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PRESENTATIONS OF OMEGA-CATEGORIES BY DIRECTED COMPLEXES

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Abstract

The theory of directed complexes is extended from free ω-categories to arbitrary ω-categories by defining presentations in which the generators are atoms and the relations are equations between molecules. Our main result relates these presentations to the more standard algebraic presentations; we also show that every ω-category has a presentation by directed complexes. The approach is similar to that used by Crans for pasting presentations.

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1. Introduction

There are at present three combinatorial structures for constructing ω-categories: pasting schemes, defined in 1988 by Johnson [8], parity complexes, introduced in 1991 by Street [16, 17] and directed complexes, given by Steiner in 1993 [15]. These structures each consist of cells which have collections of lower dimensional cells as domain and codomain; see for example Definition 2.2 below. They also have 'local' conditions on the cells, ensuring that a cell together with its bounding cells has the form of an element in an ω-category; see for example Definition 2.4 below. Finally, they have a 'global' condition called loop-freeness; the various loop-freeness conditions are however not equivalent because they require the non-existence of different types of loops.

The ω-categories so far constructed from these structures are free on a collection of generators. In 1995, Crans [5, Chapter 2] described a structure, based on the theory of pasting schemes, with which one can construct arbitrary ω-categories. His structure, called a pasting presentation, generalises the earlier approach by including relations

between pasting schemes. Here, we develop a similar structure based on the theory of directed complexes, from which it is also possible to construct all ω-categories. The result is similar to that of [5], but it is new because of the difference in the loop-freeness conditions. Moreover, our proof here is shorter, because the theory of directed complexes makes full use of the ω-category structure already present. We should also say that Verity has announced a similar construction, based on parity complexes. It is useful to have the various results, because different approaches may be appropriate for the various applications, which include homotopy theory [5, 9], non-abelian homology [14, 7], quantum physics [2, 4, 6] and computer science [13, 12].

We now outline the approach of the paper. The most important operations in an ω-category are the compositions in various directions. In practice, one often gives a composition by means of a pasting diagram; see for example [10, 3]. It is therefore desirable to describe ω-categories in terms of these diagrams. Concretely, one wants a presentation of an ω-category in which the generators are diagrams and the relations are equations between diagrams. Here we achieve this by using [15]'s directed complexes to describe the diagrams, using degenerate cells and globularisations in the same way as [5].

Thus, a generator in a 'presentation of an ω-category by directed complexes' will be an atomic directed complex (a closed cell), and a relation will be an equation between molecular directed complexes (composites of atoms); some of the atoms in the molecular directed complexes are copies of generators, and the rest are degenerate. We show that one can systematically add degenerate atoms to a directed complex, a process called globularisation, in a way which preserves loop-freeness and facilitates composition. As a result, we get an ω-category in which the elements are equivalence classes of molecular directed complexes. Our main theorem relates the theory of presentations developed here to the more standard algebraic presentations; we also show that every ω-category admits a presentation by directed complexes.

We cannot resist a final word on the relationship between directed complexes, parity complexes and pasting schemes. The theory of directed complexes makes maximal use of the ω-category structure of a diagram; in the theories of pasting schemes and parity complexes, this structure is derived from other combinatorial structures. Parity complexes have very simple loop-freeness conditions, which may however be unnecessarily restrictive; see for example [15, p. 258]. There are also more geometrical structures: Power's pasting schemes [11] and Al-Agl and Steiner's globelike sets [1]. We think it is up to the reader to decide which structure is most suitable for any particular purpose, and we hope that the present paper makes the choice easier.

2. ω-categories and directed complexes

In this section we give some preliminary definitions and results, mostly taken from [15, Section 2]. First we recall the definition of an ω-category.

DEFINITION 2.1. An *ω-category* is a set C together with unary operations d_0^-, d_0^+, d_1^-, d_1^+, ... and not-everywhere-defined binary operations $\#_0$, $\#_1$, ... such that the following hold:

(i) $x \#_n y$ is defined if and only if $d_n^+ x = d_n^- y$;

(ii) for $\alpha, \beta = \pm$ and for non-negative integers m, n,

$$d_m^\beta d_n^\alpha = \begin{cases} d_m^\beta & \text{if } m < n, \\ d_n^\alpha & \text{if } m \geq n; \end{cases}$$

(iii) $d_n^- x \#_n x = x = x \#_n d_n^+ x$ for $x \in C$ and for all n;

(iv) if $x \#_n y$ is defined, then

$$d_n^-(x \#_n y) = d_n^- x, \qquad d_n^+(x \#_n y) = d_n^+ y,$$
$$d_m^\alpha(x \#_n y) = d_m^\alpha x \#_n d_m^\alpha y \quad \text{for } m \neq n;$$

(v) $(x \#_n y) \#_n z = x \#_n (y \#_n z)$ if either side is defined;

(vi) $(x' \#_n y') \#_m (x'' \#_n y'') = (x' \#_m x'') \#_n (y' \#_m y'')$ if $m \neq n$ and both sides are defined;

(vii) for every $x \in C$ there exists n such that $d_n^- x = d_n^+ x = x$.

Conditions (i), (iii) and (v) in this definition say that the elements of C form the morphisms of a category such that $\#_n$ is composition and the $d_n^\alpha x$ are the identities which can be composed with x. Conditions (ii), (iv) and (vi) say that the category structures commute in such a way that identities under $\#_n$ are also identities under $\#_m$ for $m \geq n$. Condition (vii) (which is sometimes omitted) says that every element is an identity for some $\#_n$.

Next we work towards the definition of a directed complex, which is a higher-dimensional generalisation of a directed graph.

DEFINITION 2.2. A *directed precomplex* is a set K together with functions dim, ∂^-, ∂^+ on K such that, for $\sigma \in K$,

(i) $\dim \sigma$ is a non-negative integer, the *dimension* of σ,

(ii) the $\partial^\alpha \sigma$ are subsets of K such that $\dim \tau = \dim \sigma - 1$ for all $\tau \in \partial^\alpha \sigma$.

Thus a directed precomplex is essentially a combinatorial complex in which every p-dimensional element is provided with two sets of $(p-1)$-dimensional faces.

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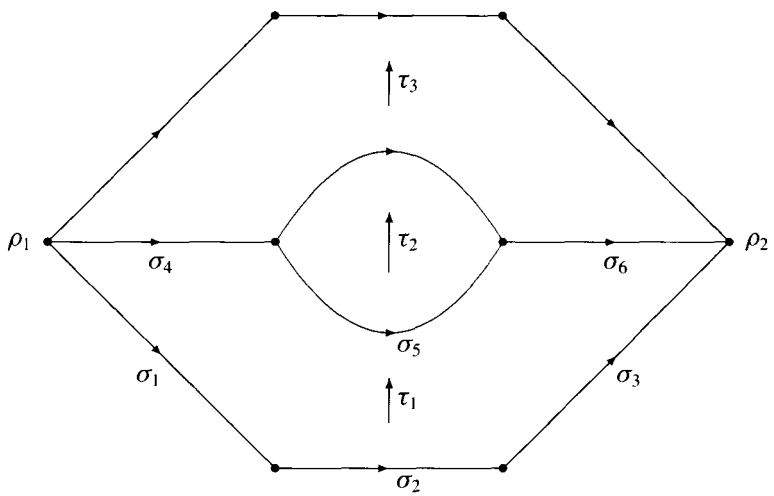

FIGURE 1. A directed precomplex

As an example for the rest of this section, we shall use the directed precomplex shown in Figure 1. It consists of three 2-dimensional elements τ_1, τ_2 and τ_3, together with 1-dimensional elements denoted by lines and 0-dimensional elements denoted by dots. The arrows run from negative to positive; thus

$$\partial^-\tau_1 = \{\sigma_1, \sigma_2, \sigma_3\}, \qquad \partial^+\tau_1 = \{\sigma_4, \sigma_5, \sigma_6\}, \qquad \partial^-\sigma_1 = \{\rho_1\}, \quad \partial^+\sigma_3 = \{\rho_2\}.$$

Let K be a directed precomplex, and let x be a subset of K. One says that x is *n-dimensional* ($\dim x = n$) if n is the maximum of the dimensions of members of x; the empty set is taken to be (-1)-dimensional. One says that x is *closed* if $\partial^\alpha\sigma \subset x$ for $\sigma \in x$ and $\alpha = \pm$; in this way, K receives a topology. The closure of x is denoted $\mathrm{Cl}\, x$. As a special case, the closure of a singleton $\{\sigma\}$ is denoted $\bar{\sigma}$ and is called an *atom*.

DEFINITION 2.3. Let K be a directed precomplex. For an integer n, a sign $\alpha = \pm$, and a subset x of K, the (α, n)-*boundary of* x, denoted $d_n^\alpha x$, is defined by

$$d_n^\alpha x = \{\sigma \in x : \dim\sigma \leq n;\ \text{if } \sigma \in \bar{\tau} \text{ with } \tau \in x \text{ and } \dim\tau = n+1 \text{ then } \sigma \in \mathrm{Cl}(\partial^\alpha\tau)\}.$$

For x, y closed subsets of K and n a non-negative integer, the *n-composite* $x \#_n y$ is defined by $x \#_n y = x \cup y$, but only when $x \cap y = d_n^+ x = d_n^- y$.

A *molecule* is a non-empty iterated composite of atoms.

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Note that $d^{\alpha}_{n}x$ is always empty for negative n.

In Figure 1, $d^{-}_{0}x$ and $d^{+}_{0}x$ are the left and right-hand end-points of x, while $d^{-}_{1}x$ and $d^{+}_{1}x$ are the lower and upper edges; for example one has

$$d^{-}_{0}\bar{\sigma}_1 = \bar{\rho}_1, \quad d^{+}_{0}\bar{\sigma}_3 = \bar{\rho}_2, \quad d^{+}_{1}\bar{\tau}_1 = d^{-}_{1}(\bar{\sigma}_4 \cup \bar{\tau}_2 \cup \bar{\sigma}_6) = \bar{\sigma}_4 \cup \bar{\sigma}_5 \cup \bar{\sigma}_6.$$

One can check that the entire directed precomplex is a molecule: it can be expressed as

$$\bar{\tau}_1 \#_1 (\bar{\sigma}_4 \#_0 \bar{\tau}_2 \#_0 \bar{\sigma}_6) \#_1 \bar{\tau}_3.$$

DEFINITION 2.4. A *directed complex* is a directed precomplex such that for $\sigma \in K$ and for $p = \dim \sigma$ the following hold:

(i) if $p > 0$ then the $d^{\alpha}_{p-1}\bar{\sigma}$ are molecules;

(ii) $d^{\beta}_{p-2}d^{\alpha}_{p-1}\bar{\sigma} = d^{\beta}_{p-2}\bar{\sigma}$ for all signs α, β.

One can check that the directed precomplex in Figure 1 is a directed complex: for example one has $d^{-}_{0}\bar{\sigma}_1 = \bar{\rho}_1$ and $d^{-}_{1}\bar{\tau}_1 = \bar{\sigma}_1 \#_0 \bar{\sigma}_2 \#_0 \bar{\sigma}_3$, which are molecules, and $d^{+}_{0}d^{-}_{1}\bar{\tau}_1 = \bar{\rho}_2 = d^{+}_{0}\bar{\tau}_1$.

We have the following result.

PROPOSITION 2.5 ([15, Proposition 2.9]). *The molecules in a directed complex satisfy the axioms for an ω-category, except that the condition $d^{+}_{n}x = d^{-}_{n}y$ may not be sufficient for $x \#_n y$ to be defined.*

In particular, if x is a molecule in a directed complex then the subsets $d^{\alpha}_{n}x$ are also molecules.

We can get genuine ω-categories from directed loop-complexes by imposing loop-freeness conditions. The first such condition that we shall consider is given by the following definition.

DEFINITION 2.6 ([15, Definition 2.14]). Let K be a directed complex and n be a non-negative integer. An *n-path in K* is a sequence $\sigma_0, \ldots, \sigma_k$ in K such that, for $1 \leq i \leq k$, either

$$\dim \sigma_{i-1} \leq n, \qquad \dim \sigma_i > n \quad \text{and} \quad \sigma_{i-1} \in d^{-}_{n}\bar{\sigma}_i \setminus (d^{-}_{n-1}\bar{\sigma}_i \cup d^{+}_{n-1}\bar{\sigma}_i)$$

or

$$\dim \sigma_{i-1} > n, \qquad \dim \sigma_i \leq n \quad \text{and} \quad \sigma_i \in d^{+}_{n}\bar{\sigma}_{i-1} \setminus (d^{-}_{n-1}\bar{\sigma}_{i-1} \cup d^{+}_{n-1}\bar{\sigma}_{i-1}).$$

A subset x of K is *loop-free* if for any n and for any n-path Π in x the elements of Π are distinct.

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For example, consider the directed complex in Figure 1. It has dimension 2, so an m-path for $m \geq 2$ consists of at most one element. Also 0-paths must run from left to right and 1-paths must run from bottom to top, so they have distinct elements too. Therefore the directed complex in Figure 1 is loop-free.

The other loop-freeness condition that we shall consider is called *total loop-freeness*. It is less natural but often easier to deal with.

DEFINITION 2.7. Let K be a directed complex. A *total path* in K is a sequence $\sigma_0, \ldots, \sigma_k$ in K such that, for $1 \leq i \leq k$, $\sigma_{i-1} \in \partial^- \sigma_i$ or $\sigma_i \in \partial^+ \sigma_{i-1}$.

A subset x of K is *totally loop-free* if the elements of any total path in x are distinct.

For example, one can verify that the directed complex in Figure 1 is totally loop-free.

As the terminology suggests, a totally loop-free directed complex is loop-free. This is a consequence of the following result, which is an improved version of [15, Proposition 5.2].

PROPOSITION 2.8. *If σ, τ is a two-element n-path in a directed complex K, then there is a total path in K from σ to τ.*

PROOF. We take the case in which $\dim \sigma \leq n$, $\dim \tau > n$, and $\sigma \in d^-_n \bar{\tau} \setminus (d^-_{n-1} \bar{\tau} \cup d^+_{n-1} \bar{\tau})$. We use induction on the positive integer $\dim \tau - \dim \sigma$. There are three cases.

Suppose firstly that $\dim \sigma = n$ and $\dim \tau = n + 1$. Since $\sigma \in d^-_n \bar{\tau}$, we must have $\sigma \in \partial^- \tau$. This means that σ, τ is itself a total path.

Next suppose that $\dim \sigma = n$ and $\dim \tau = t > n + 1$. From Proposition 2.5, $d^-_n \bar{\tau} = d^-_n d^-_{t-1} \bar{\tau}$, so we must have $\sigma \in d^-_{t-1} \bar{\tau}$. It follows that $\sigma \in \bar{\upsilon}$ for some $\upsilon \in \partial^- \tau$. From the definition of d^-_n, we must have $\sigma \in d^-_n \bar{\upsilon}$. Since $\dim \sigma = n$ we also have $\sigma \notin (d^-_{n-1} \bar{\upsilon} \cup d^+_{n-1} \bar{\upsilon})$. By the inductive hypothesis, there is a total path $\sigma, \ldots, \upsilon$, which extends to a total path $\sigma, \ldots, \upsilon, \tau$.

Finally, suppose that $\dim \sigma = s < n$. Using Proposition 2.5, we get

$$\sigma \in d^-_n \bar{\tau} \setminus d^+_{n-1} \bar{\tau} = d^-_n \bar{\tau} \setminus d^+_{n-1} d^-_n \bar{\tau}.$$

We also have

$$d^+_s d^-_n \bar{\tau} = d^+_s d^+_{n-1} d^-_n \bar{\tau} = d^+_s d^+_{n-1} \bar{\tau} \subset d^+_{n-1} \bar{\tau},$$

so $\sigma \in d^-_n \bar{\tau} \setminus d^+_s d^-_n \bar{\tau}$. It follows that $\sigma \in \bar{\upsilon} \setminus \mathrm{Cl}(\partial^+ \upsilon)$ for some $(s+1)$-dimensional element $\upsilon \in d^-_n \bar{\tau}$. We must then have $\sigma \in \partial^- \upsilon$. Now $\sigma \notin d^-_{n-1} \bar{\tau} \cup d^+_{n-1} \bar{\tau}$ and the $d^\alpha_{n-1} \bar{\tau}$ are closed (they are molecules by Proposition 2.5, so they are finite unions of atoms), so $\upsilon \notin d^-_{n-1} \bar{\tau} \cup d^+_{n-1} \bar{\tau}$. By the inductive hypothesis, there is a total path $\upsilon, \ldots, \tau$, which extends to a total path $\sigma, \upsilon, \ldots, \tau$.

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The main result on loop-free directed complexes is as follows.

THEOREM 2.9 ([15, Theorem 2.17]). *Let K be a loop-free directed complex. Then the molecules in K form an ω-category with the following presentation: the generators are the atoms; if $\bar{\sigma}$ is a p-dimensional generator then there are relations $d^{\alpha}_{p}\bar{\sigma} = \bar{\sigma}$; if $\bar{\sigma}$ is a p-dimensional generator and $p > 0$ then there are relations $d^{\alpha}_{p-1}\bar{\sigma} = c^{\alpha}(\sigma)$, where the $c^{\alpha}(\sigma)$ are arbitrarily chosen expressions for the $d^{\alpha}_{p-1}\bar{\sigma}$ as composites of atoms.*

The ω-category of molecules in a loop-free directed complex K is denoted $M(K)$.

We conclude this section with a remark about morphisms between ω-categories of molecules in loop-free directed complexes.

PROPOSITION 2.10. *Let K and L be loop-free directed complexes, and let $f: M(K) \to M(L)$ be an ω-category morphism. Then f restricts to a morphism $M(x) \to M(f(x))$ for each molecule x in K.*

PROOF. What we have to show is that $f(y) \subset f(x)$ if y is a molecule contained in x. Clearly it suffices to show that $f(\bar{\tau}) \subset f(\bar{\sigma})$ if $\bar{\tau}$ and $\bar{\sigma}$ are atoms such that $\bar{\tau} \subset \bar{\sigma}$. From the definition of closure, it suffices to show that $f(\bar{\tau}) \subset f(\bar{\sigma})$ if $\tau \in \partial^{\alpha}\sigma$ for some sign α. In this case the dimension of σ must be a positive number, p say, and $\bar{\tau}$ must clearly be a factor in the molecule $d^{\alpha}_{p-1}\bar{\sigma}$. Since f is a morphism of ω-categories, $f(\bar{\tau})$ is a factor of $f(d^{\alpha}_{p-1}\bar{\sigma})$, and we get $f(\bar{\tau}) \subset f(d^{\alpha}_{p-1}\bar{\sigma}) = d^{\alpha}_{p-1}f(\bar{\sigma}) \subset f(\bar{\sigma})$ as required.

3. Directed complex presentations

According to Theorem 2.9, certain ω-categories can be realised as the sets of molecules in loop-free directed complexes; the operations d^{α}_{n} are represented by passing to subsets, and all composites are represented by unions. We now aim to realise arbitrary ω-categories by generalising this construction: we take a *class* of loop-free directed complexes and impose an equivalence relation on their molecules. The result can be thought of as a combinatorial presentation: the directed complexes contain copies of members of a given 'generating' set G of atoms, and the equivalence relation is given by a set of subdivisions of members of G. We shall also need 'degenerate' atoms which are not copies of members of G; these are used to 'fill gaps' between equivalent molecules. The degenerate atoms help us to represent composites and to make substitutions.

We begin by formalising the notion of a directed complex in which the atoms are either copies of members of a generating set G or degenerate. We impose restrictions

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on G in order to make the copies coherent, but we cannot impose any restrictions on degenerate atoms until we consider the equivalence relation.

DEFINITION 3.1. Let K and L be directed complexes. An *embedding* of K in L is an injective function from K to L which preserves dimensions and the ∂^α operations. An embedding $f: K \to L$ is an *isomorphism* if $f(K) = L$.

DEFINITION 3.2. A *generating set* is a set G of loop-free atomic directed complexes together with a family Φ of embeddings between members of G such that the following conditions hold.

(i) For each atom $\bar{\sigma}$ in a member of G there is at most one embedding in Φ with image $\bar{\sigma}$.

(ii) For each $g \in G$ the identity embedding of g in itself belongs to Φ.

(iii) Let $\phi: g \to h$ be an embedding in Φ and let $\bar{\sigma}$ be an atom in g. If there is an embedding θ in Φ with image $\bar{\sigma}$ then $\phi \circ \theta$ is an embedding in Φ with image $\phi(\bar{\sigma})$; if there is no embedding in Φ with image $\bar{\sigma}$ then there is no embedding in Φ with image $\phi(\bar{\sigma})$.

The members of Φ are called *structural embeddings*.

DEFINITION 3.3. Let G be a generating set. A *G-complex* is a loop-free directed complex K together with a family X of embeddings of members of G in K such that the following conditions hold.

(i) For each atom $\bar{\tau}$ in K, there is at most one embedding in X with image $\bar{\tau}$.

(ii) Let $\chi: g \to K$ be an embedding in X, and let $\bar{\sigma}$ be an atom in g. If there is a structural embedding ϕ with image $\bar{\sigma}$, then $\chi \circ \phi$ is an embedding in X with image $\chi(\bar{\sigma})$; if there is no structural embedding with image $\bar{\sigma}$, then there is no embedding in X with image $\chi(\bar{\sigma})$.

The embeddings in X are called *characteristic embeddings*. If $\chi: g \to K$ is a characteristic embedding with image $\bar{\tau}$, then $\bar{\tau}$ is a *copy* of g via χ; if $\bar{\tau}$ is an atom in K which is not the image of a characteristic embedding, then $\bar{\tau}$ is called *degenerate*.

The conditions of Definition 3.2 ensure that the members of a generating set G are themselves G-complexes with the structural embeddings as characteristic embeddings, and that they are copies of themselves via the identity embeddings.

Given a generating set G, there are natural notions of embedding and isomorphism for G-complexes.

DEFINITION 3.4. Let G be a generating set, and let K and L be G-complexes. A *G-embedding* of K in L is an embedding $f: K \to L$ such that, for $\sigma \in K$:

(i) if $\bar{\sigma}$ is a copy of g via χ, then $f(\bar{\sigma})$ is a copy of g via $f \circ \chi$;

(ii) if $\bar{\sigma}$ is degenerate, then $f(\bar{\sigma})$ is degenerate.

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A G-embedding which is an isomorphism is called a *G-isomorphism.*

We note that the characteristic embeddings of G-complexes are G-embeddings; in particular, the structural embeddings of G are G-embeddings.

So far, we have analogues for generators in presentations; we will now consider analogues for relations. We call a molecular directed complex w a *subdivision* of an atomic complex g if w and g are of the same dimension and have isomorphic boundary; for example, the directed complex in Figure 1 is a subdivision of the directed complex in Figure 2. A subdivision w of g will be used to represent a relation $g = w$.

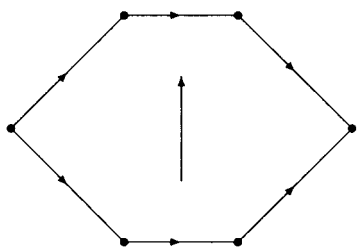

FIGURE 2. Another directed complex

Given a generating set G, the relevant subdivisions are defined as follows.

DEFINITION 3.5. Let g be a p-dimensional member of a generating set G. A *G-subdivision* of g is a p-dimensional molecular G-complex w such that that there is a G-isomorphism

$$d^-_{p-1}g \cup d^+_{p-1}g \cong d^-_{p-1}w \cup d^+_{p-1}w$$

sending $d^-_{p-1}g$ to $d^-_{p-1}w$ and $d^+_{p-1}g$ to $d^+_{p-1}w$.

In Definition 3.5, note that $d^-_{p-1}g \cup d^+_{p-1}g$ and $d^-_{p-1}w \cup d^+_{p-1}w$ are G-complexes, because they are subcomplexes of G-complexes; it therefore makes sense to talk about a G-isomorphism between them.

Let G be a generating set. If $f\colon K \to L$ is a G-embedding between G-complexes, then it induces an ω-category morphism $f\colon M(K) \to M(L)$ by Theorem 2.9 (recall that G-complexes are loop-free). Also, if g is a member of G and w is a G-subdivision of g, then there is an obvious ω-category morphism $M(g) \to M(w)$ sending g to w. We shall use these ideas to get a notion of elementary equivalence between molecular G-complexes, given a set of G-subdivisions. The idea is that members of G are sent to copies of themselves or are subdivided, while degenerate atoms are sent to degenerate atoms or collapsed.

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DEFINITION 3.6. Let $P = (G, R)$ be a pair consisting of a generating set G and a set R of G-subdivisions, and let x and y be molecular G-complexes. An *elementary P-equivalence* from x to y is an ω-category morphism $f: M(x) \to M(y)$ such that $f(x) = y$ and, for each atom $\bar{\sigma}$ in x, one of the following conditions holds:

(i) $\bar{\sigma}$ is a copy of a member g of G via χ and $f(\bar{\sigma})$ is a copy of g via $f \circ \chi$;

(ii) $\bar{\sigma}$ is a copy of a member g of G via χ, there is a G-subdivision w of g in R, there is a G-isomorphism $\theta: w \to f(\bar{\sigma})$, and the diagram

$$\begin{array}{ccc} M(g) & \xrightarrow{\chi} & M(\bar{\sigma}) \\ \downarrow & & \downarrow f \\ M(w) & \xrightarrow{\theta} & M(f(\bar{\sigma})) \end{array}$$

commutes;

(iii) $\bar{\sigma}$ is degenerate and $f(\bar{\sigma})$ is a degenerate atom of the same dimension;

(iv) $\bar{\sigma}$ is degenerate and $\dim f(\bar{\sigma}) < \dim \bar{\sigma}$.

An elementary P-equivalence *between* x and y is an elementary P-equivalence from x to y or from y to x.

Given a generating set G and a set of G-subdivisions, we can now impose conditions on degenerate atoms.

DEFINITION 3.7. Let $P = (G, R)$ be a pair consisting of a generating set G and a set R of G-subdivisions. Then the class of P-molecules is defined by induction on dimension as follows. A *P-molecule* is a molecular G-complex x such that the following conditions hold whenever $\bar{\sigma}$ is a degenerate atom and $p = \dim \sigma$: firstly, $p > 0$; secondly, $d^-_{p-1}\bar{\sigma}$ is linked to $d^+_{p-1}\bar{\sigma}$ by a chain of elementary P-equivalences between P-molecules of dimension at most $p - 1$.

An atomic P-molecule is called a *P-atom*.

To get a directed complex presentation, we take a pair $P = (G, R)$ as in Definition 3.7 and require all the molecules involved to be P-molecules.

DEFINITION 3.8. A *directed complex presentation* is a pair $P = (G, R)$ such that

(i) G is a generating set,

(ii) R is a set of G-subdivisions,

(iii) g is a P-molecule for all $g \in G$,

(iv) w is a P-molecule for all $w \in R$.

If P is a directed complex presentation, then the class of P-molecules is denoted $W(P)$.

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We note that Definition 3.8 is implicitly inductive: the allowable members of G and R in any given dimension are determined by the members of G and R in lower dimensions.

We have now constructed our classes of molecules: they are the classes of the form $W(P)$ for directed complex presentations P. The equivalence relations are to be induced by elementary P-equivalence as follows.

DEFINITION 3.9. Let P be a directed complex presentation. Then two P-molecules x and x' are *P-equivalent*, (notation $x \sim x'$), if x and x' are linked by a chain of elementary P-equivalences between P-molecules.

Let $P = (G, R)$ be a directed complex presentation. It is clear that P-equivalence is an equivalence relation on the class $W(P)$ of P-molecules. Since a G-isomorphism between P-molecules is an elementary P-equivalence, there is only a set of P-equivalence classes. The set of P-equivalence classes will be denoted $\omega(P)$, and the P-equivalence class of a P-molecule x will be denoted $[x]$.

4. Boundaries in a directed complex presentation

In this section, $P = (G, R)$ is a directed complex presentation. We aim to impose an ω-category structure on $\omega(P)$, the set of P-equivalence classes of P-molecules. Here we shall deal with the parts of the structure that do not involve composition; these parts are simply inherited from the ω-categories $M(x)$, where x is a P-molecule. Indeed, we have the following results.

PROPOSITION 4.1. *Every molecule in a P-molecule is a P-molecule.*

PROOF. Obvious.

PROPOSITION 4.2. *Let x and x' be P-molecules, and let f be an elementary P-equivalence from x to x'. Then f restricts to an elementary P-equivalence from y to $f(y)$ for every molecule y in x.*

PROOF. This follows from Proposition 2.10.

PROPOSITION 4.3. *For $n \geq 0$ there is an operation $d_n^\alpha : \omega(P) \to \omega(P)$ well-defined by the rule that $d_n^\alpha[x] = [d_n^\alpha x]$ for every P-molecule x.*

PROOF. If x is a P-molecule, then $d_n^\alpha x$ is a P-molecule by Proposition 4.1. If x and x' are P-molecules such that $[x] = [x']$, then $[d_n^\alpha x] = [d_n^\alpha x']$ by Proposition 4.2, since a P-equivalence is a chain of elementary P-equivalences.

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PROPOSITION 4.4. *Let x be a P-molecule. Then*

$$d_m^\beta d_n^\alpha[x] = \begin{cases} d_m^\beta[x] & \text{if } m < n, \\ d_n^\alpha[x] & \text{if } m \geq n, \end{cases}$$

and there exists n such that $d_n^-[x] = d_n^+[x] = [x]$.

PROOF. Since $M(x)$ is an ω-category, this follows from Definition 2.1 (ii) and (vii).

We now have all of the ω-category structure on $\omega(P)$ that does not involve composites. We conclude this section with two technical results needed in the study of composites; they also clarify the definition of a P-molecule.

PROPOSITION 4.5. *Let x and x' be P-equivalent P-molecules of dimension at most n. Then x and x' are linked by a chain of elementary P-equivalences between P-molecules of dimension at most n.*

PROOF. Since x and x' are P-equivalent, there is a chain

$$x = w_0,\ w_1,\ \ldots,\ w_k = x'$$

of P-molecules linked by elementary P-equivalences. By Proposition 4.2 the P-molecules

$$d_n^- x = d_n^- w_0,\ d_n^- w_1,\ \ldots,\ d_n^- w_k = d_n^- x'$$

are also linked by elementary P-equivalences. Since $\dim x \leq n$ and $\dim x' \leq n$, it follows from Definition 2.3 that $d_n^- x = x$ and $d_n^- x' = x'$. It also follows from Definition 2.3 that $\dim d_n^- w_i \leq n$ for each i, so x and x' are linked by a chain of elementary P-equivalences between P-molecules of dimension at most n, as required.

PROPOSITION 4.6. *Let x be a molecular G-complex such that the following conditions hold whenever $\bar{\sigma}$ is a degenerate atom and $p = \dim \sigma$: firstly, $p > 0$; secondly, $d_{p-1}^- \bar{\sigma}$ and $d_{p-1}^+ \bar{\sigma}$ are P-equivalent P-molecules. Then x is a P-molecule.*

PROOF. This follows from Proposition 4.5.

In other words, the dimensional restriction at the end of Definition 3.7 is unnecessary.

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5. Globularisations and fat composites

In this section, $P = (G, R)$ is again a directed complex presentation. We wish to define composites in $\omega(P)$. For this purpose, we shall construct a special kind of molecular directed complex called a *fat composite* and denoted $x \#_n^{\mathbf{s}} y$. We shall also construct complexes called *globularisations* and denoted $\mathrm{Gl}_n^{\mathbf{s}}(x)$. These will be used as building blocks for the fat composites; later, they will also be used to construct subdivisions.

The symbol $\mathbf{s}$ in $x \#_n^{\mathbf{s}} y$ and $\mathrm{Gl}_n^{\mathbf{s}}(x)$ denotes a $2p$-tuple

$$\mathbf{s} = (s_0^-, s_0^+, \ldots, s_{p-1}^-, s_{p-1}^+)$$

of molecular directed complexes, where $p \geq n$; the sequences that can be used are given by the following definition.

DEFINITION 5.1. Let p be a non-negative integer. An *admissible* $2p$*-tuple* is a $2p$-tuple

$$\mathbf{s} = (s_0^-, s_0^+, \ldots, s_{p-1}^-, s_{p-1}^+)$$

of disjoint molecular directed complexes such that $\dim s_i^\alpha \leq i$ for $0 \leq i < p$.

Before defining globularisations and fat composites in general, we shall describe the constructions associated to an admissible 4-tuple $\mathbf{s} = (s_0^-, s_0^+, s_1^-, s_1^+)$. For any molecular directed complex x we take $\mathrm{Gl}_0^{\mathbf{s}}(x) = x$. For x a molecular directed complex disjoint from s_0^- and s_0^+ we take $\mathrm{Gl}_1^{\mathbf{s}}(x)$ to be as illustrated in Figure 3, where the $\bar{\nu}_1^\alpha(x)$ are 1-dimensional atoms. For x a molecular directed complex disjoint from

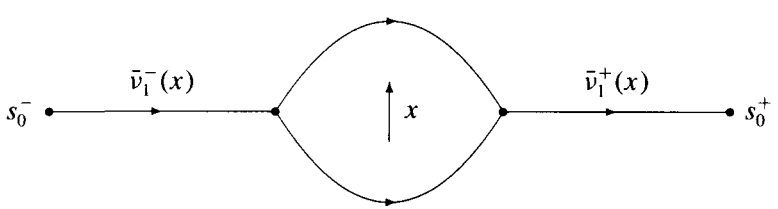

FIGURE 3. A 1-globularisation

$s_0^-, s_0^+, s_1^-, s_1^+$ we take $\mathrm{Gl}_2^{\mathbf{s}}(x)$ to be as illustrated in Figure 4, where the $\bar{\nu}_i^\alpha(z)$ are again i-dimensional atoms. Note that $\mathrm{Gl}_n^{\mathbf{s}}(x)$ really involves s_i^α only for $0 \leq i < n$.

We see that $\mathrm{Gl}_2^{\mathbf{s}}(x)$ is got from $\mathrm{Gl}_1^{\mathbf{s}}(s_1^-) \cup \mathrm{Gl}_1^{\mathbf{s}}(x) \cup \mathrm{Gl}_1^{\mathbf{s}}(s_1^+)$ by inserting two 2-dimensional atoms. For x and y molecular directed complexes disjoint from each

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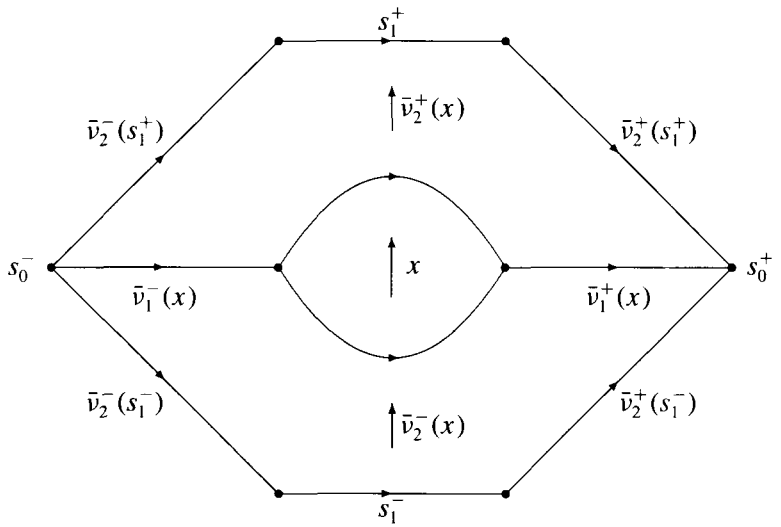

FIGURE 4. A 2-globularisation

other and from $s_0^-, s_0^+, \ldots, s_{n-1}^-, s_{n-1}^+$ we similarly construct $x \#_n^s y$ by inserting a $(n+1)$-dimensional atom $\bar{\nu}_{n+1}(x, y)$ into $\mathrm{Gl}_n^s(x) \cup \mathrm{Gl}_n^s(y)$; the case $n = 1$ is illustrated in Figure 5.

Again, $x \#_n^s y$ involves s_i^α only for $0 \leq i < n$.

We shall now describe the constructions in general. The basic result on adding atoms to directed complexes is as follows.

PROPOSITION 5.2. *Let n be a positive integer, and let $x_0, \ldots, x_k$ be molecular directed complexes such that*

(i) $d_{n-1}^\alpha x_i$ *is a union of* $(n-1)$*-dimensional atoms for* $0 \leq i \leq k$ *and* $\alpha = \pm$,
(ii) $d_{n-2}^\beta x_0 = \cdots = d_{n-2}^\beta x_k$ *for* $\beta = \pm$,
(iii) $x_i \cap x_j \subset d_{n-1}^+ x_i \cap d_{n-1}^- x_j$ *for* $i < j$.

Then there is a molecular directed complex of the form

$$x_0 \#_{n-1} \bar{\nu}_1 \#_{n-1} x_1 \#_{n-1} \cdots \#_{n-1} \bar{\nu}_k \#_{n-1} x_k,$$

where $\nu_1, \ldots, \nu_k$ are distinct n-dimensional elements not belonging to $x_0 \cup \cdots \cup x_k$.

PROOF. We use induction on k. For $k = 0$ the result is trivial. Suppose that $k > 0$. By the inductive hypothesis, there is a molecular directed complex

$$y = x_0 \#_{n-1} \bar{\nu}_1 \#_{n-1} \cdots \#_{n-1} \bar{\nu}_{k-1} \#_{n-1} x_{k-1}$$

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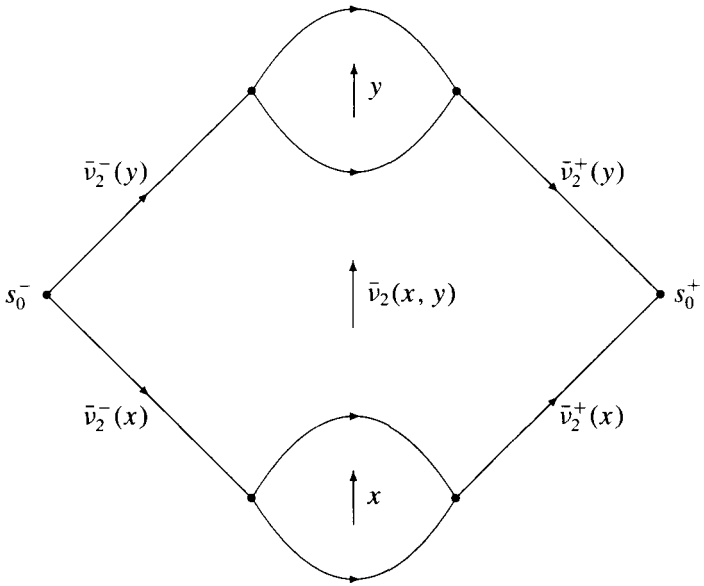

FIGURE 5. A fat composite

with $\nu_1, \ldots, \nu_{k-1}$ as required, and it suffices to construct a molecular directed complex

$$z = y \#_{n-1} \bar{\nu}_k \#_{n-1} x_k$$

with ν_k an n-dimensional element. Using Proposition 2.5, we see that $d^+_{n-1}y = d^+_{n-1}x_{k-1}$; using (i), we deduce that $d^+_{n-1}y$ and $d^-_{n-1}x_k$ are unions of $(n-1)$-dimensional atoms; using (ii), we also deduce that

$$d^\beta_{n-2}y = d^\beta_{n-2}d^+_{n-1}y = d^\beta_{n-2}d^-_{n-1}x_k = d^\beta_{n-2}x_k.$$

Now let z be the directed precomplex given by

$$z = y \cup \{\nu_k\} \cup x_k,$$

with $\dim \nu_k = n$,

$$\partial^- \nu_k = \{\sigma \in d^+_{n-1}y : \dim \sigma = n-1\},$$
$$\partial^+ \nu_k = \{\sigma \in d^-_{n-1}x_k : \dim \sigma = n-1\}.$$

To show that z is a directed complex, we must verify the conditions of Definition 2.4 for ν_k. But $d^+_{n-1}y$ and $d^-_{n-1}x_k$ are unions of $(n-1)$-dimensional atoms, so it follows

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from Definition 2.3 that

$$d^-_{n-1}\bar{v}_k = d^+_{n-1}y \quad \text{and} \quad d^+_{n-1}\bar{v}_k = d^-_{n-1}x_k;$$

therefore $d^-_{n-1}\bar{v}_k$ and $d^+_{n-1}\bar{v}_k$ are molecules. Also

$$d^\beta_{n-2}d^-_{n-1}\bar{v}_k = d^\beta_{n-2}d^+_{n-1}y = d^\beta_{n-2}d^-_{n-1}x_k = d^\beta_{n-2}d^+_{n-1}\bar{v}_k;$$

since $\bar{v}_k = d^-_{n-1}\bar{v}_k \cup \{v_k\} \cup d^+_{n-1}\bar{v}_k$ and $\dim v_k > n-1$ it follows from Definition 2.3 that

$$d^\beta_{n-2}\bar{v}_k = d^\beta_{n-2}d^-_{n-1}\bar{v}_k = d^\beta_{n-2}d^+_{n-1}\bar{v}_k$$

as well. This shows that z is a directed complex.

Next we show that $z = y \#_{n-1} \bar{v}_k \#_{n-1} x_k$. We have already seen that $d^-_{n-1}\bar{v}_k = d^+_{n-1}y$ and $d^+_{n-1}\bar{v}_k = d^-_{n-1}x_k$, and it is clear that $z = y \cup \bar{v}_k \cup x_k$; by Definition 2.3 it suffices to show that

$$y \cap \bar{v}_k \subset d^+_{n-1}y \quad \text{and} \quad (y \cup \bar{v}_k) \cap x_k \subset d^-_{n-1}x_k.$$

But this follows from (iii), since

$$(\bar{v}_k \setminus d^+_{n-1}y) \cap y \subset x_{k-1} \cap x_k \subset d^+_{n-1}x_{k-1} = d^+_{n-1}y$$

and

$$[(y \cup \bar{v}_k) \cap x_k] \setminus d^-_{n-1}x_k \subset (x_1 \cup \cdots \cup x_{k-1}) \cap x_k \subset d^-_{n-1}x_k.$$

Finally, z is a molecule, because $z = y \#_{n-1} \bar{v}_k \#_{n-1} x_k$ and y and x_k are molecules.

Next we use Proposition 5.2 to construct the globularisations $\mathrm{Gl}^{\mathbf{s}}_n(x)$. The construction is inductive; in order to get the induction to work smoothly, it is convenient to prove several properties simultaneously.

THEOREM 5.3. *There are molecular directed complexes* $\mathrm{Gl}^{\mathbf{s}}_n(x)$ *with the following properties:*

(i) $\mathrm{Gl}^{\mathbf{s}}_n(x)$ *is defined when* n *is a non-negative integer,* $\mathbf{s}$ *is an admissible* $2p$*-tuple for some* $p \geq n$*, and* x *is a molecular directed complex disjoint from* $s^-_0, s^+_0, \ldots, s^-_{n-1}, s^+_{n-1}$*;*

(ii) $\mathrm{Gl}^{\mathbf{s}}_n(x)$ *depends on* s^α_i *only for* $i < n$*;*

(iii) $\mathrm{Gl}^{\mathbf{s}}_0(x) = x$*;*

(iv) *if* $n > 0$ *then*

$$\mathrm{Gl}^{\mathbf{s}}_n(x) = \bar{v}^-_n(x) \#_{n-1} \mathrm{Gl}^{\mathbf{s}}_{n-1}(x) \#_{n-1} \bar{v}^+_n(x),$$

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where the $v_n^\alpha(x)$ *are* n*-dimensional elements depending on* α, n, s_0^-, s_0^+, $\ldots$, s_{n-2}^-, s_{n-2}^+, s_{n-1}^α, $d_{n-1}^\alpha x$;

(v)
$$d_m^\alpha \operatorname{Gl}_n^{\mathbf{s}}(x) = \begin{cases} \operatorname{Gl}_m^{\mathbf{s}}(s_m^\alpha) & \text{for } 0 \le m < n, \\ \operatorname{Gl}_n^{\mathbf{s}}(d_m^\alpha x) & \text{for } m \ge n; \end{cases}$$

(vi) $\dim \operatorname{Gl}_n^{\mathbf{s}}(x) = \max(n, \dim x)$;

(vii) *if* $\dim x \le n$ *then* $\operatorname{Gl}_n^{\mathbf{s}}(x)$ *is a union of* n*-dimensional atoms;*

(viii) *if* x *and* y *are disjoint then*
$$\operatorname{Gl}_n^{\mathbf{s}}(x) \cap \operatorname{Gl}_n^{\mathbf{s}}(y) = \begin{cases} \emptyset & \text{for } n = 0, \\ \operatorname{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^-) \cup \operatorname{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^+) & \text{for } n \ge 1. \end{cases}$$

PROOF. We use induction on n.

We define $\operatorname{Gl}_0^{\mathbf{s}}(x)$ to be x, and the results for $n = 0$ are then obvious.

From now on, suppose that $n > 0$. Let x and $\mathbf{s}$ satisfy the conditions of (i). We shall construct $\operatorname{Gl}_n^{\mathbf{s}}(x)$ by applying Proposition 5.2 to the triple
$$\operatorname{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^-), \quad \operatorname{Gl}_{n-1}^{\mathbf{s}}(x), \quad \operatorname{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^+).$$

To verify the hypotheses of Proposition 5.2, let z denote s_{n-1}^-, x or s_{n-1}^+. Then, first, $d_{n-1}^\alpha \operatorname{Gl}_{n-1}^{\mathbf{s}}(z) = \operatorname{Gl}_{n-1}^{\mathbf{s}}(d_{n-1}^\alpha z)$ by an inductive application of (v), and this is a union of $(n-1)$-dimensional atoms by an inductive application of (vii). Next, the $d_{n-2}^\beta \operatorname{Gl}_{n-1}^{\mathbf{s}}(z)$ are all empty if $n = 1$ and are all equal to $\operatorname{Gl}_{n-2}^{\mathbf{s}}(s_{n-2}^\beta)$ by an inductive application of (v) if $n > 1$. Finally, the intersection of $\operatorname{Gl}_{n-1}^{\mathbf{s}}(z)$ with any other member of the triple is equal to $d_{n-2}^- \operatorname{Gl}_{n-1}^{\mathbf{s}}(z) \cup d_{n-2}^+ \operatorname{Gl}_{n-1}^{\mathbf{s}}(z)$ by an inductive application of (v) and (viii); this intersection is equal to $d_{n-2}^- d_{n-1}^\alpha \operatorname{Gl}_{n-1}^{\mathbf{s}}(z) \cup d_{n-2}^+ d_{n-1}^\alpha \operatorname{Gl}_{n-1}^{\mathbf{s}}(z)$, and is therefore contained in $d_{n-1}^\alpha \operatorname{Gl}_{n-1}^{\mathbf{s}}(z)$ for $\alpha = \pm$.

We have now verified the hypotheses of Proposition 5.2. We therefore have a molecular directed complex $\operatorname{Gl}_n^{\mathbf{s}}(x)$ such that
$$\operatorname{Gl}_n^{\mathbf{s}}(x) = \operatorname{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^-) \#_{n-1} \bar{v}_n^-(x) \#_{n-1} \operatorname{Gl}_{n-1}^{\mathbf{s}}(x) \#_{n-1} \bar{v}_n^+(x) \#_{n-1} \operatorname{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^+)$$
with $\dim v_n^\alpha(x) = n$. It remains to verify parts (ii) and (iv)–(viii).

Part (ii) is clear from the definition and the inductive hypothesis.

Next we prove part (iv). Since $\dim s_{n-1}^\alpha \le n - 1$, it follows from an inductive application of (vi) that $\dim \operatorname{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^\alpha) = n - 1$, so $\operatorname{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^\alpha)$ is an identity for $\#_{n-1}$. The decomposition of $\operatorname{Gl}_n^{\mathbf{s}}(x)$ given in its construction therefore simplifies to the decomposition given in part (iv).

By construction, $v_n^\alpha(x)$ depends only on α, n, $\operatorname{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^\alpha)$ and $d_{n-1}^\alpha \operatorname{Gl}_{n-1}^{\mathbf{s}}(x) = \operatorname{Gl}_{n-1}^{\mathbf{s}}(d_{n-1}^\alpha x)$. By an inductive application of (ii) and (iv), $v_n^\alpha(x)$ depends only on α, n, s_0^-, s_0^+, $\ldots$, s_{n-2}^-, s_{n-2}^+, s_{n-1}^α, $d_{n-1}^\alpha x$.

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Next we prove (v). By construction, $d^{\alpha}_{n-1}\,\mathrm{Gl}^{\mathbf{s}}_{n}(x) = d^{\alpha}_{n-1}\,\mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{\alpha}_{n-1})$, and we have already seen that this is $\mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{\alpha}_{n-1})$. For $0 \leq m < n$ the inductive hypothesis now gives

$$d^{\alpha}_{m}\,\mathrm{Gl}^{\mathbf{s}}_{n}(x) = d^{\alpha}_{m}d^{\alpha}_{n-1}\,\mathrm{Gl}^{\mathbf{s}}_{n}(x) = d^{\alpha}_{m}\,\mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{\alpha}_{n-1}) = \mathrm{Gl}^{\mathbf{s}}_{m}(s^{\alpha}_{m}).$$

Now let $m \geq n$. It follows from (iv) that

$$d^{\alpha}_{m}\,\mathrm{Gl}^{\mathbf{s}}_{n}(x) = d^{\alpha}_{m}\bar{\nu}^{-}_{n}(x) \#_{n-1} d^{\alpha}_{m}\,\mathrm{Gl}^{\mathbf{s}}_{n-1}(x) \#_{n-1} d^{\alpha}_{m}\bar{\nu}^{+}_{n}(x).$$

Since $\dim \bar{\nu}^{\beta}_{n}(x) = n \leq m$ we get $d^{\alpha}_{m}\bar{\nu}^{\beta}_{n}(x) = \bar{\nu}^{\beta}_{n}(x)$. Since $d^{\beta}_{n-1}d^{\alpha}_{m}x = d^{\beta}_{n-1}x$, it follows from (iv) that $\bar{\nu}^{\beta}_{n}(x) = \bar{\nu}^{\beta}_{n}(d^{\alpha}_{m}x)$. Also, from the inductive hypothesis, $d^{\alpha}_{m}\,\mathrm{Gl}^{\mathbf{s}}_{n-1}(x) = \mathrm{Gl}^{\mathbf{s}}_{n-1}(d^{\alpha}_{m}x)$. Therefore

$$d^{\alpha}_{m}\,\mathrm{Gl}^{\mathbf{s}}_{n}(x) = \bar{\nu}^{-}_{n}(d^{\alpha}_{m}x) \#_{n-1} \mathrm{Gl}^{\mathbf{s}}_{n-1}(d^{\alpha}_{m}x) \#_{n-1} \bar{\nu}^{+}_{n}(d^{\alpha}_{m}x).$$

By (iv), $d^{\alpha}_{m}\,\mathrm{Gl}^{\mathbf{s}}_{n}(x) = \mathrm{Gl}^{\mathbf{s}}_{n}(d^{\alpha}_{m}x)$ as required.

Part (vi) is immediate from part (iv) and the inductive hypothesis.

Next we prove part (vii). Suppose that σ is an element of $\mathrm{Gl}^{\mathbf{s}}_{n}(x)$ not contained in any n-dimensional atom. By part (vi), $\dim \sigma < n$. By Definition 2.3, $\sigma \in d^{-}_{n-1}\,\mathrm{Gl}^{\mathbf{s}}_{n}(x)$. From part (iv), $d^{-}_{n-1}\,\mathrm{Gl}^{\mathbf{s}}_{n}(x) = d^{-}_{n-1}\bar{\nu}^{-}_{n}(x)$, so σ is in the n-dimensional atom $\bar{\nu}^{-}_{n}(x)$. This contradiction completes the proof of part (vii).

Finally we prove part (viii). By construction, we have

$$\mathrm{Gl}^{\mathbf{s}}_{n}(x) = \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{-}_{n-1}) \cup \{\nu^{-}_{n}(x)\} \cup \mathrm{Gl}^{\mathbf{s}}_{n-1}(x) \cup \{\nu^{+}_{n}(x)\} \cup \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{+}_{n-1}),$$
$$\mathrm{Gl}^{\mathbf{s}}_{n}(y) = \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{-}_{n-1}) \cup \{\nu^{-}_{n}(y)\} \cup \mathrm{Gl}^{\mathbf{s}}_{n-1}(y) \cup \{\nu^{+}_{n}(x)\} \cup \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{+}_{n-1}).$$

It follows that $\mathrm{Gl}^{\mathbf{s}}_{n}(x) \cap \mathrm{Gl}^{\mathbf{s}}_{n}(y)$ contains $\mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{-}_{n-1}) \cup \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{+}_{n-1})$. Since x and y are disjoint, it follows from part (iv) that the elements $\nu^{-}_{n}(x)$, $\nu^{+}_{n}(x)$, $\nu^{-}_{n}(y)$, $\nu^{+}_{n}(y)$ are all distinct. Using the inductive hypothesis, we see that elements of $\mathrm{Gl}^{\mathbf{s}}_{n}(x) \cap \mathrm{Gl}^{\mathbf{s}}_{n}(y)$ not contained in $\mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{-}_{n-1}) \cup \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{+}_{n-1})$ must be contained in $\mathrm{Gl}^{\mathbf{s}}_{n-2}(s^{-}_{n-2}) \cup \mathrm{Gl}^{\mathbf{s}}_{n-2}(s^{+}_{n-2})$ with $n \geq 2$. Since $\mathrm{Gl}^{\mathbf{s}}_{n-2}(s^{\alpha}_{n-2}) = d^{\alpha}_{n-2}\,\mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{-}_{n-1})$, we get

$$\mathrm{Gl}^{\mathbf{s}}_{n}(x) \cap \mathrm{Gl}^{\mathbf{s}}_{n}(y) = \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{-}_{n-1}) \cup \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{+}_{n-1})$$

as required.

Next we use Proposition 5.2 to construct fat composites.

THEOREM 5.4. *There are molecular directed complexes $x \#^{\mathbf{s}}_{n} y$ with the following properties:*

(i) *$x \#^{\mathbf{s}}_{n} y$ is defined when n is a non-negative integer, $\mathbf{s}$ is an admissible $2p$-tuple for some $p \geq n$, and x and y are molecular directed complexes disjoint from s^{-}_{0}, $s^{+}_{0}, \ldots, s^{-}_{n-1}, s^{+}_{n-1}$;*

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$$
\begin{array}{ccccccccc}
 & & & & s_0^-, & & & & \\
 & & & \{\nu_1^-(s_1^-)\}, & s_1^-, & \{\nu_1^+(s_1^-)\}, & & & \\
 & & \{\nu_2^-(s_2^-)\}, & \{\nu_1^-(s_2^-)\}, & s_2^-, & \{\nu_1^+(s_2^-)\}, & \{\nu_2^+(s_2^-)\}, & & \\
 & & & & \vdots & & & & \\
 & \{\nu_{n-1}^-(s_{n-1}^-)\}, & \dots, & \{\nu_1^-(s_{n-1}^-)\}, & s_{n-1}^-, & \{\nu_1^+(s_{n-1}^-)\}, & \dots, & \{\nu_{n-1}^+(s_{n-1}^-)\}, & \\
\{\nu_n^-(x)\}, & \{\nu_{n-1}^-(x)\}, & \dots, & \{\nu_1^-(x)\}, & x, & \{\nu_1^+(x)\}, & \dots, & \{\nu_{n-1}^+(x)\}, & \{\nu_n^+(x)\}, \\
 & \{\nu_{n-1}^-(s_{n-1}^+)\}, & \dots, & \{\nu_1^-(s_{n-1}^+)\}, & s_{n-1}^+, & \{\nu_1^+(s_{n-1}^+)\}, & \dots, & \{\nu_{n-1}^+(s_{n-1}^+)\}, & \\
 & & & & \vdots & & & & \\
 & & & \{\nu_2^-(s_2^+)\}, & \{\nu_1^-(s_2^+)\}, & s_2^+, & \{\nu_1^+(s_2^+)\}, & \{\nu_2^+(s_2^+)\}, & \\
 & & & & \{\nu_1^-(s_1^+)\}, & s_1^+, & \{\nu_1^+(s_1^+)\}, & & \\
 & & & & & s_0^+. & & &
\end{array}
$$

TABLE 1. The constituents of $\mathrm{Gl}_n^{\mathbf{s}}(x)$

(ii) *$x \#_n^{\mathbf{s}} y$ depends on s_i^α only for $i < n$;*

(iii) *$x \#_n^{\mathbf{s}} y = \mathrm{Gl}_n^{\mathbf{s}}(x) \#_n \bar{\nu}_{n+1}(x, y) \#_n \mathrm{Gl}_n^{\mathbf{s}}(y)$, where $\nu_{n+1}(x, y)$ is an n-dimensional element depending on $n + 1$, s_0^-, s_0^+, $\dots$, s_{n-1}^-, s_{n-1}^+, $d_n^+ x$ and $d_n^- y$.*

PROOF. We construct $x \#_n^{\mathbf{s}} y$ by applying Proposition 5.2 to the pair $\mathrm{Gl}_n^{\mathbf{s}}(x)$, $\mathrm{Gl}_n^{\mathbf{s}}(y)$; the necessary hypotheses follow from Theorem 5.3 (vii), (v), (viii). Properties (ii) and (iii) follow straightforwardly from Theorem 5.3.

We shall use Theorems 5.3 and 5.4 to construct P-molecules out of P-molecules. Since P-molecules are required to be loop-free, we must show that globularisations and fat composites are loop-free if their constituents are loop-free. We shall use the following results.

PROPOSITION 5.5. *A globularisation $\mathrm{Gl}_n^{\mathbf{s}}(x)$ is the disjoint union of the sets listed in Table* 1.

PROOF. This follows by induction from Theorem 5.3 (iii), (iv) and (viii).

PROPOSITION 5.6. *Let $\sigma, \dots, \tau$ be an n-path or a total path in a globularisation $\mathrm{Gl}_n^{\mathbf{s}}(x)$. Let S and T be the sets in Table* 1 *which contain σ and τ. Then $S = T$ or S precedes T.*

PROOF. It suffices to prove this when σ and τ are consecutive, and, because of Proposition 2.8, it suffices to consider total paths. We take the case $\sigma \in \partial^- \tau$. If

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$\tau \in s_i^\alpha$ or $\tau \in x$, then σ is clearly in the same subset, so that $S = T$. If $\tau = \nu_i^-(z)$ then σ is an $(i-1)$-dimensional element of

$$d_{i-1}^- \bar{\nu}_i^-(z) = d_{i-1}^- \mathrm{Gl}_i^{\mathbf{s}}(z) = \mathrm{Gl}_{i-1}^{\mathbf{s}}(s_{i-1}^-)$$

(see Theorem 1 (iv) and (v)), so

$$\sigma \in \{\nu_{i-1}^-(s_{i-1}^-)\} \cup s_{i-1}^- \cup \{\nu_{i-1}^+(s_{i-1}^-)\}$$

by Proposition 5.5, from which it follows that S precedes T. If $\tau = \nu_i^+(z)$ then similarly

$$\sigma \in d_{i-1}^- \bar{\nu}_i^+(z) = d_{i-1}^+ \mathrm{Gl}_{i-1}^{\mathbf{s}}(z) \subset \mathrm{Gl}_{i-1}^{\mathbf{s}}(z),$$

so $\sigma \in \{\nu_{i-1}^-(z)\} \cup z \cup \{\nu_{i-1}^+(z)\}$, and again S precedes T.

THEOREM 5.7. *Let $\mathrm{Gl}_n^{\mathbf{s}}(x)$ be a globularisation. If the molecules $s_0^-, s_0^+, \ldots, s_{n-1}^-$, s_{n-1}^+ and x are all loop-free then $\mathrm{Gl}_n^{\mathbf{s}}(x)$ is loop-free. If the molecules $s_0^-, s_0^+, \ldots, s_{n-1}^-$, s_{n-1}^+ and x are all totally loop-free then $\mathrm{Gl}_n^{\mathbf{s}}(x)$ is totally loop-free.*

PROOF. The proofs are similar in both cases; we take the totally loop-free case. We must show that a total path in $\mathrm{Gl}_n^{\mathbf{s}}(x)$ has no repeated elements. By Proposition 5.6, it suffices to show that a total path in a set in Table 1 has no repeated elements. But this is true by hypothesis for the sets s_i^α and x, and it is trivial for the singleton sets.

THEOREM 5.8. *Let $x \#_n^{\mathbf{s}} y$ be a fat composite. If the molecules $s_0^-, s_0^+, \ldots, s_{n-1}^-, s_{n-1}^+$, x and y are all loop-free then $x \#_n^{\mathbf{s}} y$ is loop-free. If the molecules $s_0^-, s_0^+, \ldots, s_{n-1}^-$, s_{n-1}^+, x and y are all totally loop-free then $x \#_n^{\mathbf{s}} y$ is totally loop-free.*

PROOF. This is similar to Theorem 5.7; we use the decomposition in Table 2.

So far we have considered globularisations and fat composites as abstract directed complexes; we shall now consider cases in which they are P-molecules, and we shall determine their P-equivalence classes.

THEOREM 5.9. *Let $\mathrm{Gl}_n^{\mathbf{s}}(x)$ be a globularisation which is a G-complex such that the s_i^α and x are P-molecules, $s_i^\alpha \sim d_i^\alpha x$ for $0 \leq i < n$, and the additional atoms $\bar{\nu}_i(z)$ are degenerate. Then:*

(i) *$\mathrm{Gl}_n^{\mathbf{s}}(x)$ is a P-molecule;*

(ii) *if there are G-isomorphisms $s_i^\alpha \cong d_i^\alpha x$ for $0 \leq i < n$, then there is an elementary P-equivalence from $\mathrm{Gl}_n^{\mathbf{s}}(x)$ to x which extends these G-isomorphisms and acts as the identity on x;*

(iii) *$\mathrm{Gl}_n^{\mathbf{s}}(x) \sim x$.*

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$$
\begin{array}{ccccccccccc}
 & & & & & s_0^-, & & & & & \\
 & & & & \{\nu_1^-(s_1^-)\}, & s_1^-, & \{\nu_1^+(s_1^-)\}, & & & & \\
 & & & \{\nu_2^-(s_2^-)\}, & \{\nu_1^-(s_2^-)\}, & s_2^-, & \{\nu_1^+(s_2^-)\}, & \{\nu_2^+(s_2^-)\}, & & & \\
 & & & & & \vdots & & & & & \\
 & & \{\nu_{n-1}^-(s_{n-1}^-)\}, & \ldots, & \{\nu_1^-(s_{n-1}^-)\}, & s_{n-1}^-, & \{\nu_1^+(s_{n-1}^-)\}, & \ldots, & \{\nu_{n-1}^+(s_{n-1}^-)\}, & & \\
\{\nu_n^-(x)\}, & \{\nu_{n-1}^-(x)\}, & \ldots, & \{\nu_1^-(x)\}, & x, & \{\nu_1^+(x)\}, & \ldots, & \{\nu_{n-1}^+(x)\}, & \{\nu_n^+(x)\}, & & \\
 & & & & \{\nu_{n+1}(x,y)\}, & & & & & & \\
\{\nu_n^-(y)\}, & \{\nu_{n-1}^-(y)\}, & \ldots, & \{\nu_1^-(y)\}, & y, & \{\nu_1^+(y)\}, & \ldots, & \{\nu_{n-1}^+(y)\}, & \{\nu_n^+(y)\}, & & \\
 & \{\nu_{n-1}^-(s_{n-1}^+)\}, & \ldots, & \{\nu_1^-(s_{n-1}^+)\}, & s_{n-1}^+, & \{\nu_1^+(s_{n-1}^+)\}, & \ldots, & \{\nu_{n-1}^+(s_{n-1}^+)\}, & & & \\
 & & & & & \vdots & & & & & \\
 & & & \{\nu_2^-(s_2^+)\}, & \{\nu_1^-(s_2^+)\}, & s_2^+, & \{\nu_1^+(s_2^+)\}, & \{\nu_2^+(s_2^+)\}, & & & \\
 & & & & \{\nu_1^-(s_1^+)\}, & s_1^+, & \{\nu_1^+(s_1^+)\}, & & & & \\
 & & & & & s_0^+. & & & & &
\end{array}
$$

TABLE 2. The constituents of $x \#_n^{\mathbf{s}} y$

PROOF. We use induction on n. The results are obvious for $n = 0$, because $\mathrm{Gl}_0^{\mathbf{s}}(x) = x$.

Suppose that $n > 0$. From Theorems 5.3 and 5.7 we know that $\mathrm{Gl}_n^{\mathbf{s}}(x)$ is a loop-free molecular directed complex, and we also know that

$$\mathrm{Gl}_n^{\mathbf{s}}(x) = \mathrm{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^-) \cup \{\nu_n^-(x)\} \cup \mathrm{Gl}_{n-1}^{\mathbf{s}}(x) \cup \{\nu_n^+(x)\} \cup \mathrm{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^+).$$

The inductive hypothesis tells us that $\mathrm{Gl}_{n-1}^{\mathbf{s}}(x)$ is a P-molecule. By Proposition 4.4, $s_i^\alpha \sim d_i^\alpha s_{n-1}^-$ for $i < n-1$, so the inductive hypothesis also tells us that $\mathrm{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^-)$ is a P-molecule. Similarly $\mathrm{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^+)$ is a P-molecule. To show that $\mathrm{Gl}_n^{\mathbf{s}}(x)$ is a P-molecule it now suffices to show that $d_{n-1}^- \bar{\nu}_n^\alpha(x) \sim d_{n-1}^+ \bar{\nu}_n^\alpha(x)$ (see Proposition 4.6).

From Theorem 5.3 we get

$$\begin{aligned} d_{n-1}^- \bar{\nu}_n^-(x) &= d_{n-1}^- \mathrm{Gl}_n^{\mathbf{s}}(x) = \mathrm{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^-), \\ d_{n-1}^+ \bar{\nu}_n^-(x) &= d_{n-1}^- \mathrm{Gl}_{n-1}^{\mathbf{s}}(x) = \mathrm{Gl}_{n-1}^{\mathbf{s}}(d_{n-1}^- x). \end{aligned}$$

Using the inductive hypothesis and the assumption on the P-equivalence class of s_{n-1}^-, we get

$$\mathrm{Gl}_{n-1}^{\mathbf{s}}(s_{n-1}^-) \sim s_{n-1}^- \sim d_{n-1}^- x \sim \mathrm{Gl}_{n-1}^{\mathbf{s}}(d_{n-1}^- x),$$

so that $d_{n-1}^- \bar{\nu}_n^-(x) \sim d_{n-1}^+ \bar{\nu}_n^-(x)$. Similarly $d_{n-1}^- \bar{\nu}_n^+(x) \sim d_{n-1}^+ \bar{\nu}_n^+(x)$. Therefore $\mathrm{Gl}_n^{\mathbf{s}}(x)$ is a P-molecule. This proves part (i).

Next we prove part (ii). Because of the inductive hypothesis, it suffices to construct an elementary P-equivalence

$$M\left(\mathrm{Gl}_n^{\mathbf{s}}(x)\right) \to M\left(\mathrm{Gl}_{n-1}^{\mathbf{s}}(x)\right)$$

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which extends the given isomorphisms on the s^{α}_{n-1} and acts as the identity on s^{-}_{0}, $s^{+}_{0}, \ldots, s^{-}_{n-2}, s^{+}_{n-2}$ and x.

The G-isomorphisms $s^{\alpha}_{n-1} \cong d^{\alpha}_{n-1}x$ clearly extend to G-isomorphisms

$$\mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{\alpha}_{n-1}) \cong \mathrm{Gl}^{\mathbf{s}}_{n-1}(d^{\alpha}_{n-1}x) = d^{\alpha}_{n-1}\,\mathrm{Gl}^{\mathbf{s}}_{n-1}(x)$$

which act as the identity on the s^{β}_{i} for $i < n-1$. Combining these with the identity of $\mathrm{Gl}^{\mathbf{s}}_{n-1}(x)$ gives a morphism

$$M\left(\mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{-}_{n-1}) \cup \mathrm{Gl}^{\mathbf{s}}_{n-1}(x) \cup \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{+}_{n-1})\right) \to M\left(\mathrm{Gl}^{\mathbf{s}}_{n-1}(x)\right).$$

We note that $d^{-}_{n-1}\bar{v}^{-}_{n}(x) = \mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{-}_{n-1})$ and $d^{+}_{n-1}\bar{v}^{-}_{n}(x) = \mathrm{Gl}^{\mathbf{s}}_{n-1}(d^{-}_{n-1}x)$ have the same image under this morphism, and similarly for $\bar{v}^{+}_{n}(x)$. Using the presentation of $M\left(\mathrm{Gl}^{\mathbf{s}}_{n}(x)\right)$ of Theorem 2.9, we get a morphism $M\left(\mathrm{Gl}^{\mathbf{s}}_{n}(x)\right) \to M\left(\mathrm{Gl}^{\mathbf{s}}_{n-1}(x)\right)$ acting in the desired way on the s^{α}_{i} and x. This morphism is an elementary P-equivalence as required (see Definition 3.6). This completes the proof of (ii).

Finally we prove (iii). Because of part (ii), it suffices to show that the P-equivalence class of $\mathrm{Gl}^{\mathbf{s}}_{n}(x)$ depends only on the P-equivalence classes of $s^{-}_{0}, s^{+}_{0}, \ldots, s^{-}_{n-1}, s^{+}_{n-1}$ and x. We must therefore consider what happens when the constituents are changed by chains of elementary P-equivalences. By Proposition 4.5, we may assume that the chain for s^{α}_{i} passes through P-molecules of dimension at most i, so we can work with admissible $2p$-tuples throughout.

It is now clearly sufficient to prove that $\mathrm{Gl}^{\mathbf{s}}_{n}(x) \sim \mathrm{Gl}^{\mathbf{t}}_{n}(y)$ when there are elementary P-equivalences $M(s^{\alpha}_{i}) \to M(t^{\alpha}_{i})$ and $M(x) \to M(y)$ all going in the same direction, and we imitate the proof of part (ii). Indeed we get elementary P-equivalences $M\left(\mathrm{Gl}^{\mathbf{s}}_{n-1}(s^{\alpha}_{n-1})\right) \to M\left(\mathrm{Gl}^{\mathbf{t}}_{n-1}(t^{\alpha}_{n-1})\right)$ and $M\left(\mathrm{Gl}^{\mathbf{s}}_{n-1}(x)\right) \to M\left(\mathrm{Gl}^{\mathbf{t}}_{n-1}(y)\right)$ by the inductive hypothesis, and these extend to an elementary P-equivalence $M\left(\mathrm{Gl}^{\mathbf{s}}_{n}(x)\right) \to M\left(\mathrm{Gl}^{\mathbf{t}}_{n}(y)\right)$ as required.

We are now in a position to define composites in $\omega(P)$.

THEOREM 5.10. *Let ξ and η be equivalence classes in $\omega(P)$ such that $d^{+}_{n}\xi = d^{-}_{n}\eta$ for some n. Then there is a well-defined composite $\xi \#_{n} \eta$ defined as follows: let x and y be disjoint representatives of ξ and η; let $\mathbf{s}$ be an admissible $2p$-tuple of P-molecules for some $p \geq n$ such that s^{α}_{i} is disjoint from x and y and $s^{\alpha}_{i} \sim d^{\alpha}_{i}x \sim d^{\alpha}_{i}y$ for $0 \leq i < n$; then*

$$\xi \#_{n} \eta = [x] \#_{n} [y] = [x \#^{\mathbf{s}}_{n} y],$$

where the additional atoms $\bar{v}^{\alpha}_{i}(z)$ and $\bar{v}_{n+1}(x, y)$ in $x \#^{\mathbf{s}}_{n} y$ are degenerate.

PROOF. We firstly show that there exist x, y and s^{α}_{i} as described. For $0 \leq i < n$ it follows from Proposition 4.4 that $d^{\alpha}_{i}\xi = d^{\alpha}_{i}d^{+}_{n}\xi = d^{\alpha}_{i}d^{-}_{n}\eta = d^{\alpha}_{i}\eta$; we can therefore

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find P-molecules x, y and s_i^α such that $[x] = \xi$, $[y] = \eta$, and $s_i^\alpha \sim d_i^\alpha x \sim d_i^\alpha y$ for $0 \leq i < n$. Clearly we can assume that all these P-molecules are disjoint and $\dim s_i^\alpha \leq i$ for $0 \leq i < n$.

It now follows from Theorem 5.4 that there is a fat composite

$$x \#_n^{\mathbf{s}} y = \mathrm{Gl}_n^{\mathbf{s}}(x) \#_n \bar{\nu}_{n+1}(x, y) \#_n \mathrm{Gl}_n^{\mathbf{s}}(y).$$

We make this into a G-complex by making the additional atoms degenerate. As in Theorem 5.9, we find that $x \#_n^{\mathbf{s}} y$ is a P-molecule whose P-equivalence class depends only on the P-equivalence classes of $s_0^-, s_0^+, \ldots, s_{n-1}^-, s_{n-1}^+, x$ and y. Since $s_i^\alpha \sim d_i^\alpha x \sim d_i^\alpha y$ for $0 \leq i < n$, the P-equivalence class of $x \#_n^{\mathbf{s}} y$ in fact depends only on the P-equivalence classes of x and y. The P-equivalence class of $x \#_n^{\mathbf{s}} y$ therefore gives a well-defined composite $\xi \#_n \eta$, as required.

6. Verification of the ω-category axioms

In this section, $P = (G, R)$ is again a directed complex presentation. In the last two sections, we have defined operations d_n^α and $\#_n$ in $\omega(P)$; we shall now show that they make $\omega(P)$ into an ω-category.

We begin with two results showing that compositions in $\omega(P)$ are induced by compositions in the ω-categories of P-molecules.

PROPOSITION 6.1. *Let $x \#_n y$ be a composite in $M(w)$, where w is a P-molecule. Then*

$$[x] \#_n [y] = [x \#_n y].$$

PROOF. We note that $d_i^\alpha x = d_i^\alpha y$ for $0 \leq i < n$. We can therefore represent $[x] \#_n [y]$ by a fat composite $x' \#_n^{\mathbf{s}} y'$ such that there are G-isomorphisms $x' \cong x$, $y' \cong y$ and $s_i^\alpha \cong d_i^\alpha x = d_i^\alpha y$ for $0 \leq i < n$. As in the proof of Theorem 5.92, these G-isomorphisms extend to an elementary P-equivalence from $x' \#_n^{\mathbf{s}} y'$ to $x \#_n y$. Therefore

$$[x] \#_n [y] = [x' \#_n^{\mathbf{s}} y'] = [x \#_n y].$$

PROPOSITION 6.2. *If z represents a composite $\xi \#_n \eta$ in $\omega(P)$, then there is a decomposition $z = x \#_n y$ such that x and y represent ξ and η.*

PROOF. By construction, z is a fat composite $x' \#_n^{\mathbf{s}} y'$ with x' and y' representing ξ and η. By Theorem 5.4,

$$z = \mathrm{Gl}_n^{\mathbf{s}}(x') \#_n \bar{\nu}_{n+1}(x', y') \#_n \mathrm{Gl}_n^{\mathbf{s}}(y').$$

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Now $d_n^+ \mathrm{Gl}_n^{\mathbf{s}}(x) = \mathrm{Gl}_n^{\mathbf{s}}(d_n^+ x')$ (Theorem 5.3 (v)) and $\nu_{n+1}(x', y')$ depends on x' only through $d_n^+ x'$ (Theorem 5.4 (iii)), so

$$z = \mathrm{Gl}_n^{\mathbf{s}}(x') \#_n \{\mathrm{Gl}_n^{\mathbf{s}}(d_n^+ x') \#_n \bar{\nu}_{n+1}(d_n^+ x', y') \#_n \mathrm{Gl}_n^{\mathbf{s}}(y')\} = \mathrm{Gl}_n^{\mathbf{s}}(x') \#_n (d_n^+ x' \#_n^{\mathbf{s}} y') = x \#_n y,$$

where $x = \mathrm{Gl}_n^{\mathbf{s}}(x')$ and $y = d_n^+ x' \#_n^{\mathbf{s}} y'$. By Theorem 5.9, $[x] = [x'] = \xi$; it remains to show that $[y] = \eta$. But $[d_n^+ x'] = d_n^+ \xi = d_n^- \eta = [d_n^- y']$, so y represents the composite $[d_n^- y'] \#_n [y']$. Since the composite $d_n^- y' \#_n y'$ exists in $M(y')$, we get $[y] = [d_n^- y' \#_n y'] = [y'] = \eta$ by Proposition 6.1.

We shall also need the following result on the globularisations of composites.

PROPOSITION 6.3. *If $\mathrm{Gl}_n^{\mathbf{s}}(w)$ is a globularisation and there is a decomposition $w = x \#_m y$ such that x and y are molecules and $m \geq n$, then*

$$\mathrm{Gl}_n^{\mathbf{s}}(w) = \mathrm{Gl}_n^{\mathbf{s}}(x \#_m y) = \mathrm{Gl}_n^{\mathbf{s}}(x) \#_m \mathrm{Gl}_n^{\mathbf{s}}(y).$$

PROOF. The proof is by induction on n. The result is obvious for $n = 0$, since $\mathrm{Gl}_0^{\mathbf{s}}(w) = w$, etcetera.

Suppose that $n > 0$. By Theorem 5.3,

$$\mathrm{Gl}_n^{\mathbf{s}}(w) = \bar{\nu}_n^-(w) \#_{n-1} \mathrm{Gl}_{n-1}^{\mathbf{s}}(w) \#_{n-1} \bar{\nu}_n^+(w),$$

where $\nu_n^\alpha(w)$ depends on w only through $d_{n-1}^\alpha w$. Since $w = x \#_m y$ and $m > n - 1$, we get $d_{n-1}^\alpha w = d_{n-1}^\alpha x = d_{n-1}^\alpha y$, so that $\nu_n^\alpha(w) = \nu_n^\alpha(x) = \nu_n^\alpha(y)$. Since $\dim \bar{\nu}_n^\alpha(w) = n \geq m$, we find that $\bar{\nu}_n^\alpha(w)$ acts as an identity for $\#_n$; we therefore have $\bar{\nu}_n^\alpha(w) = \bar{\nu}_n^\alpha(x) \#_m \bar{\nu}_n^\alpha(y)$. Also, by the inductive hypothesis,

$$\mathrm{Gl}_{n-1}^{\mathbf{s}}(w) = \mathrm{Gl}_{n-1}^{\mathbf{s}}(x) \#_m \mathrm{Gl}_{n-1}^{\mathbf{s}}(y).$$

It follows that

$$\begin{aligned}
\mathrm{Gl}_n^{\mathbf{s}}(w) &= \bar{\nu}_n^-(w) \#_{n-1} \mathrm{Gl}_{n-1}^{\mathbf{s}}(w) \#_{n-1} \bar{\nu}_n^+(w), \\
&= \{\bar{\nu}_n^-(x) \#_m \bar{\nu}_n^-(y)\} \#_{n-1} \{\mathrm{Gl}_{n-1}^{\mathbf{s}}(x) \#_m \mathrm{Gl}_{n-1}^{\mathbf{s}}(y)\} \#_{n-1} \{\bar{\nu}_n^+(x) \#_m \bar{\nu}_n^+(y)\} \\
&= \{\bar{\nu}_n^-(x) \#_{n-1} \mathrm{Gl}_{n-1}^{\mathbf{s}}(x) \#_{n-1} \bar{\nu}_n^+(x)\} \#_m \{\bar{\nu}_n^-(y) \#_{n-1} \mathrm{Gl}_{n-1}^{\mathbf{s}}(y) \#_{n-1} \bar{\nu}_n^+(y)\} \\
&= \mathrm{Gl}_n^{\mathbf{s}}(x) \#_m \mathrm{Gl}_n^{\mathbf{s}}(y),
\end{aligned}$$

as required.

We can now prove the main theorem of this section.

THEOREM 6.4. *The operations d_n^α and $\#_n$ make $\omega(P)$ into an ω-category.*

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PROOF. We must verify the axioms of Definition 2.1. Axiom (i) holds by Theorem 5.10. Axioms (ii) and (vii) hold by Proposition 4.4. To verify axiom (iii) ($d_n^- \xi \#_n \xi = \xi = \xi \#_n d_n^+ \xi$) we use Proposition 6.1: if x represents ξ, then $d_n^- x \#_n x = x = x \#_n d_n^+ x$ in $M(x)$, so

$$d_n^- \xi \#_n \xi = [d_n^- x] \#_n [x] = [d_n^- x \#_n x] = [x] = \xi,$$

and so on.

Next we verify axiom (iv), which gives the value of $d_m^\alpha(\xi \#_n \eta)$. By Proposition 6.2, $\xi \#_n \eta$ is represented by a P-molecule z such that $z = x \#_n y$ with $[x] = \xi$ and $[y] = \eta$. Inside $M(z)$ we have $d_m^\alpha z = d_m^\alpha x \#_n d_m^\alpha y$ for $m \neq n$, etcetera. Applying Proposition 6.1 shows that $d_m^\alpha(\xi \#_n \eta) = d_m^\alpha \xi \#_n d_m^\alpha \eta$ for $m \neq n$, and so on, as required.

Next we verify axiom (v), the associativity of $\#_n$. Consider a product $(\xi \#_n \eta) \#_n \zeta$ in $\omega(P)$. According to Proposition 6.2, $\xi \#_n \eta$ can be represented by a genuine composite; therefore $(\xi \#_n \eta) \#_n \zeta$ is represented by a composite of the form

$$(x \#_n y) \#_n^{\mathbf{s}} z,$$

where x, y and z represent ξ, η and ζ. By Theorem 5.4 and Proposition 6.3, this composite can be expressed as

$$\begin{aligned} \mathrm{Gl}_n^{\mathbf{s}}(x \#_n y) \#_n \bar{\nu}_{n+1}(x \#_n y, z) \#_n \mathrm{Gl}_n^{\mathbf{s}}(z) &= \mathrm{Gl}_n^{\mathbf{s}}(x) \#_n \mathrm{Gl}_n^{\mathbf{s}}(y) \#_n \bar{\nu}_{n+1}(y, z) \#_n \mathrm{Gl}_n^{\mathbf{s}}(z) \\ &= \mathrm{Gl}_n^{\mathbf{s}}(x) \#_n (y \#_n^{\mathbf{s}} z) \end{aligned}$$

(it follows from Theorem 5.4 that $\nu_{n+1}(x \#_n y, z) = \nu_{n+1}(y, z)$ because $d_n^+(x \#_n y) = d_n^+ y$). Now $\mathrm{Gl}_n^{\mathbf{s}}(x)$ represents ξ by Theorem 5.9 and $y \#_n^{\mathbf{s}} z$ represents $\eta \#_n \zeta$ by definition, so $\mathrm{Gl}_n^{\mathbf{s}}(x) \#_n (y \#_n^{\mathbf{s}} z)$ represents $\xi \#_n (\eta \#_n \zeta)$ by Proposition 6.1. Therefore $(\xi \#_n \eta) \#_n \zeta = \xi \#_n (\eta \#_n \zeta)$.

We conclude by verifying axiom (vi), which states that

$$(\xi' \#_n \eta') \#_m (\xi'' \#_n \eta'') = (\xi' \#_m \xi'') \#_n (\eta' \#_m \eta'')$$

for $m \neq n$. It suffices to take the case $m < n$. We argue as for axiom (v). We can choose representatives x', y', x'', y'' for ξ', η', ξ'', η'' such that $(\xi' \#_n \eta') \#_m (\xi'' \#_n \eta'')$ is represented by a composite of the form $(x' \#_n y') \#_m^{\mathbf{s}} (x'' \#_n y'')$. We note that

$$\nu_{m+1}(x' \#_n y', x'' \#_n y'') = \nu_{m+1}(x', y') = \nu_{m+1}(x'', y''),$$

and we write ν_{m+1} for the common value. We also note that $\bar{\nu}_{m+1} = \bar{\nu}_{m+1} \#_n \bar{\nu}_{m+1}$ because $\dim \bar{\nu}_{m+1} = m + 1 \leq n$. We now get

$$\begin{aligned} &(x' \#_n y') \#_m^{\mathbf{s}} (x'' \#_n y'') \\ &\quad = \mathrm{Gl}_m^{\mathbf{s}}(x' \#_n y') \#_m \bar{\nu}_{m+1} \#_m \mathrm{Gl}_m^{\mathbf{s}}(x'' \#_n y'') \\ &\quad = \{\mathrm{Gl}_m^{\mathbf{s}}(x') \#_n \mathrm{Gl}_m^{\mathbf{s}}(y')\} \#_m \{\bar{\nu}_{m+1} \#_n \bar{\nu}_{m+1}\} \#_m \{\mathrm{Gl}_m^{\mathbf{s}}(x'') \#_n \mathrm{Gl}_m^{\mathbf{s}}(y'')\} \\ &\quad = \{\mathrm{Gl}_m^{\mathbf{s}}(x') \#_m \bar{\nu}_{m+1} \#_m \mathrm{Gl}_m^{\mathbf{s}}(x'')\} \#_n \{\mathrm{Gl}_m^{\mathbf{s}}(y') \#_m \bar{\nu}_{m+1} \#_m \mathrm{Gl}_m^{\mathbf{s}}(y'')\} \\ &\quad = (x' \#_m^{\mathbf{s}} x'') \#_n (y' \#_m^{\mathbf{s}} y''), \end{aligned}$$

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and this final composite represents $(\xi' \#_m \xi'') \#_n (\eta' \#_m \eta'')$. Therefore $(\xi' \#_n \eta') \#_m (\xi'' \#_n \eta'') = (\xi' \#_m \xi'') \#_n (\eta' \#_m \eta'')$ as required.

7. An algebraic presentation for $\omega(P)$

In this section, $P = (G, R)$ is again a directed complex presentation. We shall find an algebraic presentation for the ω-category $\omega(P)$.

We first aim to show that $\omega(P)$ is generated by the P-equivalence classes of members of G. As well as copies of members of G, there are also degenerate P-atoms; we use the following result to eliminate the degenerate P-atoms.

THEOREM 7.1. *Let $\bar{\sigma}$ be a degenerate P-atom of dimension p. Then $\bar{\sigma} \sim d^-_{p-1}\bar{\sigma} \sim d^+_{p-1}\bar{\sigma}$.*

PROOF. Recall from Definition 3.7 that $p > 0$ and $d^-_{p-1}\bar{\sigma} \sim d^+_{p-1}\bar{\sigma}$. Since $\dim d^\alpha_{p-1}\bar{\sigma} \leq p-1$, the common P-equivalence class acts as an identity for $\#_{p-1}$ in $\omega(P)$; in other words,

$$[d^-_{p-1}\bar{\sigma}] \#_{p-1} [d^+_{p-1}\bar{\sigma}] = [d^-_{p-1}\bar{\sigma}] = [d^+_{p-1}\bar{\sigma}].$$

It follows that there is a fat composite $x^- \#^{\mathbf{s}}_{p-1} x^+$ with $x^\alpha \cong d^\alpha_{p-1}\bar{\sigma}$, and we get

$$x^- \#^{\mathbf{s}}_{p-1} x^+ \sim d^-_{p-1}\bar{\sigma} \sim d^+_{p-1}\bar{\sigma}.$$

It therefore suffices to show that $x^- \#^{\mathbf{s}}_{p-1} x^+ \sim \bar{\sigma}$.

We note that $d^\alpha_i d^-_{p-1}\bar{\sigma} = d^\alpha_i \bar{\sigma} = d^\alpha_i d^+_{p-1}\bar{\sigma}$ for $0 \leq i < p-1$. We can therefore choose the s^α_i to be G-isomorphic to $d^\alpha_i\bar{\sigma}$ for $0 \leq i < p-1$ (see Theorem 5.10). Now consider the decomposition

$$x^- \#^{\mathbf{s}}_{p-1} x^+ = \mathrm{Gl}^{\mathbf{s}}_{p-1}(x^-) \#_{p-1} \bar{\nu}_p(x^-, x^+) \#_{p-1} \mathrm{Gl}^{\mathbf{s}}_{p-1}(x^+).$$

According to Theorem 5.9 there are elementary P-equivalences

$$M\left(\mathrm{Gl}^{\mathbf{s}}_{p-1}(x^\alpha)\right) \to M(x^\alpha) \cong M(d^\alpha_{p-1}\sigma)$$

which extend the given G-isomorphisms $s^\beta_i \cong d^\beta_i\bar{\sigma}$. By Theorem 5.9 we can extend these elementary P-equivalences to a morphism $M(x^- \#^{\mathbf{s}}_{p-1} x^+) \to M(\bar{\sigma})$ such that $\bar{\nu}_p(x^-, x^+) \mapsto \bar{\sigma}$, and this extension is clearly an elementary P-equivalence. Therefore $x^- \#^{\mathbf{s}}_{p-1} x^+ \sim \bar{\sigma}$, as required.

We now have the following result.

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THEOREM 7.2. *Every member of $\omega(P)$ is a composite of P-equivalence classes of members of G.*

PROOF. Every P-molecule is a composite of P-atoms. Since composition in P-molecules induces composition in $\omega(P)$ (Proposition 6.1), every member of $\omega(P)$ is a composite of P-equivalence classes of P-atoms. Now some P-atoms are copies of members of G and the others are degenerate. It is clear that a copy of a member g of G is P-equivalent to g, while a degenerate P-atom is P-equivalent to a lower-dimensional P-molecule by Theorem 7.1. The result follows by induction on dimension.

We now give an algebraic presentation for $\omega(P)$.

THEOREM 7.3. *The ω-category $\omega(P)$ has the following algebraic presentation: there is a generator $[g]$ for each member g of G; if $[g]$ is a p-dimensional generator then there are relations $d^{\alpha}_{p}[g] = [g]$; if $[g]$ is a p-dimensional generator and $p > 0$ then there are relations $d^{\alpha}_{p-1}[g] = c^{\alpha}(g)$, where the $c^{\alpha}(g)$ are arbitrarily chosen expressions for the $d^{\alpha}_{p-1}[g]$ as composites of generators; if w is a subdivision in R of a member g of G then there is a relation $[g] = c(w)$, where $c(w)$ is an arbitrarily chosen expression for $[w]$ as a composite of generators.*

PROOF. By Definition 3.8, the members of G are P-molecules, so $\omega(P)$ does contain P-equivalence classes $[g]$ for $g \in G$. If w is a subdivision in R of g, then the morphism $M(g) \to M(w)$ is clearly an elementary P-equivalence, so $[g] = [w]$. It is clear that the other relations in the statement of the theorem are valid as well.

It remains to show that the generators and relations are sufficient to determine the structure of $\omega(P)$. From Propositions 4.1–4.3, 6.1 and 6.2 we see that $\omega(P)$ is generated by the ω-categories $M(w)$ (where w is a P-molecule), subject to the relations given by elementary P-equivalences. We must show that the generators and relations are sufficient to determine the structure of the $M(w)$ (where w is a P-molecule) and also sufficient to account for elementary P-equivalences. Throughout this argument, we assume that members of $\omega(P)$ have been expressed as composites of P-equivalence classes of members of G, as is permitted by Theorem 7.2.

First we consider the structure of $M(w)$, where w is a P-molecule. According to Theorem 2.9, $M(w)$ is generated by its atoms $\bar{\sigma}$, subject to the following relations: if $\dim \sigma = p$ then $d^{\alpha}_{p}\bar{\sigma} = \bar{\sigma}$; if $\dim \sigma = p > 0$ then $d^{\alpha}_{p-1}\bar{\sigma}$ is equal to some composite. Let $\bar{\sigma}$ be an atom in w. If $\bar{\sigma}$ is a copy of a member g of G, then we certainly have $[\bar{\sigma}] = [g]$, so we get the generator $[\bar{\sigma}]$ and the associated relations from the presentation given in the statement of the theorem. On the other hand, if $\bar{\sigma}$ is degenerate, then we can express $[\bar{\sigma}]$ as a composite of lower-dimensional generators by Theorems 7.1 and 7.2, so the generator $[\bar{\sigma}]$ and the associated relations

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are redundant. This shows that the presentation in the statement of the theorem is sufficient to give the structure of $M(w)$.

Now we consider an elementary P-equivalence $f: M(w) \to M(w')$. We must show that the relations in the statement of the theorem imply that $[x] = [f(x)]$ for every molecule x in $M(w)$. We use induction on $\dim x$. Clearly it suffices to take the case in which x is an atom $\bar{\sigma}$. We consider the various cases of Definition 3.6.

First, suppose that $\bar{\sigma}$ is a copy of a member g of G. If $f(\bar{\sigma})$ is a copy of the same member, then we have expressed both $[\bar{\sigma}]$ and $[f(\bar{\sigma})]$ in the form $[g]$, so we get $[\bar{\sigma}] = [f(\bar{\sigma})]$ automatically. If $f(\bar{\sigma})$ is a subdivision w of g, then the necessary relation is $[g] = c(w)$ as included in the statement of the theorem.

Now suppose that $\bar{\sigma}$ is degenerate. Let $\dim \sigma = p$. If $f(\bar{\sigma})$ is a degenerate p-dimensional atom $\bar{\tau}$, then the expressions for $[\bar{\sigma}]$ and $[\bar{\tau}]$ as composites of generators equate $[\bar{\sigma}]$ and $[\bar{\tau}]$ with $[d^-_{p-1}\bar{\sigma}]$ and $[d^-_{p-1}\bar{\tau}]$, so $[\bar{\sigma}] = [\bar{\tau}]$ by the inductive hypothesis. If $\dim f(\bar{\sigma}) < p$ then $f(\bar{\sigma}) = d^-_{p-1} f(\bar{\sigma}) = f(d^-_{p-1}\bar{\sigma})$, so $[\bar{\sigma}] = [f(\bar{\sigma})]$ similarly.

8. A directed complex presentation for an arbitrary ω-category

In this section, we shall prove the following result, which says that every ω-category has a directed complex presentation.

THEOREM 8.1. *If C is an ω-category, then there is a directed complex presentation P such that $C \cong \omega(P)$.*

In the proof of Theorem 8.1 we shall use the subsets $C_0, C_1, \ldots$ of C defined by

$$C_p = \{ x \in C : d^-_p x = d^+_p x = x \}.$$

These subsets have the following properties.

PROPOSITION 8.2. (i) *The sets $C_0, C_1, \ldots$ are sub-ω-categories of C such that $C_0 \subset C_1 \subset C_2 \subset \cdots$ and $C = \bigcup_p C_p$.*

(ii) *If $n \geq p$ then every element of C_p acts as an identity for $\#_n$.*

(iii) *Let S be a subset of C_p, where $p > 0$, and let r be a member of the ω-category generated by $C_{p-1} \cup S$. For $n < p$ the value of $d^\alpha_n r$ is determined by the structure of C_{p-1} and the values of d^α_{p-1} on S.*

PROOF. Parts (i) and (ii) follow straightforwardly from Definition 2.1. As to part (iii), it is clear that $d^\alpha_n r$ is determined by the structure of C_{p-1} and the values of d^α_n on S. But for $s \in S$ we have $d^\alpha_n s = d^\alpha_n d^\alpha_{p-1} s$ with $d^\alpha_{p-1} s \in C_{p-1}$. Therefore $d^\alpha_n r$ is determined by the structure of C_{p-1} and the values of d^α_{p-1} on S, as required.

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PROPOSITION 8.3. (i) *The ω-category C_0 has an algebraic presentation of the following form: for every generator x there are relations $d_0^- x = d_0^+ x = x$; every other relation has the form $x = r$ such that x is a generator and r is a word not involving the composition operators $\#_n$.*

(ii) *As an extension of C_{p-1}, the ω-category C_p (where $p > 0$) has an algebraic presentation of the following form: for every generator x there are relations $d_p^- x = d_p^+ x = x$ and relations equating the $d_{p-1}^\alpha x$ to elements of C_{p-1}; every other relation has the form $x = r$ such that x is a generator and r is a word not involving $\#_n$ for $n \geq p$.*

PROOF. (i) It follows from Proposition 8.2 (ii) that we can construct an algebraic presentation of C_0 without using the operators $\#_n$. In this presentation we can replace every relation $r = r'$ by two relations $x = r$ and $x = r'$, where x is an additional generator. For every generator x (original or additional) we can add a relation $d_0^- x = d_0^+ x = x$. This will produce a presentation of the required form.

(ii) This is similar.

PROOF OF THEOREM 8.1. Because of Proposition 8.2 (i) we can construct a directed complex presentation P for C inductively; that is to say, we construct a directed complex presentation P_0 for C_0, extend it to a presentation P_1 for C_1, and so on, and then take the union of the P_p.

To construct P_0, we use an algebraic presentation for C_0 of the form given in Proposition 8.3 (i). According to Theorem 7.3, it suffices to take a loop-free 0-dimensional atom g_x for each algebraic generator x, and a subdivision w_r of g_x such that w_r represents r for each algebraic relation $x = r$.

For the g_x, we can clearly take directed complexes consisting of a single 0-dimensional element.

Now consider a relation $x = r$. We get r from some generator y by applying operations d_n^α. Since these operations actually leave y unchanged, we can take $w_r = g_y$. It is then clear that w_r is a subdivision of g_x; in fact w_r is isomorphic to g_x.

It remains to construct P_p as an extension of P_{p-1} for $p > 0$. We use an algebraic presentation for C_p as an extension of C_{p-1} of the form given in Proposition 8.3 (ii). According to Theorem 7.3, it suffices to take a loop-free p-dimensional atom g_x such that $d_{p-1}^\alpha g_x$ represents $d_{p-1}^\alpha x$ for each generator x, and a subdivision w_r of g_x such that w_r represents r for each algebraic relation $x = r$.

To construct g_x, we take a fat composite $s_{p-1}^- \#_{p-1}^{\mathbf{s}} s_{p-1}^+$, where $\mathbf{s}$ is an admissible $2p$-tuple of P_{p-1}-molecules such that s_i^α represents $d_i^\alpha x$ (it is clear that there are such $2p$-tuples). By Theorems 5.4 and 5.8, g_x is a loop-free molecular directed complex and

$$g_x = \mathrm{Gl}_{p-1}^{\mathbf{s}}(s_{p-1}^-) \#_{p-1} \bar{\nu}_p(s_{p-1}^-, s_{p-1}^+) \#_{p-1} \mathrm{Gl}_{p-1}^{\mathbf{s}}(s_{p-1}^+),$$

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where $\bar{\nu}_p(s^-_{p-1}, s^+_{p-1})$ is a p-dimensional atom. By Theorem 5.3 (vi), $\dim \mathrm{Gl}^{\mathbf{s}}_{p-1}(s^\alpha_{p-1}) = p - 1$, so g_x is actually a p-dimensional atom. By Theorem 5.9, the molecule $d^\alpha_{p-1} g_x = \mathrm{Gl}^{\mathbf{s}}_{p-1}(s^\alpha_{p-1})$ is a P_{p-1}-molecule, so g_x is permissible for P_p (this being an extension of P_{p-1}). It also follows from Theorem 5.9 that $\mathrm{Gl}^{\mathbf{s}}_{p-1}(s^\alpha_{p-1}) \sim s^\alpha_{p-1}$, so $d^\alpha_{p-1} g_x$ represents $d^\alpha_{p-1} x$.

Now we consider a relation $x = r$. Since r does not involve $\#_n$ for $n \geq p$, the compatibility conditions required for r to exist involve the values of d^α_n only for $n < p$. By Proposition 8.2 (iii), we have sufficient data to construct a P_p-molecule u representing the word r. Since d^-_p acts as the identity on C_p, the P_p-molecule $v = d^-_p u$ also represents r, and we have $\dim v \leq p$. Finally, we let $w_r = \mathrm{Gl}^{\mathbf{s}}_p(v)$, where $g_x = s^-_{p-1} \#^{\mathbf{s}}_{p-1} s^+_{p-1}$. Since s^α_i represents $d^\alpha_i x = d^\alpha_i r$ for $i < p$ (we are working in $\omega(P_{p-1}) \cong C_{p-1}$ here), it follows from Theorem 5.9 that $\mathrm{Gl}^{\mathbf{s}}_p(v)$ is a P_p-molecule.

It remains to show that w_r is a subdivision of the p-dimensional atom g_x. But $\dim v \leq p$, so $\dim w_r = p$ by Theorem 5.3 (vi), and

$$d^-_{p-1} w_r \cup d^+_{p-1} w_r = \mathrm{Gl}^{\mathbf{s}}_{p-1}(s^-_{p-1}) \cup \mathrm{Gl}^{\mathbf{s}}_{p-1}(s^+_{p-1}) = d^-_{p-1} g_x \cup d^+_{p-1} g_x$$

by Theorem 5.3 (v).

REMARK 8.4. In the proof of Theorem 8.1 we have used 0-dimensional molecules, globularisations and fat composites. Now 0-dimensional molecules are obviously totally loop-free, and globularisations and fat composites are totally loop-free if their constituents are totally loop-free (Theorems 5.7 and 5.8). So the proof of Theorem 8.1 actually provides a directed complex presentation in which all the generators and subdivisions are totally loop-free. One could therefore develop the theory of directed complex presentations with totally loop-free molecules throughout.

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