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ON THE IMPLICIT COMPLEMENTARITY PROBLEM IN HILBERT SPACES

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We consider in this paper the implicit complementarity problem imposed by quasi-variational inequalities and stochastic optimal control. The principal result is an existence theorem for the implicit complementarity problem in Hilbert spaces.

1.

We consider in this paper the implicit complementarity problem in infinite dimensional spaces.

Let $<E,E^*>$ be a dual system of locally convex spaces and let $K \subset E$ be a convex cone.

We denote by K^* the dual cone of K, that is, $K^* = \{u \in E^* | <x,u> \geq 0 \ ; \ \forall \ x \in K\}$ and if $f : K \to E^*$, $g : K \to E$ are given, then the implicit complementarity problem with respect to f and g is,

$$\text{(I.C.P.)} \quad \left\| \begin{array}{l} \text{find } x_o \in E \text{ such that} \\ g(x_o) \in K, \ f(x_o) \in K^* \text{ and } <g(x_o),f(x_o)> = 0 \, . \end{array} \right.$$

If $g(x) = x \ ; \ \forall \ x \in K$ we have the complementarity problem (C.P.) (or the explicit complementarity problem (E.C.P.)).

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The extensive literature on complementarity problems is motivated by interesting applications in areas as : Optimization Theory, Structural Mechanics, Lubrication Theory, Elasticity Theory, Economical Equilibrium Theory, Equilibrium Theory on Networks, Mathematical Economics Variational Calculus and Stochastic Optimal Control.

The complementarity problems, till now, have been intensively studied in finite dimensional spaces, but not much in infinite dimensional spaces.

We can find the principal results on complementarity problems in infinite dimensional spaces in, [1], [2], [9], [15], [17-18], [19-21], [22], [25], [28], [31].

The implicit complementarity problem was imposed by some special problems in Stochastic Optimal Control and it was considered by Bensoussan, Lions, Dolcetta, Mosco and Pang in [3-7], [13], [16], [27], [29-30].

It is well known that the theory of explicit complementarity problem is strongly supported by the fixed-point theory.

Recently, in our papers [20-21] we proved, that the theory of coincidence equations on convex cones gives an unified study of the explicit and implicit complementarity problem.

Now, in this paper we consider the implicit complementarity problem in Hilbert spaces and we prove that a generalization of Banach's contraction theorem to coincidences, implies an interesting existence theorem for implicit complementarity problems.

2.

Let $(E, \|\ \|)$ be a Banach space and consider, $D \subset E$ a closed subset.

We say that $G : D \to E$ is a proper mapping if the inverse image of any compact subset of $G(D)$ under G is compact.

The following coincidence theorem is fundamental for our principal result.

THEOREM. [*Coincidence*]

Let $(E, \|\ \|)$ be a Banach space and consider $D \subset E$ a closed subset.

Assume that the continuous mappings $F, G : D \to E$ satisfy the following assumptions:

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1) G is proper,

2) $F(D) \subset G(D)$,

3) there exists a constant $0 < \rho < 1$ such that

$$\|F(x) - F(y)\| \leq \rho \, \| G(x) - G(y)\| \quad , \forall \, x,y \in D.$$

Then, there exists $x_* \in D$ such that $F(x_*) = G(x_*)$.

Moreover, if F or G is one to one then the coincidence point x_* is unique.

Proof. From (2) there exists a sequence $\{x_n\}_{n \in N}$ defined by $G(x_{n+1}) = F(x_n) \; ; \forall \, n \in N$, where x_o is an arbitrary point of D.

By recurrence we have,

$$\| G(x_n) - G(x_{n+1})\| = \| F(x_{n-1}) - F(x_n)\| \leq \rho^n \, \|G(x_o) - G(x_1)\|$$

$\forall \, n \in N$, which implies,

$$\| G(x_n) - G(x_{n+p})\| \leq \frac{\rho^n}{1-\rho} \|G(x_o) - G(x_1)\| \; ; \forall \, n,p = 1,2 \ldots$$

and, hence, $\{G(x_n)\}_{n \in N}$ is a Cauchy sequence in E.

Since E is complete $\{G(x_n)\}_{n \in N}$ is convergent and because G is a proper mapping the sequence $\{x_n\}_{n \in N}$ has a convergent subsequence $\{x_{n_k}\}_{k \in N}$.

If $x_* = \lim_{k \to \infty} x_{n_k}$, then from continuity and the definition of $\{x_n\}_{n \in N}$ we have, $F(x_*) = G(x_*)$.

If x_* and x_{**} are two elements of D, such that, $F(x_*) = G(x_*)$ and $F(x_{**}) = G(x_{**})$ then from the following relations,

$$\|F(x_*) - F(x_{**})\| \leq \rho \, \|G(x_*) - G(x_{**})\| = \rho \, \| F(x_*) - F(x_{**})\| \, ,$$

$$\| G(x_*) - G(x_{**})\| = \| F(x_*) - F(x_{**})\| \leq \rho \, \| G(x_*) - G(x_{**})\| \, ,$$

the assumption that F or G is one to one implies that $x_* = x_{**}$// □

REMARK. A very important class of proper mappings for practical problems is the class of the form, $G = L + \mathbb{C}$, where L is a linear Fredholm operator and $\mathbb{C}$ is a completely continuous nonlinear operator

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in the case where E is a Hilbert space [8].

Consider now, $(H,\langle ,\rangle)$ a Hilbert space and $K \subset H$ a closed convex cone.

If $f,g : K \to H$ are given, consider again the implicit complementarity problem,

$$(\text{I.C.P.}) \quad \left\| \begin{array}{l} \text{find } x_o \in H \text{ such that,} \\ g(x_o) \in K,\ f(x_o) \in K^* \text{ and } \langle g(x_o), f(x_o)\rangle = 0. \end{array} \right.$$

DEFINITIONS. 1) We say that f is *k-Lipschitz with respect to* g if,

$$\|f(x) - f(y)\| \leq k\, \| g(x) - g(y) \| \ ;\ \forall\ x,y \in K\ ;\ (k \in \mathbb{R}_+ \setminus \{0\}).$$

2) The mapping f is called *c-strongly monotone with respect to* g if,

$\langle f(x) - f(y),\ g(x) - g(y)\rangle \geq c\ \|g(x) - g(y)\|^2\ ;\ \forall\ x,y \in K$; where $c \in \mathbb{R}_+\setminus\{0\}$.

THEOREM. 1. *Let* $(H,\langle ,\rangle)$ *be a Hilbert space and* $K \subset H$ *a closed convex cone. If* $f,g : K \to H$ *satisfy the following assumptions:*

1) f *is k-Lipschitz with respect to* g,
2) f *is c-strongly monotone with respect to* g,
3) g *is a continuous proper mapping and* $g(K) = K$,
4) $k^2 < 2c$,

then there is a solution $y_o \in K$ *of problem (I.C.P.).*

Moreover, if g *is one to one on* K, *then the solution* y_o *is unique.*

Proof. Since K is a closed, convex subset in H then for any $y \in K$ there is an unique element $\phi(y) \in K$ such that,

$$(1)\ :\qquad \|\phi(y) - g(y) + f(y)\| \leq \|z - g(y) + f(y)\|\ ;\ \forall\ z \in K.$$

Consider the function $\Psi : [0,1] \to \mathbb{R}_+$ defined by,

$\Psi(\lambda) = \frac{1}{2}\| g(y) - f(y) - (1-\lambda)\, \phi(y) - \lambda z\|^2$, where z is an element in K.

Obviously, Ψ is a C^1-function and we have,

$$\Psi'(\lambda) = \langle g(y) - f(y) - (1-\lambda)\, \phi(y) - \lambda z,\ \phi(y) - z\rangle.$$

Since $\phi(y)$ is the unique element satisfying (1) we have,

$$(2)\ :\qquad \Psi'(0) \geq 0$$

and considering the function Ψ for any $z \in K$, from (2) we obtain,

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(3) : $\langle g(y) - f(y) - \phi(y), \phi(y) - z\rangle \geq 0 : \forall z \in K.$

Consider now, $y_1, y_2 \in K$, so that $y_1 \neq y_2$ and substituting in (3), y by y_1 and z by $\phi(y_2)$ we have,

(4) : $\langle g(y_1) - f(y_1) - \phi(y_1), \phi(y_1) - \phi(y_2)\rangle \geq 0$

and also, substituting $y = y_2$ and $z = \phi(y_1)$, we obtain,

(5) : $\langle g(y_2) - f(y_2) - \phi(y_2), \phi(y_2) - \phi(y_1)\rangle \geq 0.$

By addition from (4) and (5) changing the signs in (5) we get,

$$\langle g(y_1) - f(y_1) - g(y_2) + f(y_2), \phi(y_1) - \phi(y_2)\rangle \geq$$

$$\geq \langle \phi(y_1) - \phi(y_2), \phi(y_1) - \phi(y_2)\rangle = \|\phi(y_1) - \phi(y_2)\|$$

which implies,

$$\|\phi(y_1) - \phi(y_2)\|^2 \leq |\langle g(y_1) - f(y_1) - g(y_2) + f(y_2), \phi(y_1) - \phi(y_2)\rangle| \leq$$

$$\leq \|g(y_1) - f(y_1) - g(y_2) + f(y_2)\| . \|\phi(y_1) - \phi(y_2)\|$$

and finally,

(6) : $\|\phi(y_1) - \phi(y_2)\|^2 \leq \|f(y_1) - f(y_2) - g(y_1) + g(y_2)\|^2.$

From assumptions 1), 2) and formula (6) we obtain,

$$\|\phi(y_1) - \phi(y_2)\|^2 \leq \|f(y_1) - f(y_2)\|^2 + \|g(y_1) - g(y_2)\|^2 -$$

$$-2\langle f(y_1) - f(y_2), g(y_1) - g(y_2)\rangle \leq k^2 \|g(y_1) - g(y_2)\|^2$$

$$+ \|g(y_1) - g(y_2)\|^2 - 2c\|g(y_1) - g(y_2)\|^2 .$$

Hence we have the formula,

(7) : $\|\phi(y_1) - \phi(y_2)\|^2 \leq [k^2 + 1 - 2c]\|g(y_1) - g(y_2)\|^2 .$

Observe now that we can suppose

(8) : $k^2 < 2c < k^2 + 1.$

Indeed, if $2c \geq k^2 + 1$, we can change c to $c_1 > 0$ in definition 2), such that $k^2 < 2c_1 < k^2 + 1$, since if f is c-strongly monotone, it is also c_1-strongly monotone with respect to g.

Hence, assuming formula (8), we put $\beta^2 = k^2 + 1 - 2c$ and we obtain that ϕ is a β-contraction with respect to g (since $0 < \beta < 1$).

Also, ϕ is a continuous mapping (since g is a continuous mapping) and we can use the coincidence theorem for ϕ and g. Thus there is an element $y_0 \in K$ such that $\phi(y_0) = g(y_0)$.

If we now put $y = y_0$ in formula (3) we get,

(9) : $$<f(y_0), z - g(y_0)> \geq 0 \ ; \forall z \in K$$

which implies (since $0 \in K$),

(10) : $$<f(y_0) , g(y_0)> \leq 0 \ .$$

Since K is a cone, $2g(y_0) \in K$ and if we put $z = 2g(y_0)$ in (9) we obtain,

(11) : $$<f(y_0) , g(y_0)> \geq 0$$

and hence, $<f(y_0) , g(y_0)> = 0$.

To finish the first part of the proof, we must prove that $f(y_0) \in K^*$.

Indeed, let $x \in K$ be an arbitrary element and since $g(y_0) \in K$, we have $x + g(y_0) \in K$ and substituting $z = x + g(y_0)$ in formula (9), we obtain $<f(y_0), x> \geq 0 \ ; \forall x \in K$, that is, $f(y_0) \in K^*$.

Assume now that g is one to one on K and consider two solutions y_0 and y_* of problem (I.C.P.).

The definition of problem (I.C.P.) and assumption 2) imply,

$$0 \geq <f(y_0),g(y_0)> - <f(y_*),g(y_0)> - <f(y_0),g(y_*)> + <f(y_*),g(y_*)> =$$
$$= <f(y_0) - f(y_*) , g(y_0) - g(y_*)> \geq c \|g(y_0) - g(y_1)\|^2$$

and hence it is necessary to have $y_0 = y_*$.// □

COROLLARY 1. *Let $(H, <,>)$ be a Hilbert space and $K \subset H$ a closed, convex cone. If $f,g : K \to H$ satisfy the following assumptions:*

1) f is k-Lipschitz,

2) f is c-strongly monotone,

3) $k^2 < 2c$,

then the explicit complementarity problem (E.C.P.) associated with f has a unique solution.

We say that $g : K \to H$ *is expansive* if, there exists $\rho \geq 1$ such that,

$$\rho\|x - y\| \leq \|g(x) - g(y)\| \ ; \forall x,y \in K.$$

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COROLLARY 2. *Let* $(H,<,>)$ *be a Hilbert space and* $K \subset H$ *a closed convex cone. If* $f,g : K \to H$ *satisfy the following assumptions:*

1) g *is a continuous, proper expansion such that* $g(K) = K$,

2) f *is c-strongly monotone with respect to* g *and k-Lipschitz,*

3) $k^2 < 2c$,

then there exists a unique solution of problem (I.C.P.).

Proof. It is sufficient to observe that

$$\|f(x) - f(y)\| \le \frac{k}{\rho}\|g(x) - g(y)\| \le k\ \|g(x) - g(y)\|\ ;\ \forall\ x,y\ \ K.//$$

An interesting case for applications is the case of *accretive mappings*, introduced by Kato [23] and Browder [11] and intensively studied by Browder [10-12], Crandal and Pazy [14], Minty [26], Webb [33], Schöneberg [32], etc.

The study of accretive mappings is motivated by its applications to nonlinear functional equations and to evolution equations.

If $(E,\| \ \|)$ is a Banach space, $D \subset E$ a subset and $h : D \to E$, then h is said to be *accretive*, if and only if,

$$\|x - y\| \le \|(x - y) + \lambda(h(x) - h(y))\|\ , \text{ for all } x,y \in D$$

and $\lambda \ge 0$.

Also, $U : D \to E$ is said to be *pseudo-contractive*, if and only if, for all $x,y \in D$ and all $\lambda > 0$ we have,

$$\|x - y\| \le \|(1 + \lambda)(x - y) - (U(x) - U(y))\|.$$

An interesting study of pseudo-contractive mappings is [24].

A known result of Kato and Browder [11] is that if $g = Id - U$, where $U : E \to E$, then U is pseudo-contrative if and only if g is accretive.

COROLLARY 3. *Let* $(H,<,>)$ *be a Hilbert space and* $K \subset H$ *a closed convex convex cone. If* $f,g : K \to H$ *satisfy the following assumptions:*

1) g *is a continuous, proper mapping such that,* $g(K) = K$,

2) $g-\rho I$ *is accretive for some* $\rho > 0$ *on* K,

3) f *is c-strongly monotone with respect to* g *and k-Lipschitz,*

4) $k^2 < 2c\rho^2$,

then problem (I.C.P.) has a solution y_o.

Moreover, if $\rho \ge 1$, *then the solution* y_o *is unique.*

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Proof. Indeed, in this case we have,

$$\|f(x) - f(y)\| \leq \frac{k}{\rho}\|g(x) - g(y)\| \quad ; \forall\, x,y \in K,$$

since assumption *2)* implies

$$\|x - y\| \leq \|(x - y) + \rho^{-1}(g(x) - \rho x - g(y) + \rho y\| =$$

$$= \rho^{-1}\|g(x) - g(y))\| ,$$

and we have the corollary using theorem 1.

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