta}, c_{2\theta}; \theta)$ in (1) is homothetic, condition (2) holds, leading to the result that the wealth elasticity of consumption at either period is unity. Together with (4), the share of annuity purchase to wealth is constant. Putting $t = \frac{w^h}{w}$ in (2) where w^h is defined in (A12), it can be shown that

$$\frac{v_\theta \left(\widehat{V}^*, w^h, 0 \right)}{v_\theta \left(\widehat{V}^*, w, 0 \right)} = \frac{w^h}{w} = 1 + \left(\frac{G^*}{\widehat{V}^*} - 1 \right) \frac{m}{w} \tag{A13}$$

does not depend on θ .

Combining (A11) and (A13), it can be shown that

$$\frac{v_\theta \left(\widehat{V}^*, w, m \right)}{\widehat{v}_\theta^*} = \left[\frac{v_\theta \left(\widehat{V}^*, w, m \right)}{v_\theta \left(\widehat{V}^*, w, m \right) + \frac{G^*}{\widehat{V}^*} m} \right] \frac{v_\theta \left(\widehat{V}^*, w^h, 0 \right)}{v_\theta \left(\widehat{V}^*, w, 0 \right)} = \frac{1 + \left(\frac{G^*}{\widehat{V}^*} - 1 \right) \frac{m}{w}}{1 + \frac{G^*}{\widehat{V}^*} \frac{m}{v_\theta \left(\widehat{V}^*, w, m \right)}} \tag{A14}$$

is strictly increasing in θ for $\theta > \frac{(1+\rho)u'(w-m)}{\widehat{V}^*u'(G^*m)}$, because $v_\theta(\widehat{V}^*, w, m)$ is increasing in θ for interior solutions. Together with $\frac{v_\theta(\widehat{V}^*, w, m)}{\widehat{v}_\theta^*} = 0$ for $\theta \leq \frac{(1+\rho)u'(w-m)}{\widehat{V}^*u'(G^*m)}$, we conclude that the Chebyshev's Sum

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} (x - y) \left[\frac{v_x \left(\widehat{V}^*, w, m \right)}{\widehat{v}_x^*} - \frac{v_y \left(\widehat{V}^*, w, m \right)}{\widehat{v}_y^*} \right] \widehat{v}_x^* \widehat{v}_y^* dF(x) dF(y) \tag{A15}$$

is positive, leading to³¹

$$K \left(\widehat{V}^*, w, m \right) = \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} v_\theta \left(\widehat{V}^*, w, m \right) dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta v_\theta \left(\widehat{V}^*, w, m \right) dF(\theta)} < \frac{(1+r) \int_{\underline{\theta}}^{\bar{\theta}} \widehat{v}_\theta^* dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} \theta \widehat{v}_\theta^* dF(\theta)} = \widehat{V}^*. \tag{A16}$$

Combining (A9), (A10), (A16) and the continuity of the $K(V, w, m)$ function, we conclude that there exists an equilibrium value of V^* which is smaller than \widehat{V}^* . This proves part (b) of Proposition 1.

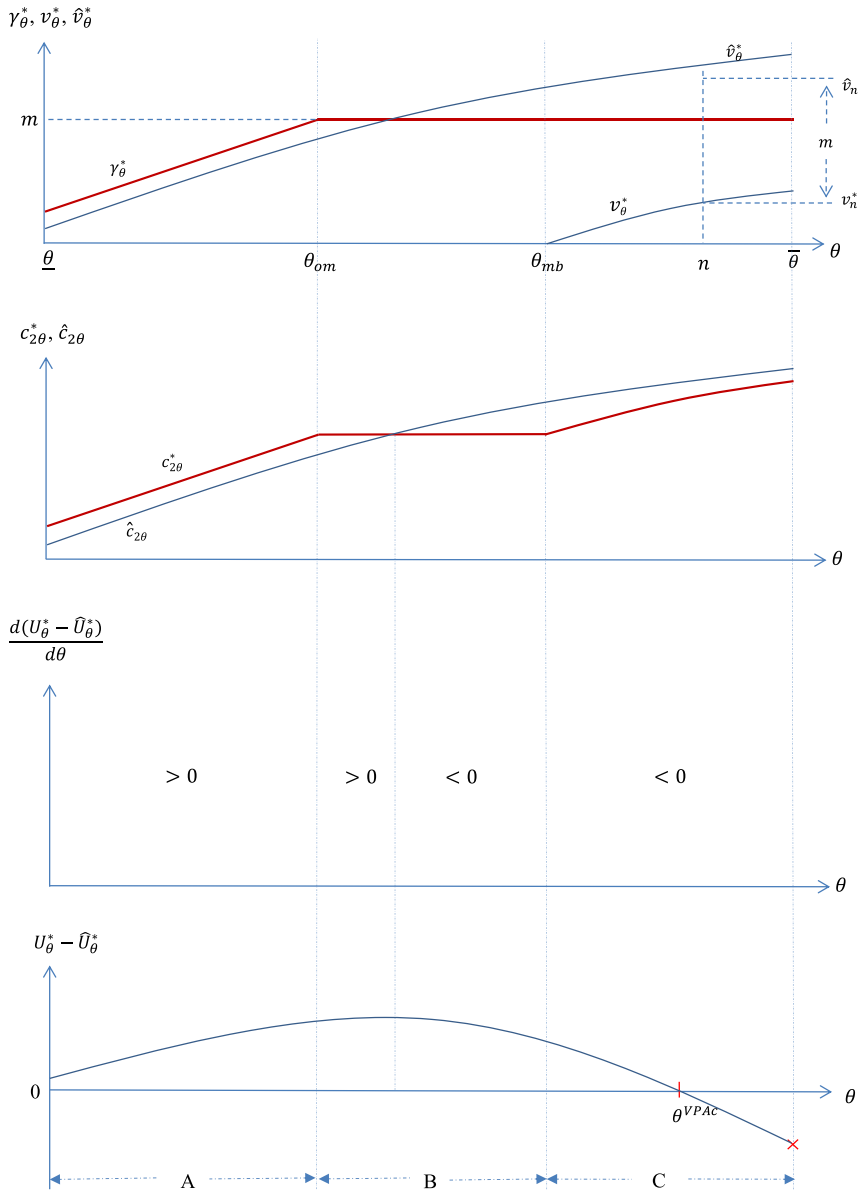


Figure A2. Utility comparison under the VPAC plan.

Appendix B: Proof of Proposition 3

For the VPAC plan, it is convenient to label $\theta \in [\underline{\theta}, \theta_{om})$ as Region A, $\theta \in [\theta_{om}, \theta_{mb})$ as Region B, and $\theta \in [\theta_{mb}, \bar{\theta}]$ as Region C (See Figure A2.)

The following lemma is useful for the analysis regarding Region C. The proof is given in the Online Appendix.

Lemma A1. In Region C, $c_{2\theta}^*$ and $\hat{c}_{2\theta}^*$ either intersect once or do not intersect.

In the following, we first prove Lemma 1. We then use this result, as well as Lemmas 2 and A1, to prove Proposition 3.

B.1. Proof of Lemma 1

Differentiating (5) with respect to θ , evaluated at the optimal choices, and using the first-order condition (6), we obtain (36).

The proof for the economy after the introduction of the VPAC plan is similar but slightly more tedious. Differentiating (1) with respect to θ , evaluated at the optimal choices, and using (23), we obtain (37) in Region A. Similarly, differentiating (1) with respect to θ , evaluated at the optimal choices, and using (15), we obtain (37) in Region C. The analysis for Region B is simpler because the optimal choice of $c_{1\theta}^*$ is the same for all buyers in this region. As a result, (37) holds.

Combining (36) and (37) leads to (35).

B.2. Proof of Proposition 3

First, we focus on Region A. The choices $\widehat{c}_{1\theta}^*$ and $\widehat{c}_{2\theta}^*$ are feasible under the VPAC plan because of (31), but they are not optimal since they do not satisfy the first-order condition (23). Moreover, the combination of $c_{1\theta}^* \leq \widehat{c}_{1\theta}^*$ and $c_{2\theta}^* \leq \widehat{c}_{2\theta}^*$ is inconsistent with the two lifetime budget constraints (with and without the PA plan),³² and the combination of $c_{1\theta}^* > \widehat{c}_{1\theta}^*$ and $c_{2\theta}^* \leq \widehat{c}_{2\theta}^*$ is inconsistent with the two relevant first-order conditions (6) and (23).³³ Combining the above results, we conclude that $c_{2\theta}^* > \widehat{c}_{2\theta}^*$ and $U_\theta^* > \widehat{U}_\theta^*$ in Region A. This is Result 1.

Applying Lemma 1, we conclude that $\frac{\partial(U_\theta^* - \widehat{U}_\theta^*)}{\partial\theta} > 0$ in Region A; thus, $U_\theta^* - \widehat{U}_\theta^*$ is increasing in this region. Combining it with $U_\theta^* > \widehat{U}_\theta^*$ in Region A, we conclude that $U_\theta^* - \widehat{U}_\theta^*$ starts from a positive value when $\theta = \underline{\theta}$, and it increases and continues to be positive in Region A. This is Result 2.

Next, it is easy to see that $c_{2\theta}^*$ is constant and $\widehat{c}_{2\theta}^*$ is increasing in Region B. Together with Result 1, we conclude that there are two possible cases in Region B: (a) $\widehat{c}_{2\theta}^*$ is increasing and passes the constant level of $c_{2\theta}^*$ once, and as a result, $U_\theta^* - \widehat{U}_\theta^*$ is first increasing and then decreasing after reaching a maximum; or (b) $\widehat{c}_{2\theta}^*$ is increasing but still below $c_{2\theta}^*$, and as a result, $U_\theta^* - \widehat{U}_\theta^*$ is increasing (and is always positive). Call them Case (a) and Case (b), respectively.

Regarding Region C, we show that the combination of $c_{1\theta}^* < \widehat{c}_{1\theta}^*$ and $c_{2\theta}^* \geq \widehat{c}_{2\theta}^*$ is inconsistent with (32) and the first-order conditions (6) and (15). On the other hand, the combination of $c_{1\theta}^* \geq \widehat{c}_{1\theta}^*$ and $c_{2\theta}^* \geq \widehat{c}_{2\theta}^*$ is inconsistent with (38). Therefore, we conclude that $c_{2\theta}^* < \widehat{c}_{2\theta}^*$. This is Result 3.

Combining Results 1 to 3 and Lemma A1, we end up with two possibilities. First, with Case (a), we conclude that $c_{2\theta}^*$ and $\widehat{c}_{2\theta}^*$ do not intersect in Region C, since intersecting once in Region C will lead to an inconsistency with Result 3. Therefore, there is one threshold, labeled as θ^{VPAC} , which appears in either Region B or C. (Some key features are shown in Figure A2, which is based on Case (a).)

On the other hand, with Case (b), we conclude that $c_{2\theta}^*$ and $\widehat{c}_{2\theta}^*$ intersect once in Region C, since no intersection will lead to an ever-increasing (and positive) $U_\theta^* - \widehat{U}_\theta^*$, which is inconsistent with (38) in Lemma 2. Therefore, there is one threshold in Region C.

Cite this article: Lau S-HP and Zhang Q (2024). "Severity reduction and private market distortion effects of voluntary and mandatory public annuity plans." *Macroeconomic Dynamics* 28, 403–425. <https://doi.org/10.1017/S1365100523000093>