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A NEW q-ANALOGUE OF VAN HAMME'S (A.2) SUPERCONGRUENCE

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Abstract

We give a new q-analogue of the (A.2) supercongruence of Van Hamme. Our proof employs Andrews' multiseries generalisation of Watson's ${}_8\phi_7$ transformation, Andrews' terminating q-analogue of Watson's ${}_3F_2$ summation, a q-Watson-type summation due to Wei–Gong–Li and the creative microscoping method, developed by the author and Zudilin ['A q-microscope for supercongruences', *Adv. Math.* **346** (2019), 329–358]. As a conclusion, we confirm a weaker form of Conjecture 4.5 by the author ['Some generalizations of a supercongruence of van Hamme', *Integral Transforms Spec. Funct.* **28** (2017), 888–899].

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1. Introduction

India's great mathematician Ramanujan mentioned the formula

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}=\frac{2}{\Gamma(\frac{3}{4})^4} \tag{1.1}$$

in his second letter to Hardy on February 27, 1913. Here $\Gamma(x)$ stands for the Gamma function and $(a)_k=a(a+1)\cdots(a+k-1)$ is the rising factorial. In 1997, Van Hamme [15] observed that thirteen Ramanujan-type formulae possess neat p-adic analogues. For instance, the formula (1.1) corresponds to the supercongruence

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv\begin{cases}-\dfrac{p}{\Gamma_p(\frac{3}{4})^4} \pmod{p^3} & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^3} & \text{if } p\equiv 3 \pmod 4,\end{cases} \tag{1.2}$$

(tagged as (A.2) in [15]). Here and in what follows, p always denotes an odd prime and $\Gamma_p(x)$ is Morita's p-adic Gamma function (see, for example, [12, Ch. 7]). The supercongruence (1.2) was first confirmed by McCarthy and Osburn [11]. Swisher [13] further proved that (1.2) is true modulo p^5 for $p \equiv 1 \pmod 4$ and $p > 5$. Liu [10] extended (1.2) for $p \equiv 3 \pmod 4$ to a congruence modulo p^4. Recently, among other things, Wei [18] gave the following generalisation of the second case of (1.2):

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) \frac{(\frac{1}{2})_k^5}{k!^5} \equiv p^2 \frac{(\frac{3}{4})_{(p-1)/2}}{(\frac{5}{4})_{(p-1)/2}} \pmod{p^5} \quad \text{for } p \equiv 3 \pmod 4.$$

During the past few years, there has been considerable interest in q-analogues of supercongruences. In particular, using the creative microscoping method introduced by the author and Zudilin [7], Wang and Yue [16], together with the author [5], gave a q-analogue of (1.2): modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{M} (-1)^k [4k+1] \frac{(q;q^2)_k^4 (q^2;q^4)_k}{(q^2;q^2)_k^4 (q^4;q^4)_k} q^k \equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}[n] & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4, \end{cases} \tag{1.3}$$

where $M = (n-1)/2$ or $n-1$. Wei [17, 18] further generalised (1.3) to the moduli $[n]\Phi_n(q)^3$ and $[n]\Phi_n(q)^4$.

We now need to familiarise ourselves with the standard q-notation. The *q-shifted factorial* is defined by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$ and $(a;q)_0 = 1$. For simplicity, we also use the abbreviated notation $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n$ for $n \geqslant 0$. The *q-integer* is $[n] = [n]_q = (1-q^n)/(1-q^n)$. The nth *cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(k,n)=1}} (q - \zeta^k),$$

where ζ is a primitive nth root of unity.

Letting $n = p \equiv 1 \pmod 4$ and taking $q \to 1$ in (1.3), we obtain

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) \frac{(\frac{1}{2})_k^5}{k!^5} \equiv \frac{(\frac{1}{2})_{(p-1)/4}^2}{(1)_{(p-1)/4}^2} p = \binom{-1/2}{(p-1)/4}^2 p \pmod{p^3}. \tag{1.4}$$

From [14, Theorem 3], we know that

$$\binom{-1/2}{(p-1)/4} \equiv \frac{\Gamma_p(\frac{1}{4})^2}{\Gamma_p(\frac{1}{2})} \pmod{p^2}.$$

Since $\Gamma_p(\frac{1}{2})^2 = -1$ for $p \equiv 1 \pmod 4$, by the identity $\Gamma_p(\frac{1}{4})^4 \Gamma_p(\frac{3}{4})^4 = 1$, we see that the supercongruence (1.4) is just (1.2) for $p \equiv 1 \pmod 4$. This implies that (1.3) for $M = (n-1)/2$ really is a q-analogue of the (A.2) supercongruence of Van Hamme.

Note that supercongruences may have different q-analogues. See [8] for such examples. In this note, we shall give the following new q-analogue of (1.2).

THEOREM 1.1. *Let $n > 1$ be an odd integer. Then, modulo $[n]_{q^2}\Phi_n(q^2)^2$,*

$$\sum_{k=0}^{M}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4(q^4;q^8)_k}{(q^4;q^4)_k^4(q^8;q^8)_k}q^{-2k}$$

$$\equiv\begin{cases}-\dfrac{2q(q^4;q^8)_{(n-1)/4}^2}{(1+q^2)(q^8;q^8)_{(n-1)/4}^2}[n]_{q^2} & \text{if } n\equiv 1 \pmod 4,\\[2ex] \dfrac{(1+q)^2(q^4,q^{12};q^8)_{(n-3)/4}}{(1+q^2)(1+q^4)(q^8,q^{16};q^8)_{(n-3)/4}}[n]_{q^2} & \text{if } n\equiv 3 \pmod 4,\end{cases} \tag{1.5}$$

where $M = (n-1)/2$ or $n-1$.

For n prime, letting $q \to -1$ in Theorem 1.1, we get (1.2). However, for n prime and $q \to 1$ in Theorem 1.1, we arrive at

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)^3\frac{(\frac12)_k^5}{k!^5}\equiv\begin{cases}\dfrac{p}{\Gamma_p(\frac34)^4} \pmod{p^3} & \text{if } p\equiv 1 \pmod 4,\\[2ex] \dfrac{(\frac12)_{(p-3)/4}(\frac32)_{(p-3)/4}\,p}{(\frac{p-3}{4})!\,(\frac{p+1}{4})!} \pmod{p^3} & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{1.6}$$

Thus, Theorem 1.1 may be considered as a common q-analogue of (1.2) and (1.6).

Letting n be an odd prime power and $q \to 1$ in (1.3) and (1.5), we are led to the following results. If $p^r \equiv 1 \pmod 4$, then

$$\sum_{k=0}^{(p^r-1)/d}(-1)^k(4k+1)\frac{(\frac12)_k^5}{k!^5}\equiv\binom{(p^r-1)/2}{(p^r-1)/4}^2\frac{p^r}{2^{p^r-1}} \pmod{p^{r+2}}, \tag{1.7}$$

$$\sum_{k=0}^{(p^r-1)/d}(-1)^k(4k+1)^3\frac{(\frac12)_k^5}{k!^5}\equiv-\binom{(p^r-1)/2}{(p^r-1)/4}^2\frac{p^r}{2^{p^r-1}} \pmod{p^{r+2}}, \tag{1.8}$$

where $d = 1$ or 2. Since $4 + 1 + (4k+1)^3 = 2(4k+1)(8k^2+4k+1)$, combining (1.7) and (1.8), we obtain the following conclusion.

COROLLARY 1.2. *If $p^r \equiv 1 \pmod 4$, then*

$$\sum_{k=0}^{(p^r-1)/d}(-1)^k(4k+1)(8k^2+4k+1)\frac{(\frac12)_k^5}{k!^5}\equiv 0 \pmod{p^{r+2}}, \tag{1.9}$$

where $d = 1$ or 2.

Note that the author [4, Conjecture 4.5] conjectured that (1.9) is true modulo p^{3r} for $p \equiv 1 \pmod 4$.

We shall prove Theorem 1.1 in the next section. In Section 3, we raise two related conjectures on supercongruences.

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2. Proof of Theorem 1.1

We first give the following q-congruence. See [6, Lemma 3.1] for a short proof.

LEMMA 2.1. *Let n be a positive odd integer. Then, for $0 \leqslant k \leqslant (n-1)/2$,*

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.$$

We will use a powerful transformation of Andrews (see [1, Theorem 4]), which can be stated as follows:

$$\begin{aligned}
&\sum_{k\geqslant 0}\frac{(a,q\sqrt{a},-q\sqrt{a},b_1,c_1,\ldots,b_m,c_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b_1,aq/c_1,\ldots,aq/b_m,aq/c_m,aq^{N+1};q)_k}\Big(\frac{a^mq^{m+N}}{b_1c_1\cdots b_mc_m}\Big)^k\\
&\quad=\frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N}\sum_{j_1,\ldots,j_{m-1}\geqslant 0}\frac{(aq/b_1c_1;q)_{j_1}\cdots(aq/b_{m-1}c_{m-1};q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}}\\
&\qquad\times\frac{(b_2,c_2;q)_{j_1}\cdots(b_m,c_m;q)_{j_1+\cdots+j_{m-1}}}{(aq/b_1,aq/c_1;q)_{j_1}\cdots(aq/b_{m-1},aq/c_{m-1};q)_{j_1+\cdots+j_{m-1}}}\\
&\qquad\times\frac{(q^{-N};q)_{j_1+\cdots+j_{m-1}}}{(b_mc_mq^{-N}/a;q)_{j_1+\cdots+j_{m-1}}}\frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1}q^{j_1+\cdots+j_{m-1}}}{(b_2c_2)^{j_1}\cdots(b_{m-1}c_{m-1})^{j_1+\cdots+j_{m-2}}}.
\end{aligned}\tag{2.1}$$

It should be pointed out that Andrews' transformation is a multiseries generalisation of Watson's ${}_8\phi_7$ transformation:

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e, & q^{-n}\\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix};q,\ \frac{a^2q^{n+2}}{bcde}\right]\\
&\quad=\frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,\ d,\ e,\ q^{-n}\\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix};q,\ q\right]
\end{aligned}$$

(see [3, Appendix (III.18)]), where the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_r\end{matrix};q,\ z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_k}{(q,b_1,\ldots,b_r;q)_k}z^k.$$

We shall also use Andrews' terminating q-analogue of Watson's ${}_3F_2$ summation (see [2] or [3, (II.17)]):

$${}_4\phi_3\left[\begin{matrix}q^{-n},\ a^2q^{n+1},\ c,\ -c\\ aq,\ -aq,\ c^2\end{matrix};q,q\right]=\begin{cases}0 & \text{if } n \text{ is odd},\\ \dfrac{c^n(q,a^2q^2/c^2;q^2)_{n/2}}{(a^2q^2,\ c^2q;q^2)_{n/2}} & \text{if } n \text{ is even},\end{cases}\tag{2.2}$$

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and the following q-Watson-type summation due to Wei *et al.* [19, Corollary 5]:

$$
{}_4\phi_3\left[\begin{matrix} q^{-n},\ a^2q^{n+1},\ c,\ -cq \\ aq,\ -aq,\ c^2q \end{matrix}; q, q\right] = \begin{cases} \dfrac{c^n(q;q^2)_{(n+1)/2}(a^2q^2/c^2;q^2)_{(n-1)/2}}{(a^2q^2;q^2)_{(n-1)/2}(c^2q;q^2)_{(n+1)/2}} & \text{if } n \text{ is odd,} \\ \dfrac{c^n(q,a^2q^2/c^2;q^2)_{n/2}}{(a^2q^2,\ c^2q;q^2)_{n/2}} & \text{if } n \text{ is even.} \end{cases} \tag{2.3}
$$

We first prove the following parametric version of Theorem 1.1.

THEOREM 2.2. *Let $n>1$ be an odd integer. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$
\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(aq^2,q^2/a;q^4)_k(q^2;q^4)_k^2(q^4;q^8)_k}{(aq^4,q^4/a;q^4)_k(q^4;q^4)_k^2(q^8;q^8)_k}q^{-2k}
$$
$$
\equiv \begin{cases} \Big(1-\dfrac{(1+q)(1-aq^2)(1-q^2/a)}{(1-q)(1-q^4)}\Big)\dfrac{(q^4;q^8)^2_{(n-1)/4}}{(q^8;q^8)^2_{(n-1)/4}}[n]_{q^2} & \textit{if } n\equiv 1 \pmod 4, \\ \dfrac{(1+q)(1-aq^2)(1-q^2/a)(q^4,q^{12};q^8)_{(n-3)/4}}{(1-q)(1-q^8)(q^8,q^{16};q^8)_{(n-3)/4}}[n]_{q^2} & \textit{if } n\equiv 3 \pmod 4. \end{cases} \tag{2.4}
$$

PROOF. For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (2.4) may be written as

$$
\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^{2-2n},q^{2+2n};q^4)_k(q^2;q^4)_k^2(q^4;q^8)_k}{(q^{4-2n},q^{4+2n};q^4)_k(q^4;q^4)_k^2(q^8;q^8)_k}q^{-2k}.
$$

Letting $m=3$, $q\mapsto q^4$, $a=q^2$, $b_1=c_1=q^5$, $b_2=c_2=q^2$, $b_3=-q^2$, $c_3=q^{2+2n}$ and $N=(n-1)/2$ in (2.1), we see that the above summation is equal to

$$
\begin{aligned}
&\frac{(q^6,-q^{2-2n};q^4)_{(n-1)/2}}{(-q^4,q^{4-2n};q^4)_{(n-1)/2}} \\
&\quad\times\sum_{j_1,j_2\geqslant 0}\frac{(q^{-4};q^4)_{j_1}(q^2;q^4)_{j_2}(q^2,q^2;q^4)_{j_1}(-q^2,q^{2+2n},q^{2-2n};q^4)_{j_1+j_2}}{(q^4;q^4)_{j_1}(q^4;q^4)_{j_2}(q,q;q^4)_{j_1}(q^4,q^4,-q^4;q^4)_{j_1+j_2}}q^{6j_1+4j_2} \\
&=(-1)^{(n-1)/2}q^{1-n}[n]_{q^2}\sum_{j_2=0}^{(n-1)/2}\frac{(q^2,-q^2,q^{2+2n},q^{2-2n};q^4)_{j_2}}{(q^4,q^4,q^4,-q^4;q^4)_{j_2}}q^{4j_2} \\
&\quad+(-1)^{(n+1)/2}q^{3-n}[n]_{q^2}(1+q)^2\sum_{j_2=0}^{(n-3)/2}\frac{(q^2;q^4)_{j_2}(-q^2,q^{2+2n},q^{2-2n};q^4)_{j_2+1}}{(q^4;q^4)j_2(q^4,q^4,-q^4;q^4)_{j_2+1}}q^{4j_2},
\end{aligned} \tag{2.5}
$$

where we have used the fact that $(q^{-4};q^4)_{j_1}=0$ for $j_1>1$.

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Taking $q \mapsto q^4$, $a = 1$, $c = q^2$ and $n \mapsto (n-1)/2$ in (2.2), we have

$$\sum_{j_2=0}^{(n-1)/2} \frac{(q^2, -q^2, q^{2+2n}, q^{2-2n}; q^4)_{j_2}}{(q^4, q^4, q^4, -q^4; q^4)_{j_2}} q^{4j_2} = \begin{cases} q^{n-1} \dfrac{(q^4; q^8)^2_{(n-1)/4}}{(q^8; q^8)^2_{(n-1)/4}} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Similarly, taking $q \mapsto q^4$, $a = q^4$, $c = q^2$ and $n \mapsto (n-3)/2$ in (2.3), we get

$$\begin{aligned} &\sum_{j_2=0}^{(n-3)/2} \frac{(q^2; q^4)_{j_2} (-q^2, q^{2+2n}, q^{2-2n}; q^4)_{j_2+1}}{(q^4; q^4) j_2 (q^4, q^4, -q^4; q^4)_{j_2+1}} q^{4j_2} \\ &= \frac{(1+q^2)(1-q^{2+2n})(1-q^{2-2n})}{(1-q^4)^2(1+q^4)} \sum_{j_2=0}^{(n-3)/2} \frac{(q^2, -q^6, q^{6+2n}, q^{6-2n}; q^4)_{j_2}}{(q^4, q^8, q^8, -q^8; q^4)_{j_2}} q^{4j_2} \\ &= \begin{cases} q^{n-3} \dfrac{(1+q^2)(1-q^{2+2n})(1-q^{2-2n})(q^4; q^8)^2_{(n-1)/4}}{(1-q^4)^2 (q^8; q^8)^2_{(n-1)/4}} & \text{if } n \equiv 1 \pmod 4, \\ q^{n-3} \dfrac{(1+q^2)(1-q^{2+2n})(1-q^{2-2n})(q^4, q^{12}; q^8)_{(n-3)/4}}{(1-q^4)^2(1+q^4)(q^8, q^{16}; q^8)_{(n-3)/4}} & \text{if } n \equiv 3 \pmod 4. \end{cases} \end{aligned}$$

Substituting the above two identities into (2.5), we obtain

$$\begin{aligned} &\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1]_{q^2} [4k+1]^2 \frac{(q^{2-2n}, q^{2+2n}; q^4)_k (q^2; q^4)_k^2 (q^4; q^8)_k}{(q^{4-2n}, q^{4+2n}; q^4)_k (q^4; q^4)_k^2 (q^8; q^8)_k} q^{-2k} \\ &= \begin{cases} \left(1 - \dfrac{(1+q)(1-q^{2+2n})(1-q^{2-2n})}{(1-q)(1-q^4)}\right) \dfrac{(q^4; q^8)^2_{(n-1)/4}}{(q^8; q^8)^2_{(n-1)/4}} [n]_{q^2} & \text{if } n \equiv 1 \pmod 4, \\ \dfrac{(1+q)(1-q^{2+2n})(1-q^{2-2n})(q^4, q^{12}; q^8)_{(n-3)/4}}{(1-q)(1-q^8)(q^8, q^{16}; q^8)_{(n-3)/4}} [n]_{q^2} & \text{if } n \equiv 3 \pmod 4. \end{cases} \end{aligned}$$

This proves that both sides of (2.4) are equal when $a = q^{\pm 2n}$. Namely, the q-congruence (2.4) holds modulo $1 - aq^{2n}$ or $a - q^{2n}$.

Moreover, in view of Lemma 2.1, we can verify that the kth and $((n-1)/2 - k)$th summands cancel each other modulo $\Phi_n(q^2)$ for any positive odd integer n. It follows that

$$\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1]_{q^2} [4k+1]^2 \frac{(aq^2, q^2/a; q^4)_k (q^2; q^4)_k^2 (q^4; q^8)_k}{(aq^4, q^4/a; q^4)_k (q^4; q^4)_k^2 (q^8; q^8)_k} q^{-2k} \equiv 0 \pmod{\Phi_n(q^2)}. \tag{2.6}$$

Noticing that $[n]_{q^2} \equiv 0 \pmod{\Phi_n(q^2)}$ for $n > 1$, we conclude that the q-congruence (2.4) also holds modulo $\Phi_n(q)$.

Since $1 - aq^{2n}$, $a - q^{2n}$ and $\Phi_n(q^2)$ are pairwise relatively prime polynomials in q, we complete the proof of the theorem. □

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Proof of Theorem 1.1. It is easy to see that the denominators on both sides of (2.4) when $a=1$ are relatively prime to $\Phi_n(q^2)$. However, when $a=1$, the polynomial $(1-aq^{2n})(a-q^{2n})$ contains the factor $\Phi_n(q^2)^2$. Therefore, the $a=1$ case of (2.4) implies that (1.5) is true modulo $\Phi_n(q^2)^3$ for $M=(n-1)/2$. Furthermore, since $(q^2;q^4)_k^4(q^4;q^8)_k/((q^4;q^4)_k^4(q^8;q^8)_k)\equiv 0 \pmod{\Phi_n(q^2)^5}$ for $(n-1)/2<k\leqslant n-1$, we see that (1.5) is also true modulo $\Phi_n(q^2)^3$ for $M=n-1$.

It remains to prove the following two q-congruences:

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4(q^4;q^8)_k}{(q^4;q^4)_k^4(q^8;q^8)_k}q^{-2k}\equiv 0 \pmod{[n]_{q^2}}, \tag{2.7}$$

$$\sum_{k=0}^{n-1}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4(q^4;q^8)_k}{(q^4;q^4)_k^4(q^8;q^8)_k}q^{-2k}\equiv 0 \pmod{[n]_{q^2}}. \tag{2.8}$$

For $n>1$, let $\zeta\neq 1$ be an nth root of unity, possibly not primitive. Suppose ζ is a primitive root of unity of odd degree d satisfying $d\mid n$. Let $c_q(k)$ be the kth term on the left-hand side of the congruences (2.7) and (2.8). Then

$$c_q(k)=(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4(q^4;q^8)_k}{(q^4;q^4)_k^4(q^8;q^8)_k}q^{-2k}.$$

Observe that (2.6) is true for any odd $n>1$. Thus, letting $a=1$ and $n=d$ in (2.6), we obtain

$$\sum_{k=0}^{(d-1)/2}c_\zeta(k)=\sum_{k=0}^{d-1}c_\zeta(k)=0 \quad\text{and}\quad \sum_{k=0}^{(d-1)/2}c_{-\zeta}(k)=\sum_{k=0}^{d-1}c_{-\zeta}(k)=0.$$

Noticing that

$$\frac{c_\zeta(\ell d+k)}{c_\zeta(\ell d)}=\lim_{q\to\zeta}\frac{c_q(\ell d+k)}{c_q(\ell d)}=c_\zeta(k),$$

we have

$$\sum_{k=0}^{n-1}c_\zeta(k)=\sum_{\ell=0}^{n/d-1}\sum_{k=0}^{d-1}c_\zeta(\ell d+k)=\sum_{\ell=0}^{n/d-1}c_\zeta(\ell d)\sum_{k=0}^{d-1}c_\zeta(k)=0,$$

and

$$\sum_{k=0}^{(n-1)/2}c_\zeta(k)=\sum_{\ell=0}^{(n/d-3)/2}c_\zeta(\ell d)\sum_{k=0}^{d-1}c_\zeta(k)+\sum_{k=0}^{(d-1)/2}c_\zeta((n-d)/2+k)=0.$$

This means that both the sums $\sum_{k=0}^{n-1}c_q(k)$ and $\sum_{k=0}^{(n-1)/2}c_q(k)$ are divisible by $\Phi_d(q)$. Similarly, we can show that they are also divisible by $\Phi_d(-q)$. Since d can be any

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divisor of n larger than 1, we deduce that each of them is congruent to 0 modulo

$$\prod_{d|n,\, d>1} \Phi_d(q)\Phi_d(-q) = [n]_{q^2},$$

thus establishing (2.7) and (2.8). $\square$

3. Two open problems

Swisher [13] proposed many interesting conjectures on generalisations of Van Hamme's supercongruences (A.2)–(L.2). Recently, the author and Zudilin [9] have proved some conjectures of Swisher by establishing their q-analogues. Here we would like to propose a similar conjecture.

CONJECTURE 3.1. Let $p \equiv 1 \pmod 4$ and let $r, s \geqslant 1$. Then

$$\sum_{k=0}^{(p^r-1)/d} (-1)^k (4k+1)^{2s+1} \frac{(\frac{1}{2})_k^5}{k!^5} \equiv -p\Gamma_p(\tfrac{1}{4})^4 \sum_{k=0}^{(p^{r-1}-1)/d} (-1)^k (4k+1)^{2s+1} \frac{(\frac{1}{2})_k^5}{k!^5} \pmod{p^{3r-2}}, \tag{3.1}$$

where $d = 1$ or 2.

For $s = 0$, Swisher [13, (A.3)] and the author [5, Conjecture 4.1] conjectured that (3.1) holds modulo p^{5r} for $p > 5$. From (1.8), we can easily see that (3.1) is true modulo p^r for $s = 1$.

Finally, motivated by [4, Conjecture 4.5], we believe that the following generalisation of Corollary 1.2 for p of the form $4k + 3$ should be true.

CONJECTURE 3.2. Let $p \equiv 3 \pmod 4$ and let $r \geqslant 2$ be even. Then

$$\sum_{k=0}^{(p^r-1)/d} (-1)^k (4k+1)(8k^2+4k+1) \frac{(\frac{1}{2})_k^5}{k!^5} \equiv 0 \pmod{p^{2r}},$$

where $d = 1$ or 2.

References

[1] G. E. Andrews, 'Problems and prospects for basic hypergeometric functions', in *Theory and Application for Basic Hypergeometric Functions*, Mathematics Research Center, the University of Wisconsin–Madison, 35 (ed. R. A. Askey) (Academic Press, New York, 1975), 191–224.

[2] G. E. Andrews, 'On q-analogues of the Watson and Whipple summations', *SIAM J. Math. Anal.* **7** (1976), 332–336.

[3] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd edn, Encyclopedia of Mathematics and Its Applications, 96 (Cambridge University Press, Cambridge, 2004).

[4] V. J. W. Guo, 'Some generalizations of a supercongruence of Van Hamme', *Integral Transforms Spec. Funct.* **28** (2017), 888–899.

[5] V. J. W. Guo, 'A q-analogue of the (A.2) supercongruence of Van Hamme for primes $p \equiv 1 \pmod 4$', *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **114** (2020), Article no. 123.

https://doi.org/10.1017/S0004972722000478 Published online by Cambridge University Press

[6] V. J. W. Guo and M. J. Schlosser, 'Some q-supercongruences from transformation formulas for basic hypergeometric series', *Constr. Approx.* **53** (2021), 155–200.
[7] V. J. W. Guo and W. Zudilin, 'A q-microscope for supercongruences', *Adv. Math.* **346** (2019), 329–358.
[8] V. J. W. Guo and W. Zudilin, 'A common q-analogue of two supercongruences', *Results Math.* **75** (2020), Article no. 46.
[9] V. J. W. Guo and W. Zudilin, 'A q-microscope for supercongruences', *Adv. Math.* **346** (2019), 329–358.
[10] J.-C. Liu, 'On Van Hamme's (A.2) and (H.2) supercongruences', *J. Math. Anal. Appl.* **471** (2019), 613–622.
[11] D. McCarthy and R. Osburn, 'A p-adic analogue of a formula of Ramanujan', *Arch. Math. (Basel)* **91** (2008), 492–504.
[12] A. M. Robert, *A Course in p-Adic Analysis*, Graduate Texts in Mathematics, 198 (Springer-Verlag, New York, 2000).
[13] H. Swisher, 'On the supercongruence conjectures of Van Hamme', *Res. Math. Sci.* **2** (2015), Article no. 18.
[14] L. Van Hamme, 'Proof of a conjecture of Beukers on Apéry numbers', in *Proceedings of the Conference on p-Adic Analysis, Houthalen, 1987* (eds. N. De Grande-De Kimpe and L. Van Hamme) (Vrije Universiteit Brussel, Brussels, 1986), 189–195.
[15] L. Van Hamme, 'Some conjectures concerning partial sums of generalized hypergeometric series', in *p-Adic Functional Analysis, Nijmegen, 1996*, Lecture Notes in Pure and Applied Mathematics, 192 (eds. W. H. Schikhof, C. Perez-Garcia and J. Kakol) (Dekker, New York, 1997), 223–236.
[16] X. Wang and M. Yue, 'A q-analogue of the (A.2) supercongruence of Van Hamme for any prime $p \equiv 3 \pmod 4$', *Int. J. Number Theory* **16** (2020), 1325–1335.
[17] C. Wei, 'Some q-supercongruences modulo the fourth power of a cyclotomic polynomia', *J. Combin. Theory Ser. A* **182** (2021), Article no. 105469.
[18] C. Wei, 'Some q-supercongruences modulo the fifth and sixth powers of a cyclotomic polynomial', Preprint, 2021, arXiv:2104.07025.
[19] C. Wei, D. Gong and J. Li, 'Summation formulae for q-Watson type ${}_4\phi_3$-series', *Discrete Math.* **313** (2013), 1589–1593.

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