AXIOMS FOR ABSOLUTE GEOMETRY. II

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Introduction. In this paper I continue the process, begun in (**2**), of reducing and weakening the axioms of congruence needed for absolute geometry. The congruence axioms **C1*–C4***, **C4****, and **C5a–C5c** (frequently referred to below) can all be found in (**2**) and will not be quoted again here. (This paper should be read in conjunction with (**2**); any attempt to make it self-contained would result in the repetition of large parts of (**2**).) The notation of (**2**) will be used throughout the paper.

The main result here is that axiom **C5c** is unnecessary. This is shown in § 1. In § 2 I discuss three other points arising from (**2**).

Note added in proof. Since writing this paper, I have constructed examples of (a) Archimedean planes satisfying **C1*–C4*** in which not all points are isometric, (b) non-Archimedean planes satisfying **C1*–C4*** but not **C4****, and (c) one-dimensional geometries in which 2.1 (with "plane" replaced by "line") is false. These examples are relevant to various remarks in § 2. The various examples of planes will appear in (**3**), and the remaining examples at a later date.

1. The existence and construction of perpendicular lines. Throughout this section (which replaces part of (**2**, § 5)) we shall consider an absolute plane π satisfying the axioms of order and axioms **C1*–C4***, **C4****, **C5a**, **C5b**. *All the points and lines referred to lie in* π. We shall show, without using axiom **C5c**, that perpendicular lines exist and can be constructed.

Let l be a line. If, through every point, either on l or not on l, there exists a line perpendicular to l, we shall say that *all perpendiculars to l exist.*

1.1. *If there exists a line perpendicular to a line l, then all perpendiculars to l exist.*

This is not an exact re-statement of (**2**, 4.5(i) and 4.7). We do not assume here that *we can construct* a line perpendicular to l, but the proofs of (**2**, 4.5(i) and 4.7) can be used to prove 1.1.

1.2. *If there exists a line perpendicular to a line l, and if m is a line that meets l, then all perpendiculars to m exist.*

Proof (see Figure 1A). Let $l \cap m = O$. If $m \perp l$, then $l \perp m$; hence all

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perpendiculars to m exist (1.1). If not, let $P \in m$, $P \neq O$, and let PQ be the perpendicular from P to l (1.1), where $Q \in l$. Then $Q \neq O$. There exists $R \in l$ such that Q bisects OR. There exists $X \in$ ray OQ such that $OP \equiv OX$ and there exist $Y, Z \in$ ray OP such that $OQ \equiv OY, OR \equiv OZ$ (so that Y bisects OZ). Then $XZ \equiv PR$ (**C5b**) $\equiv PO$ (by the definition of perpendicularity) $\equiv XO$. Hence $XY \perp m$ (**2**, 4.1). Hence all perpendiculars to m exist (1.1).

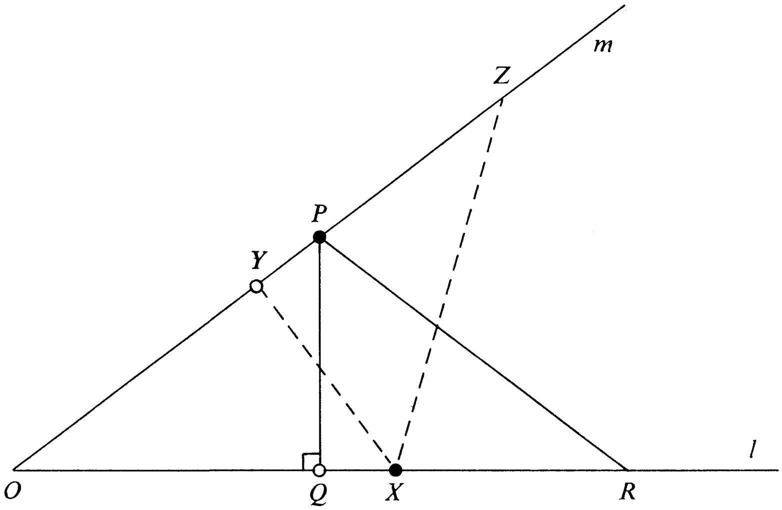

FIGURE 1A

1.3. *If there exists a pair of perpendicular lines, then all perpendiculars to all lines exist.*

Proof. Let l be one of the pair of perpendicular lines. Then all perpendiculars to l exist (1.1). Let n be any other line, and let m be any line joining a point of l to a point of n. Then m meets l, so that all perpendiculars to m exist (1.2), and n meets m, so that all perpendiculars to n exist (1.2).

COROLLARY. *Either all perpendiculars to all lines exist, or there exists no pair of perpendicular lines.*

1.4. *If there exists no pair of perpendicular lines, then there exists no pair of congruent triangles ABC and ABC' with C, C' on opposite sides of AB.*

Proof (see Figure 1B). Suppose that such a pair of triangles exists. Let $CC' \cap AB = X$, and suppose, without loss of generality, that $X \neq A$. Then $XC \equiv XC'$ (**C5a**). Also $AC \equiv AC'$; thus $AX \perp CC'$ (**2**, 4.1), a contradiction.

1.5. *All perpendiculars to all lines exist.*

Proof (see Figure 1C). Suppose the contrary; then there exists no pair of perpendicular lines (1.3, Corollary). Let A, M be distinct points. There exists

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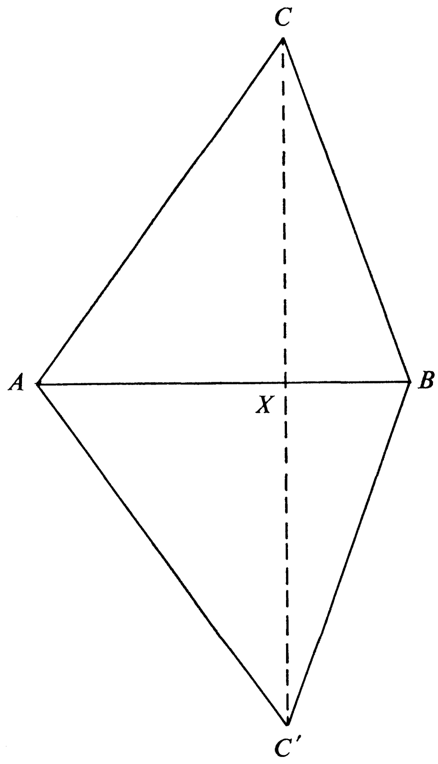

FIGURE 1B

$Q \in$ line AM such that M bisects AQ. Let O be any point not on line AM. Then $OA \not\equiv OQ$, for otherwise $OM \perp AQ$, a contradiction. There exists $P \in$ ray OA such that $OQ \equiv OP$, and there exists $B \in$ ray OQ such that $OA \equiv OB$. Then $P \neq A$, $Q \neq B$, and either $[OPA]$ and $[OQB]$ (as shown) or $[OAP]$ and $[OBQ]$. In either case, seg AQ meets seg PB at S, say. The points A, Q, B, P are all isometric; hence $AP \equiv BQ \equiv QB$. Also $AQ \equiv BP$ (**C5b**).

There exists $X \in$ ray SA such that $SB \equiv SX$. Now $SA \not\equiv SB$, for otherwise $\triangle OSA \equiv \triangle OSB$, contradicting 1.4; hence $X \neq A$. Suppose $[SXA]$; the proof is easily adapted if $[SAX]$. There exists $Y \in$ ray Q/A such that $AX \equiv QY$. Since X and Y are isometric to A, B, P, and Q, we easily see that $XY \equiv AQ$. Hence $XY \equiv BP$. Also $SX \equiv SB$; thus $SY \equiv SP$. Hence

$$PX \equiv YB \text{ } (\mathbf{C5b}) \equiv BY.$$

Hence $\triangle AXP \equiv \triangle QYB$.

There exists $Y' \in AQ$ such that A bisects XY'. Then M bisects YY'. Let B' be the reflection of B in M, and let B'' be the reflection of B' in A. The reflections in M and A are isometries (**2**, 3.5); thus $\triangle QYB \equiv \triangle AY'B' \equiv \triangle AXB''$. Hence $\triangle AXP \equiv \triangle AXB''$. However, P, B'' lie on opposite sides of AX. This contradicts 1.4. Hence our supposition is incorrect. Hence all perpendiculars to all lines exist.

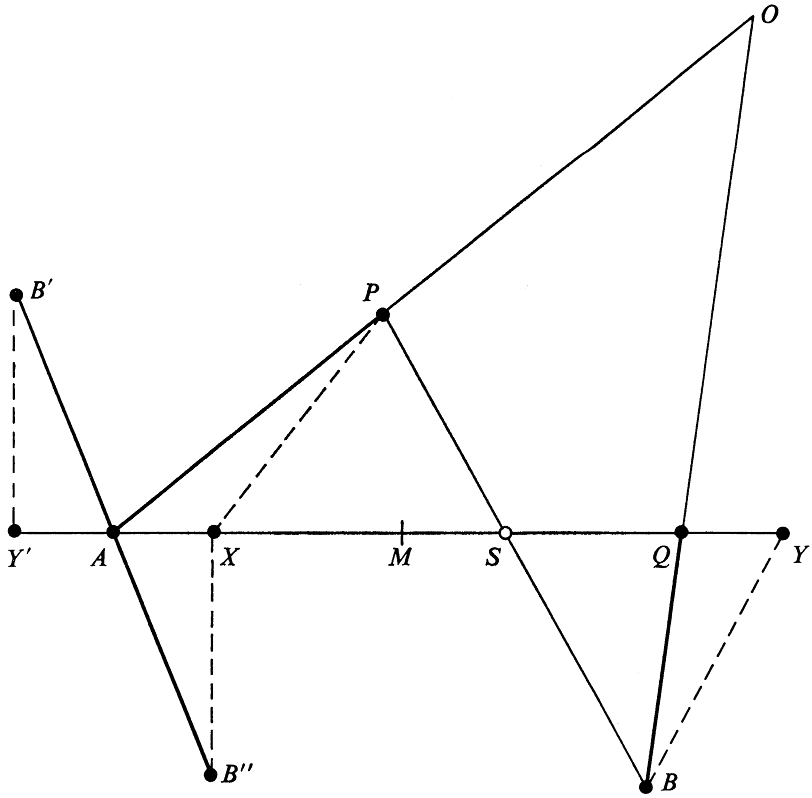

FIGURE 1C

1.6. *The reflection in every line exists and is an isometry.*

Proof. The existence of the reflection follows from **1.5**. For the rest of the proof, see (**2**, 4.6).

1.7. *If OA and OB are distinct congruent segments, then* (i) *the mid-point of AB exists, and* (ii) *the mid-point is constructible.*

Proof (i) (see Figure 1D). If O, A, and B are collinear, then O is the mid-point of AB. If not, then there exists a line through O perpendicular to AB (1.5) meeting AB at P, say. Let O' be the reflection of O in AB. Then $AO' \equiv AO \equiv BO \equiv BO'$. If $P = A$, then the collinear points O, A, and O' are congruent to the triangle OBO'. This contradicts (**2**, 3.2); thus $P \neq A$. Similarly, $P \neq B$. Hence $\triangle OO'A \equiv \triangle OO'B$. Hence $PA \equiv PB$ (**C5a**); thus P is the mid-point of AB.

(ii) (see Figure 1E). If O, A, and B are collinear, then there is nothing more to construct. If not, let Y be a point between O and A, and construct $Z \in$ ray OB such that $OY \equiv OZ$. Let M be the mid-point of AB, and let $OM \cap AZ = S$. Since $OM \perp AB$ by the proof of (i), the reflection in OM interchanges A and B. Hence this reflection maps Z onto Y (1.6 and **2**, 3.4). However, A, S, and Z are

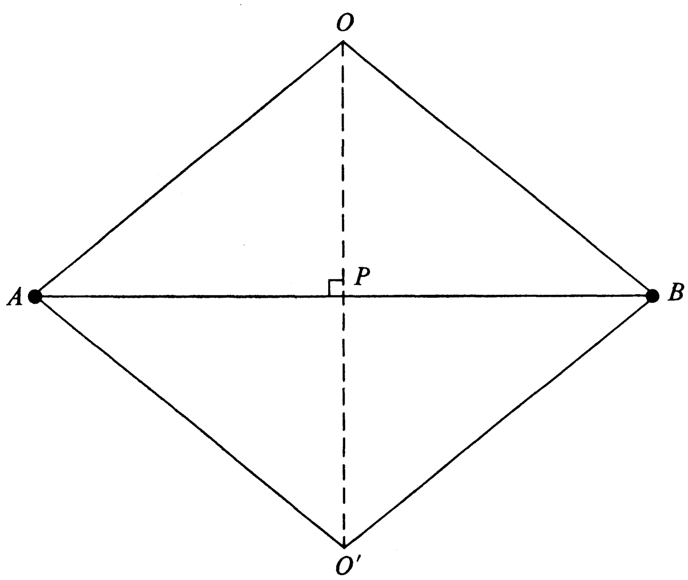

FIGURE 1D

collinear, therefore their reflections B, S, and Y are collinear (1.6 and **2**, 3.4). Hence $S = AZ \cap BY$; thus S is constructible. Therefore $M = OS \cap AB$ is constructible.

We can now prove (**2**, 5.3, 5.4 and 5.6), on the construction of perpendiculars. We can also prove **C5c**, though this is unnecessary, on the same lines as the proof of 1.7(ii).

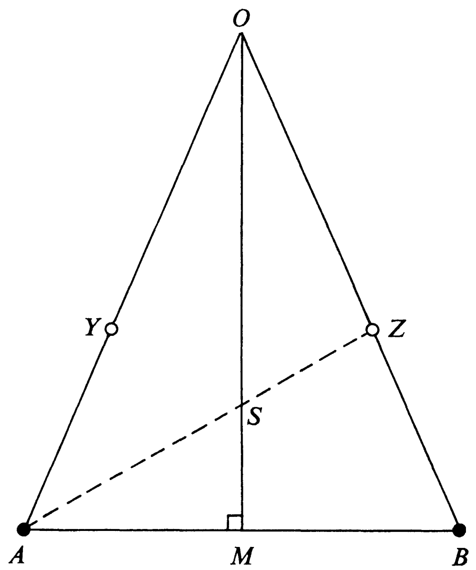

FIGURE 1E

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2. Further remarks.

Remark (a). In the comments on the proof of (**2**, 6.5), I stated that theorem 2.1 below was easily proved. This is not so; the simple proof that holds when all points are isometric cannot be used. I do not know whether 2.1 is true in a one-dimensional geometry.

2.1. *If A, B, and C are collinear points in an absolute plane, and if AB and BC have mid-points, then AC has a mid-point* (*unless $A = C$*).

Proof (see Figure 2A). Let M and N be the mid-points of AB and BC, and assume that $A \neq C$. Denote the reflections in M, B, and N by ρ_M, ρ_B, ρ_N, and write $\rho_M\rho_B\rho_N = \rho$. Then $A\rho = C$. Also ρ maps seg AC onto seg CA', say, where $AC \equiv CA'$ and AC, CA' have opposite senses (**1**, p. 75 *et seq.*). However, A and C are isometric, therefore $AC \equiv CA$; hence $A' = A$. Hence $C\rho = A$.

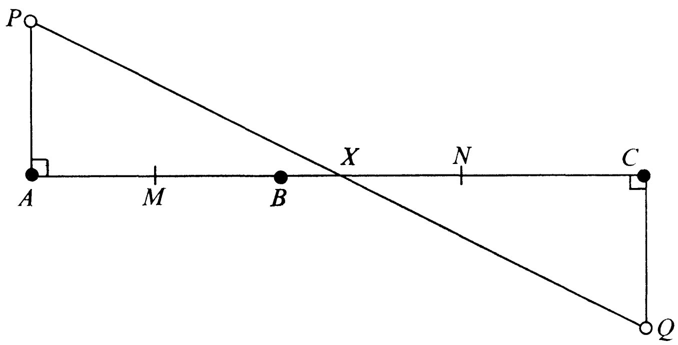

FIGURE 2A

Let P and Q be points on the perpendiculars to ABC at A and C such that P and Q lie on opposite sides of ABC and $AP \equiv CQ$. Then $P\rho = Q$ and $Q\rho = P$ since ρ is an isometry (using **2**; 4.8).

Let $PQ \cap AC = X$. Then $X\rho = (PQ)\rho \cap (AC)\rho = QP \cap CA = X$. Hence $XA \equiv (XA)\rho = XC$; thus X is the mid-point of AC.

Remark (b). I am still unable to deduce axiom **C4**** from the previous axioms alone, but it can be deduced if we also assume the *axiom of Archimedes* (**1**, p. 221) suitably reworded in terms of isometric points. Thus, to show that **C4**** cannot be deduced from the previous axioms we should have to produce a non-Archimedean counterexample.

A. (THE AXIOM OF ARCHIMEDES). *If A and A_1 are distinct isometric points, and if $P \in$ ray AA_1, then there exists a positive integer n and a sequence of points $A, A_1, A_2, \ldots, A_n$ all isometric to A, such that $[AA_1A_2 \ldots A_n]$,*

$$AA_1 \equiv A_1A_2 \equiv \ldots \equiv A_{n-1}A_n$$

and $[APA_n]$.

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2.2. *Assuming* axioms **C1*–C4*** *and* **A**, *if AB and CD are segments and if $AB \equiv CD$, then $BA \equiv DC$.*

Proof (see Figure 2B). It is clear that A, C, A_1, C_1, A_2, C_2, . . . (as defined below) are isometric, and so are B, D. The notation "$DC < BA$" means that if X is the point on ray BA such that $DC \equiv BX$, then $[BXA]$.

Suppose that $BA \not\equiv DC$, and suppose, without loss of generality, that $DC < BA$. Then there exists A_1 such that $[BA_1A]$ and $BA_1 \equiv DC$. Since $AB \equiv CD$, there exists C_1 such that $[CC_1D]$ and $AA_1 \equiv CC_1$. Then $A_1B \equiv C_1D$. Since $BA_1 \equiv DC$, there exists A_2 such that $[BA_2A_1]$ and $BA_2 \equiv DC_1$. Then $A_2A_1 \equiv C_1C \equiv A_1A$; hence $A_1A_2 \equiv AA_1$. Since $A_1B \equiv C_1D$, there exists C_2 such that $[C_1C_2D]$ and $A_1A_2 \equiv C_1C_2$. Then $A_2B \equiv C_2D$. Proceeding in this

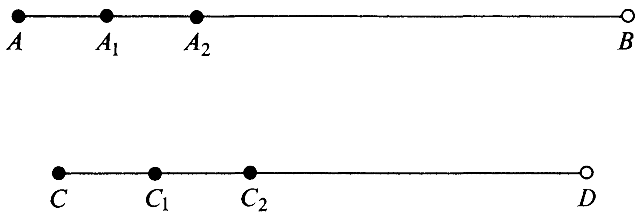

FIGURE 2B

way, we obtain an infinite sequence A, A_1, A_2, . . . such that $[AA_nB]$ for every positive integer n. This contradicts **A**. Hence $BA \equiv DC$.

Remark (c). At the end of (**2**, § 1) I gave an example of a one-dimensional geometry satisfying axioms **C1*–C4*** and **C4****, in which not all points are isometric and not every segment has a mid-point. (See also the remarks at the end of **2**, § 7.) This is an absolute geometry, since **C5a** is satisfied vacuously and **C5b** trivially.

I have found no example of a *plane* satisfying **C1*–C4*** in which not all points are isometric, but it is possible to construct examples in which not every segment has a mid-point, in the following way.

Let the points of a plane π consist of all ordered pairs (a, b) of rational numbers, and let the lines of π consist of all points satisfying rational linear equations. We define geometrical order in the obvious way. To be more precise, if (a_1, b_1), (a_2, b_2), (a_3, b_3) are collinear, then (a_2, b_2) lies between the other two points if (i) $a_1 < a_2 < a_3$ or $a_1 > a_2 > a_3$, or if (ii) $a_1 = a_2 = a_3$, and $b_1 < b_2 < b_3$ or $b_1 > b_2 > b_3$. It is easily verified that the axioms of order (**1**, Chapter II) are satisfied.

Let S denote the set of all rational numbers of the form $p/3^r$, p an integer, r a non-negative integer (cf. the example at the end of **2**, § 1). Then both S and Q (the set of all rational numbers) are countable and totally ordered. Neither has a least or a greatest element, and between any two distinct elements of S (or Q)

there lies another element of S (or Q). Hence there exists a one-to-one order-preserving mapping from Q onto S (**4**, pp. 209, 202). If $a \in Q$, denote the corresponding element of S by a'.

Define the *distance* between (a_1, b_1) and (a_2, b_2) to be (i) $|a_1' - a_2'|$ if $a_1' \neq a_2'$ (i.e., if $a_1 \neq a_2$), (ii) $|b_1' - b_2'|$ if $a_1' = a_2'$ (i.e., if $a_1 = a_2$). If we then define congruence in terms of distance in the obvious way, we find that axioms **C1***–**C4*** and **C4**** are satisfied. All points are isometric, and we need only discuss **C3***, as the verification of the other axioms is simple. Suppose that we redefine the geometrical order in π in the following way. If (a_1, b_1), (a_2, b_2), and (a_3, b_3) are collinear, then (a_2, b_2) lies between the other two points if (i) $a_1' < a_2' < a_3'$ or $a_1' > a_2' > a_3'$, or if (ii) $a_1' = a_2' = a_3'$, and $b_1' < b_2' < b_3'$ or $b_1' > b_2' > b_3'$. Then this definition yields the same geometrical order as before, because of the order-preserving mapping from Q onto S. If we think of geometrical order in terms of this new definition, it is clear that axiom **C3*** is satisfied.

Since the line $y = 0$ is isomorphic to the line in the example at the end of (**2**, § 1), we see that not every segment has a mid-point.

An alternative possible definition of the distance between (a_1, b_1) and (a_2, b_2) is $\max(|a_1' - a_2'|, |b_1' - b_2'|)$.

(The reader's attention is drawn again to the note at the end of the introduction.)

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