

ON A NEW PARADIGM OF OPTIMAL REINSURANCE: A STOCHASTIC STACKELBERG DIFFERENTIAL GAME BETWEEN AN INSURER AND A REINSURER

BY

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ABSTRACT

This paper proposes a new continuous-time framework to analyze optimal reinsurance, in which an insurer and a reinsurer are two players of a stochastic Stackelberg differential game, i.e., a stochastic leader-follower differential game. This allows us to determine optimal reinsurance from joint interests of the insurer and the reinsurer, which is rarely considered in the continuous-time setting. In the Stackelberg game, the reinsurer moves first and the insurer does subsequently to achieve a Stackelberg equilibrium toward optimal reinsurance arrangement. Speaking more precisely, the reinsurer is the leader of the game and decides on an optimal reinsurance premium to charge, while the insurer is the follower of the game and chooses an optimal proportional reinsurance to purchase. Under utility maximization criteria, we study the game problem starting from the general setting with generic utilities and random coefficients to the special case with exponential utilities and constant coefficients. In the special case, we find that the reinsurer applies the variance premium principle to calculate the optimal reinsurance premium and the insurer's optimal ceding/retained proportion of insurance risk depends not only on the risk aversion of itself but also on that of the reinsurer.

KEYWORDS

Stackelberg game, proportional reinsurance, stochastic Hamilton-Jacobi-Bellman equation, backward stochastic differential equation, variance premium principle.

1. INTRODUCTION

The nature of insurance is to pool risk from a large number of policyholders. By doing this, an insurer receives insurance premiums from the policyholders and promises to cover policyholders' claims. In the meantime, due to solvency

capital requirement and business goal, the insurer usually needs to cede extra insurance risk from the balance sheet to one or multiple reinsurers. Thus, how to transfer insurance risk through reinsurance agreement is an important theme on the insurer's agenda. With massive insurance premiums on hand, another important task of the insurer is to generate investment income from these premiums. Therefore, research problems on optimal investment and reinsurance from the perspective of insurers have been receiving considerable attention. There is a vast literature on optimal reinsurance in the continuous-time setting. Højgaard and Taksar (1998) found the optimal proportional policy to maximize a discounted return function. Under the criterion of minimizing the ruin probability, Promislow and Young (2005) studied an optimal investment and reinsurance problem. See also Yang and Zhang (2005) for the insurer's investment only problems. Further investigations to optimal investment and reinsurance problems can be found in Zhang and Siu (2009), Zeng and Li (2011), Lin *et al.* (2012), Chen and Yam (2013), Shen and Zeng (2014, 2015) and Zhao *et al.* (2016), just to name a few.

In the aforementioned works, optimal reinsurance problems are treated from the insurer's point of view. In literature, little attention has been paid to determining optimal reinsurance from the reinsurer's point of view or joint interests of the insurer and the reinsurer. However, since any reinsurance policy is obviously a mutual agreement between two parties—the insurer and the reinsurer, optimal reinsurance constructed with only one party is somewhat unreasonable. In reality, an optimal reinsurance treaty obtained by taking into account the interest of only one party may be unacceptable to the other party. In the discrete-time single-period setting, Cai *et al.* (2016) studied optimal reinsurance designs from the perspectives of both the insurer and the reinsurer with a view to minimizing the corresponding VaR risk measures. Although optimal reinsurance with joint interests of the insurer and the reinsurer has been discussed in discrete-time single-period models under different criteria (refer to Cai *et al.* (2016) and the references therein for a survey), formal investigation to the question of optimal reinsurance designs in the dynamic setting is rare. The objective of this paper is to provide an answer to the question in a continuous-time model. To be more specific, this paper proposes a new paradigm to analyze optimal reinsurance from the perspectives of an insurer and a reinsurer in a stochastic differential game setting.

Optimal reinsurance has already been studied in the context of stochastic differential games, where two or more insurers act as players of the games. This direction of research follows largely the original idea of the stochastic differential portfolio game proposed by Browne (2000). For instance, Zeng (2010) and Taksar and Zeng (2011) studied zero-sum stochastic differential games between two insurance companies that compete with each other on their surplus processes and apply reinsurance to reduce risk exposure; Jin *et al.* (2013) discussed the case for regime-switching jump-diffusion models and developed a Markov chain approximation method to solve the game problem numerically; Bensoussan *et al.* (2014) considered a non-zero-sum stochastic differential game under

the regime-switching diffusion approximation model and obtained explicit solutions for constant absolute risk aversion (CARA) insurers; Meng *et al.* (2015) solved a reinsurance game between two insurance companies with nonlinear surplus processes, which contain quadratic control terms (i.e., retained proportions). Though the insurer and the reinsurer obviously play interactive roles in reinsurance agreements, attention has been seldom paid to stochastic differential games involving both parties. This induces us to contemplate how the insurer and the reinsurer could be included in a new game theoretic framework, which reflects both parties' interests in achieving an optimal agreement toward reinsurance arrangement.

While hundreds of thousands of insurance companies co-exist in insurance markets across different countries, the global reinsurance business is monopolized by several giant reinsurance companies, such as Munich Re, Swiss Re, Hannover Re and China Re. This fact reminds us if ceding insurance risk from an insurer to a reinsurer is regarded as a game, the unequal power of the two parties in the insurance market reveals that they would have unequal dominance over the game. In this regard, it is natural to put the insurer and the reinsurer in the framework of a Stackelberg game (i.e., a leader-follower game) and assume that the reinsurer and the insurer are the leader and the follower of the game, respectively. The Stackelberg game was introduced by and named after the German economist Heinrich von Stackelberg (refer to von Stackelberg (1934)). It is described as a strategic game model in which the leading player moves first and then the following player(s) move sequentially. The stochastic differential form of Stackelberg games was considered by Yong (2002), Bensoussan *et al.* (2015) and Shi *et al.* (2016), which discussed the games from a theoretical perspective. On the other hand, the framework of stochastic Stackelberg differential games has found extensive applications in supply chain management. For instance, one may refer to He *et al.* (2009), Chutani and Sethi (2012) and Øksendal *et al.* (2013). Owing to the leadership and followership roles played by reinsurers and insurers, the stochastic Stackelberg differential game would be an ideal framework to model the activity of transferring insurance risk through reinsurance contracts.

This paper aims to study the optimal reinsurance problem in a stochastic Stackelberg differential game, where the insurer and the reinsurer are two players of the game. As a starting point, we concentrate on proportional reinsurance treaties and assume that the objectives of the insurer and the reinsurer are to maximize their respective expected utilities of terminal surpluses. We describe the insurer's and the reinsurer's surplus processes by diffusion approximation models, the idea of which originates from Grandell (1990). In terms of surplus models and utility functions, our paper proceeds from general cases to special ones. Specifically speaking, we begin our investigation by modeling surplus processes with extended diffusion approximation models whose coefficients are given by general adapted processes. This extension makes our problem not only more theoretically interesting but also of greater practical values. Unlike some existing works on optimal investment and reinsurance, we focus on

determining optimal reinsurance and leave the consideration of investment opportunities to the future research. In the general setting, we discuss the problem via the stochastic Hamilton–Jacobi–Bellman (HJB) equation approach. When the insurer’s and reinsurer’s preferences are specified by exponential utilities, we represent optimal strategies and value functions of the insurer and the reinsurer by unique solutions to two quadratic backward stochastic differential equations (BSDEs). Indeed, in this step we obtain unique solutions to stochastic HJB equations, which are seldom solved explicitly in the literature. To do this, we apply a small amount of the general theory of stochastic processes, including the martingale representation theorem and the Doob–Meyer decomposition theorem. Finally, when model coefficients are deterministic, we manage to solve the BSDEs explicitly, and obtain closed-form solutions to the problem. In the special case with exponential utilities and constant coefficients, an interesting finding is that the reinsurer achieves optimality by charging variance reinsurance premium. This complements the well-known result in the literature: given the variance reinsurance premium principle, the optimal reinsurance among all reinsurance type for the insurer is the proportional reinsurance. It is shown in our numerical examples that the reinsurer’s characteristics have significant impacts on optimal strategies; the insurer and the reinsurer affect optimal solutions to the game in different patterns. This supports our intuition that ignoring the reinsurer in previous research may make the obtained “optimal” reinsurance and reinsurance premium unreasonable and inaccurate.

The remainder of the paper is organized as follows. Section 2 sets up the model dynamics and introduces basic notations. In Section 3, we adopt the stochastic HJB equation approach to study the game problem in the setting with general utility functions and random parameters. In Section 4, we apply BSDEs to derive optimal strategies and value functions, when the insurer’s and reinsurer’s preferences are specified by exponential utility functions. Section 5 provides closed-form solutions to the Stackelberg game for the case with exponential utilities and constant parameters. In Section 6, we illustrate the results of Section 5 by numerical examples. The final section concludes the paper with some remarks. In Appendix A, we provide an alternative approach, i.e., the HJB equation approach, to solve the problem with exponential utilities and constant coefficients. In Appendix B, we discuss the game problem for the case of power utilities.

2. STATEMENT OF PROBLEM

In this section, we first introduce some basic notations, including various Banach spaces on a probability space, which are commonly used in the literature of stochastic HJB equations and BSDEs. Then, we present the dynamics of surplus processes of an insurer and a reinsurer, and formulate an optimal reinsurance problem as a stochastic Stackelberg differential game.

To begin with, we fix a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ indexed by a finite time horizon $[0, T]$, where $\mathbb{F} := \{\mathcal{F}_t | t \in [0, T]\}$ is a right-continuous, \mathbb{P} -complete filtration generated by a one-dimensional standard Brownian motion $\{W(t) | t \in [0, T]\}$. We denote by $\mathbb{E}[\cdot]$ an expectation taken under \mathbb{P} and $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$ a conditional expectation given $\mathcal{F}(t)$ under \mathbb{P} . Let $k, K > 0$ and $m \geq 1$ be generic constants, n be a generic natural number, and $\varphi(\cdot) := \{\varphi(t) | t \in [0, T]\}$ be a generic process, which may be different from line to line. Let $|\cdot|$ denote the Euclidean norm on \mathbb{R}^n . Throughout the paper, we denote $f_t(t, x) := \frac{\partial f(t, x)}{\partial t}$, $f_x(t, x) := \frac{\partial f(t, x)}{\partial x}$ and $f_{xx}(t, x) := \frac{\partial^2 f(t, x)}{\partial x^2}$ as corresponding partial derivatives for any differentiable function $f(t, x)$.

For later use, we define the following Banach spaces on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

- $\mathcal{S}_{\mathbb{F}, \mathbb{P}}^m(0, T; \mathbb{R}^n)$: the space of \mathbb{R}^n -valued, continuous, \mathbb{F} -adapted processes $\varphi(\cdot)$ such that

$$\|\varphi(\cdot)\|_{\mathcal{S}_{\mathbb{F}, \mathbb{P}}^m(0, T; \mathbb{R}^n)} := \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |\varphi(t)|^m \right] \right\}^{\frac{1}{m}} < \infty;$$

- $\mathcal{S}_{\mathbb{F}, \mathbb{P}}^\infty(0, T; \mathbb{R}^n)$: the space of \mathbb{R}^n -valued, continuous, essentially bounded, \mathbb{F} -adapted processes under \mathbb{P} ;
- $\mathcal{L}_{\mathcal{F}_t, \mathbb{P}}^m(\mathbb{R}^n)$: the space of \mathbb{R}^n -valued, \mathcal{F}_t -measurable random variables ξ such that

$$\|\xi\|_{\mathcal{L}_{\mathcal{F}_t, \mathbb{P}}^m(\mathbb{R}^n)} := \left\{ \mathbb{E}[|\xi|^m] \right\}^{\frac{1}{m}} < \infty;$$

- $\mathcal{L}_{\mathcal{F}_t, \mathbb{P}}^\infty(\mathbb{R}^n)$: the space of \mathbb{R}^n -valued, essentially bounded, \mathcal{F}_t -measurable random variables under \mathbb{P} ;
- $\mathcal{L}_{\mathbb{F}, \mathbb{P}}^m(0, T; \mathbb{R}^n)$: the space of \mathbb{R}^n -valued, \mathbb{F} -adapted processes $\varphi(\cdot)$ such that

$$\|\varphi(\cdot)\|_{\mathcal{L}_{\mathbb{F}, \mathbb{P}}^m(0, T; \mathbb{R}^n)} := \left\{ \mathbb{E} \left[\left(\int_0^T |\varphi(t)|^2 dt \right)^{\frac{m}{2}} \right] \right\}^{\frac{1}{m}} < \infty;$$

- $\mathcal{L}_{\mathbb{F}, \mathbb{P}}^{m, \text{loc}}(0, T; \mathbb{R}^n)$: the space of \mathbb{R}^n -valued, \mathbb{F} -adapted processes $\varphi(\cdot)$ such that

$$\mathbb{P} \left(\int_0^T |\varphi(t)|^m dt < \infty \right) = 1;$$

- $\text{BMO}_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R})$: the space of real-valued, bounded mean oscillation (BMO) martingales on (\mathbb{F}, \mathbb{P}) , i.e., (\mathbb{F}, \mathbb{P}) -martingales $\varphi(\cdot)$ satisfying

$$\|\varphi(\cdot)\|_{\text{BMO}_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R})} := \sup_{\tau \in \mathcal{T}} \left\| \left\{ \mathbb{E}[|\varphi(T) - \varphi(\tau)|^2 | \mathcal{F}_\tau] \right\}^{\frac{1}{2}} \right\|_\infty < \infty,$$

where \mathcal{T} denotes the set of all \mathbb{F} -stopping times on $[0, T]$ and $\|\cdot\|_\infty$ is a shorthand notation for the essential supremum under the probability measure \mathbb{P} ;

- $\mathcal{H}_{\mathbb{F}, \mathbb{P}}^{2, \text{BMO}}(0, T; \mathbb{R})$: the space of real-valued, \mathbb{F} -adapted processes $\varphi(\cdot)$ such that

$$\|\varphi(\cdot)\|_{\mathcal{H}_{\mathbb{F}, \mathbb{P}}^{2, \text{BMO}}(0, T; \mathbb{R})} := \left\| \int_0^\cdot \varphi(s) dW(s) \right\|_{\text{BMO}_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R})} < \infty.$$

We next generalize the diffusion approximation model proposed by Grandell (1990) to describe the insurance risk. In the classical risk model, insurance claims are modeled by the compound Poisson process. Grandell (1990) proposed to approximate the claim process $C(\cdot) := \{C(t) | t \in [0, T]\}$ by a diffusion-type model as follows:

$$dC(t) = a dt - \sigma dW(t), \tag{2.1}$$

where the two constants $a > 0$ and $\sigma > 0$ are the mean and volatility coefficients of the insurance risk model, respectively. Under the expected value premium principle, the insurance premium is then determined by

$$c := (1 + \theta)a, \tag{2.2}$$

where the constant $\theta > 0$ denotes a relative security loading of the insurer. Therefore, the insurer’s surplus process $R(\cdot) := \{R(t) | t \in [0, T]\}$ is governed by a Brownian motion with drift:

$$\begin{aligned} dR(t) &= c dt - dC(t) \\ &= \theta a dt + \sigma dW(t). \end{aligned} \tag{2.3}$$

Now, we extend the diffusion approximation model (2.3) to one with time-varying and random coefficients. To be more precise, we assume that the claim process $C(\cdot) = \{C(t) | t \in [0, T]\}$ is governed by a generalized diffusion-type model:

$$dC(t) = a(t) dt - \sigma(t) dW(t), \tag{2.4}$$

where $a(\cdot) := \{a(t) | t \in [0, T]\}$ and $\sigma(\cdot) := \{\sigma(t) | t \in [0, T]\}$ are positive, uniformly bounded, \mathbb{F} -adapted processes. Furthermore, we assume that the insurer’s relative security loading is also modeled by a positive, uniformly bounded, \mathbb{F} -adapted process $\theta(\cdot) := \{\theta(t) | t \in [0, T]\}$. Then, the insurance premium rate at time t is random and satisfies

$$c(t) := (1 + \theta(t))a(t) > 0, \quad \forall t \in [0, T].$$

Let $\rho_F(t) > 0$ be a uniformly bounded, deterministic function representing an instantaneous interest rate at which the insurer’s surplus is debited/credited.

Then, the insurer's surplus process $X_F(\cdot) := \{X_F(t) | t \in [0, T]\}$ follows:

$$\begin{aligned} dX_F(t) &= [\rho_F(t)X_F(t) + c(t)]dt - dC(t) \\ &= [\rho_F(t)X_F(t) + \theta(t)a(t)]dt + \sigma(t)dW(t), \quad X_F(0) = x_{0F} > 0. \end{aligned} \quad (2.5)$$

Here, the insurer earns credit interest when the surplus is positive, and pays out debit interest when the surplus is negative. The reader may refer to Cai *et al.* (2006) for more discussions on the necessity of introducing debit and credit interest to insurance risk models.

In this paper, we focus on proportional reinsurance, and leave the investigation to non-proportional reinsurance treaties, say, excess-of-loss reinsurance, to our future research. Suppose that the insurer has the option to cede a proportion of the insurance risk to the reinsurer. To do this, the insurer should simultaneously divert the corresponding proportion of the insurance premium as the reinsurance premium to the reinsurer. Denote by $p(t)$ and $q(t)$, the premium charged by the reinsurer and the retained proportion of the insurance risk at time t , respectively. In accordance with practice, we assume that $p(t) \in [c(t), g(t)]$ and $q(t) \in [0, 1]$, where $g(t) := (1 + \eta(t))a(t)$ and $\eta(\cdot) := \{\eta(t) | t \in [0, T]\}$ can be regarded as an upper bound of the reinsurer's relative security loading, which is also assumed to be a positive, uniformly bounded, \mathbb{F} -adapted process such that $\eta(t) \geq \theta(t)$, for any $t \in [0, T]$. Denote by $P(t) := [c(t), g(t)]$ and $Q := [0, 1]$, being a random and a constant intervals, respectively. Here, the constraints on $p(t)$ imply that the reinsurance is non-cheap and the reinsurer cannot set the reinsurance premium as much as it desires; those on $q(t)$ indicate that the insurer cannot borrow or short-sell the insurance risk. Therefore, by taking into consideration of the ceded insurance risk and the paid reinsurance premium, the insurer's surplus process $X_F(\cdot)$ is governed by the following stochastic differential equation (SDE):

$$\begin{aligned} dX_F(t) &= [\rho_F(t)X_F(t) + c(t) - (1 - q(t))p(t)]dt - q(t)dC(t) \\ &= [\rho_F(t)X_F(t) + \theta(t)a(t) - (1 - q(t))(p(t) - a(t))]dt \\ &\quad + q(t)\sigma(t)dW(t), \quad X_F(0) = x_{0F}. \end{aligned} \quad (2.6)$$

On the other hand, we let $\rho_L(t) > 0$ be a uniformly bounded, deterministic interest rate function at which the reinsurer's surplus is debited/credited. Since the reinsurer receives the premium at the rate of $(1 - q(t))p(t)$ from the insurer and is responsible for $100(1 - q(t))\%$ of the insurance risk, the reinsurer's surplus process $X_L(\cdot) := \{X_L(t) | t \in [0, T]\}$ is governed by the following SDE:

$$\begin{aligned} dX_L(t) &= [\rho_L(t)X_L(t) + (1 - q(t))p(t)]dt - (1 - q(t))dC(t) \\ &= [\rho_L(t)X_L(t) + (1 - q(t))(p(t) - a(t))]dt + (1 - q(t))\sigma(t)dW(t), \\ X_L(0) &= x_{0L} > 0. \end{aligned} \quad (2.7)$$

Throughout this paper, we call $p(\cdot) := \{p(t)|t \in [0, T]\}$ a reinsurance premium strategy and $q(\cdot) := \{q(t)|t \in [0, T]\}$ a reinsurance strategy. Indeed, $p(\cdot)$ is the control process chosen by the reinsurer, while $q(\cdot)$ is that chosen by the insurer. The interactions between $p(\cdot)$ and $q(\cdot)$ and $X_L(\cdot)$ and $X_F(\cdot)$ motivate us to formulate the problem of determining optimal reinsurance as a game.

In what follows, we simplify our presentation by denoting $X(\cdot) := \{X(t)|t \in [0, T]\} = \{(X_L(t), X_F(t))^\top | t \in [0, T]\}$ as a vector-valued controlled state process of the leader and the follower (i.e., the reinsurer and the insurer), which follows a two-dimensional SDE:

$$dX(t) = [A(t)X(t) + B(t, p(t), q(t))]dt + D(t, q(t))dW(t),$$

$$X(0) = x_0 := (x_{0L}, x_{0F})^\top, \tag{2.8}$$

where

$$A(t) := \begin{pmatrix} \rho_L(t) & 0 \\ 0 & \rho_F(t) \end{pmatrix},$$

$$B(t, p(t), q(t)) := ((1 - q(t))(p(t) - a(t)), \theta(t)a(t) - (1 - q(t))(p(t) - a(t)))^\top,$$

and

$$D(t, q(t)) := ((1 - q(t))\sigma(t), q(t)\sigma(t))^\top.$$

Definition 2.1. *A pair of strategies $(p(\cdot), q(\cdot))$ is said to be admissible if*

- i. $p(\cdot)$ is an \mathbb{F} -adapted process, such that $p(t) \in [c(t), g(t)] = P(t)$, for any $t \in [0, T]$;
- ii. $q(\cdot)$ is an \mathbb{F} -adapted process, such that $q(t) \in [0, 1] = Q$, for any $t \in [0, T]$;
- iii. the state equation (2.8) associated with $(p(\cdot), q(\cdot))$ has a unique strong solution $X(\cdot) \in S_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R}^2)$.

Let $\mathcal{A} := \mathcal{A}_L \times \mathcal{A}_F$ be the space of all admissible strategies, where \mathcal{A}_L denotes the space of all admissible reinsurance premium strategies and \mathcal{A}_F denotes the space of all admissible reinsurance strategies.

In what follows, we introduce the stochastic Stackelberg differential game between the reinsurer and the insurer. We assume that the preferences of the reinsurer and the insurer are modeled by general utility functions $U_L(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ and $U_F(\cdot) : \mathbb{R} \mapsto \mathbb{R}$, respectively, which are continuously differentiable, strictly increasing and strictly concave and satisfy the Inada conditions:

$$U'_F(-\infty) = +\infty, \quad U'_F(\infty) = 0,$$

and

$$U'_L(-\infty) = +\infty, \quad U'_L(\infty) = 0.$$

Some special utility functions are power utility, exponential utility, log utility, etc. The cost functionals of the reinsurer and the insurer are the expected utilities of their respective terminal surpluses, i.e.,

$$J_L(x_0; p(\cdot), q(\cdot)) := \mathbb{E}[U_L(X_L(T))], \tag{2.9}$$

and

$$J_F(x_0; p(\cdot), q(\cdot)) := \mathbb{E}[U_F(X_F(T))]. \tag{2.10}$$

The Stackelberg game is also called the leader-follower game in some literature. The nature of this game allows us to find the Stackelberg equilibrium by solving the leader’s and follower’s optimization problems sequentially, which is a standard procedure to solve Stackelberg games (see Yong (2002) and the references therein). This is different from the way to find the Nash equilibrium in general stochastic differential games, where optimization problems of different players are solved simultaneously. The procedure of solving the Stackelberg game uses the idea of backward induction. To be more specific, in the Stackelberg game, the leader (reinsurer) moves first by announcing its strategy $p(\cdot) \in \mathcal{A}_L$; the follower (insurer) observes the reinsurer’s strategy and follows to choose its own strategy $q^*(\cdot) = \alpha^*(\cdot, p(\cdot)) \in \mathcal{A}_F$ as a response to $p(\cdot)$ so as to maximize $J_F(x_0; p(\cdot), q(\cdot))$; knowing that the insurer would execute $\alpha^*(\cdot, p(\cdot))$, the reinsurer then decides on a strategy $p^*(\cdot) \in \mathcal{A}_L$ to maximize its own cost $J_L(x_0; p(\cdot), \alpha^*(\cdot, p(\cdot)))$. In the above procedure of backward induction, the follower’s optimal strategy is found in the first step, and the leader’s is obtained in the second step. The Stackelberg equilibrium is determined by the pair $(p^*(\cdot), \alpha^*(\cdot, p^*(\cdot)))$, and in the equilibrium the reinsurer plays $p^*(\cdot)$, while the insurer plays $\alpha^*(\cdot, p^*(\cdot))$. Throughout this paper, we call the reinsurer (resp., the insurer) and the leader (resp., the follower) interchangeably. The Stackelberg game is referred to as

Definition 2.2. *The insurer’s problem is the following stochastic optimization problem: for any $p(\cdot) \in \mathcal{A}_L$, find a map $q^*(\cdot) = \alpha^*(\cdot, p(\cdot)) : [0, T] \times \Omega \times \mathcal{A}_L \rightarrow \mathcal{A}_F$ such that*

$$J_F(x_0; p(\cdot), \alpha^*(\cdot, p(\cdot))) = \max_{q(\cdot) \in \mathcal{A}_F} J_F(x_0; p(\cdot), q(\cdot)), \tag{2.11}$$

while the reinsurer’s problem is the following stochastic optimization problem: find a $p^*(\cdot) \in \mathcal{A}_L$ such that

$$J_L(x_0; p^*(\cdot), \alpha^*(\cdot, p^*(\cdot))) = \max_{p(\cdot) \in \mathcal{A}_L} J_L(x_0; p(\cdot), \alpha^*(\cdot, p(\cdot))). \tag{2.12}$$

The pair $(p^*(\cdot), \alpha^*(\cdot, p^*(\cdot)))$ is called a solution to the stochastic Stackelberg game.

To define the value functions of the problems (2.11) and (2.22), we set the time- t value of the state process as

$$X(t) = (X_L(t), X_F(t))^T = x = (x_L, x_F)^T, \tag{2.13}$$

and consider the dynamic cost functionals:

$$J_L(t, x; p(\cdot), q(\cdot)) := \mathbb{E}_{t,x}[U_L(X_L(T))], \tag{2.14}$$

and

$$J_F(t, x; p(\cdot), q(\cdot)) := \mathbb{E}_{t,x}[U_F(X_F(T))], \tag{2.15}$$

where $\mathbb{E}_{t,x}[\cdot]$ denotes the conditional expectation taken under \mathbb{P} and given $\{\mathcal{F}_t, X(t) = x\}$.

Definition 2.3. *The value function of the insurer’s problem is*

$$\begin{aligned} V^F(t, x; p(\cdot)) &= J_F(t, x; p(\cdot), \alpha^*(\cdot, p(\cdot))) \\ &= \max_{q(\cdot) \in \mathcal{A}_F} J_F(t, x; p(\cdot), q(\cdot)), \quad \forall p(\cdot) \in \mathcal{A}_L, \end{aligned} \tag{2.16}$$

while the value function of the reinsurer’s problem is

$$V^L(t, x) = J_L(t, x; p^*(\cdot), \alpha^*(\cdot, p^*(\cdot))) = \max_{p(\cdot) \in \mathcal{A}_L} J_L(t, x; p(\cdot), \alpha^*(\cdot, p(\cdot))). \tag{2.17}$$

Furthermore, if there is no risk of confusion, when the solution of the game $(p^*(\cdot), \alpha^*(\cdot, p^*(\cdot)))$ is adopted,

$$V^F(t, x) = V^F(t, x; p^*(\cdot)) = J_F(t, x; p^*(\cdot), \alpha^*(\cdot, p^*(\cdot))). \tag{2.18}$$

is also called the value function of the insurer’s problem.

3. DISCUSSIONS ON GENERAL UTILITY CASE

In this section, under appropriate regularity conditions, we discuss how the Stackelberg game (2.11)–(2.12) in Definition 2.2 can be solved via stochastic HJB equations. For general utility functions, these regularity conditions are very difficult, if not impossible, to be verified. So, the discussions in this section are only heuristic.

First of all, we introduce a stochastic HJB equation as follows:

$$\left\{ \begin{aligned} -d\Phi^F(t, x) &= \sup_{q(\cdot) \in \mathcal{A}_F} \left\{ \Phi_x^F(t, x)^\top [A(t)x + B(t, p(t), q(t))] \right. \\ &\quad + \frac{1}{2} \text{tr}[\Phi_{xx}^F(t, x) D(t, q(t)) D(t, q(t))^\top] \\ &\quad \left. + \Psi_x^F(t, x)^\top D(t, q(t)) \right\} dt - \Psi^F(t, x) dW(t), \quad t \in [0, T] \\ \Phi^F(T, x) &= U_F(x_F), \end{aligned} \right. \tag{3.1}$$

which will be used to characterize the insurer’s optimal strategy and value function. Here a pair of real-valued, \mathbb{F} -adapted random fields $(\Phi^F(t, x), \Psi^F(t, x))$ is said to be a solution to the stochastic HJB equation (3.1) if the equation is satisfied \mathbb{P} -a.s.. In fact, the stochastic HJB equation is a special backward stochastic partial differential equations or infinite-dimensional BSDE. Peng (1992) studied the solvability of stochastic HJB equations in one special case, where the diffusion component of the state equation is degenerate in the control. The general theory of stochastic HJB equations is still lacking.

To simplify our presentation, in the sequel, we denote the Hamiltonian by

$$\begin{aligned} H(t, x, p, q, \Phi_x^F(t, x), \Phi_{xx}^F(t, x), \Psi_x^F(t, x)) \\ := \Phi_x^F(t, x)^\top [A(t)x + B(t, p, q)] \\ + \frac{1}{2} \text{tr}[\Phi_{xx}^F(t, x) D(t, q) D(t, q)^\top] + \Psi_x^F(t, x)^\top D(t, q). \end{aligned} \tag{3.2}$$

Under some regularity conditions on $(\Phi^F(t, x), \Psi^F(t, x))$, we can characterize the optimal solution to the insurer’s optimization problem by the stochastic HJB equation (3.1) as follows:

Proposition 3.1. *Suppose that $(\Phi^F(t, x), \Psi^F(t, x))$ is a solution to the stochastic HJB equation (3.1), such that the following regularity conditions are satisfied*

- i. $\{\Phi^F(t, X(t)) | t \in [0, T]\}$ is uniformly integrable;
- ii. $\Phi_x^F(\cdot, X(\cdot)) \in \mathcal{L}_{\mathbb{F}, \mathbb{P}}^{2,loc}(0, T; \mathbb{R}^2)$ and $\Psi^F(\cdot, X(\cdot)) \in \mathcal{L}_{\mathbb{F}, \mathbb{P}}^{2,loc}(0, T; \mathbb{R})$.

For any $p(\cdot) \in \mathcal{A}_L$, if the supremum in the drift of (3.1) is achieved at $q^*(\cdot) := \alpha^*(\cdot, p(\cdot)) \in \mathcal{A}_F$, then $V^F(t, x; p(\cdot)) = \Phi^F(t, x)$ is the value function and $q^*(\cdot) = \alpha^*(\cdot, p(\cdot))$ is the optimal strategy of Problem (2.11).

Proof. For any $(p(\cdot), q(\cdot)) \in \mathcal{A}_L \times \mathcal{A}_F$, we know that the corresponding state equation (2.8) has a unique strong solution $X(\cdot) \in \mathcal{S}_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R}^2)$. From the regularity condition (ii) and the boundedness of $D(t, q(t))$, we know that $\Phi_x^F(\cdot, X(\cdot))^\top D(\cdot, q(\cdot)) + \Psi_x^F(\cdot, X(\cdot)) \in \mathcal{L}_{\mathbb{F}, \mathbb{P}}^{2,loc}(0, T; \mathbb{R})$, thereby $\int_0^\cdot [\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi_x^F(s, X(s))] dW(s)$ is an (\mathbb{F}, \mathbb{P}) -local martingale.

Therefore, there exists a localizing sequence $\{\tau_n | n = 1, 2, \dots\}$ for this local martingale. Without loss of generality, we can define $\{\tau_n | n = 1, 2, \dots\}$ as

$$\tau_n := \inf \left\{ t > 0 : \int_0^t |\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi^F(s, X(s))|^2 ds \geq n \right\}. \tag{3.3}$$

Applying Itô-Kunita’s formula (see Kunita (1981)) to $\Phi^F(s, X(s))$ and integrating from 0 to $u \wedge \tau_n$, we obtain

$$\begin{aligned} & \Phi^F(u \wedge \tau_n, X(u \wedge \tau_n)) - \Phi^F(0, x_0) \\ &= - \int_0^{u \wedge \tau_n} \Delta^H(s, X(s), p(s), q(s)) ds \\ & \quad + \int_0^{u \wedge \tau_n} [\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi^F(s, X(s))] dW(s), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} & \Delta^H(s, X(s), p(s), q(s)) \\ &:= \sup_{q(\cdot) \in \mathcal{A}_F} [H(s, X(s), p(s), q(s), \Phi_x^F(s, X(s)), \Phi_{xx}^F(s, X(s)), \Psi_x^F(s, X(s)))] \\ & \quad - H(s, X(s), p(s), q(s), \Phi_x^F(s, X(s)), \Phi_{xx}^F(s, X(s)), \Psi_x^F(s, X(s))). \end{aligned}$$

Setting $u = T$ and $u = t$ in (3.4), for any $t \in [0, T]$, and taking the difference between these two cases give

$$\begin{aligned} & \Phi^F(T \wedge \tau_n, X(T \wedge \tau_n)) - \Phi^F(t \wedge \tau_n, X(t \wedge \tau_n)) \\ &= - \int_0^{T \wedge \tau_n} \Delta^H(s, X(s), p(s), q(s)) ds \\ & \quad + \int_0^{t \wedge \tau_n} \Delta^H(s, X(s), p(s), q(s)) ds \\ & \quad + \int_0^{T \wedge \tau_n} [\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi^F(s, X(s))] dW(s) \\ & \quad - \int_0^{t \wedge \tau_n} [\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi^F(s, X(s))] dW(s). \end{aligned}$$

Conditioning both sides of the above equation on \mathcal{F}_t makes the stochastic integrals with respect to the Brownian motion vanish. Thus, we obtain

$$\begin{aligned} \mathbb{E}_t[\Phi^F(T \wedge \tau_n, X(T \wedge \tau_n))] &= \Phi^F(t \wedge \tau_n, X(t \wedge \tau_n)) \\ &\quad - \mathbb{E}_t \left[\int_0^{T \wedge \tau_n} \Delta^H(s, X(s), p(s), q(s)) ds \right] \\ &\quad + \int_0^{t \wedge \tau_n} \Delta^H(s, X(s), p(s), q(s)) ds. \end{aligned} \tag{3.5}$$

In (3.5), $\Phi^F(\cdot, X(\cdot))$ is uniformly integrable (regularity condition (i)) and the integrand $\Delta^H(s, X(s), p(s), q(s))$ is non-negative. So, the dominated convergence theorem and the monotone convergence theorem apply to the conditional expectation terms on the left-hand side and the right-hand side of (3.5), respectively. Therefore, sending n to ∞ on both sides of (3.5), we obtain

$$\begin{aligned} \mathbb{E}_t[U_F(X_F(T))] &= \mathbb{E}_t[\Phi^F(T, X(T))] \\ &= \Phi^F(t, x) - \mathbb{E}_t \left[\int_t^T \Delta^H(s, X(s), p(s), q(s)) ds \right] \leq \Phi^F(t, x). \end{aligned} \tag{3.6}$$

Then, taking supremum with respect to $q(\cdot)$ over \mathcal{A}_F in (3.6) gives

$$V^F(t, x; p(\cdot)) \leq \Phi^F(t, x). \tag{3.7}$$

Since $q^*(\cdot) = \alpha^*(\cdot, p(\cdot))$ is achieved within the admissible set \mathcal{A}_F , we have that for any $p(\cdot) \in \mathcal{A}_L$, the corresponding state equation, i.e.,

$$dX(t) = [A(t)X(t) + B(t, p(t), \alpha^*(t, p(\cdot)))]dt + D(t, \alpha^*(t, p(\cdot)))dW(t), \tag{3.8}$$

admits a unique strong solution $X(\cdot) \in \mathcal{S}_{\mathbb{R}, \mathbb{P}}^2(0, T; \mathbb{R}^2)$. If we set $q(\cdot)$ to be $q^*(\cdot)$ in (3.6), the last inequality becomes an equality and $V^F(t, x; p(\cdot)) = \mathbb{E}_t[U_F(X_F^{(p(\cdot), q^*(\cdot))}(T))]$. Thus, $V^F(t, x; p(\cdot)) = \Phi^F(t, x)$. This completes the proof. ■

Similarly, the reinsurer’s optimal strategy and value function can be characterized by the following stochastic HJB equation:

$$\left\{ \begin{aligned} -d\Phi^L(t, x) &= \sup_{p(\cdot) \in \mathcal{A}_L} \left\{ \Phi_x^L(t, x)^\top [A(t)x + B(t, p(t), \alpha^*(t, p(\cdot)))] \right. \\ &\quad + \frac{1}{2} \text{tr}[\Phi_{xx}^L(t, x) D(t, \alpha^*(t, p(\cdot))) D(t, \alpha^*(t, p(\cdot)))^\top] \\ &\quad \left. + \Psi_x^L(t, x)^\top D(t, \alpha^*(t, p(\cdot))) \right\} dt - \Psi^L(t, x) dW(t), \quad t \in [0, T] \\ \Phi^L(T, x) &= U_L(x_L). \end{aligned} \right. \tag{3.9}$$

The solution to the stochastic HJB equation (3.9) can be defined similarly as that to (3.1). Under similar regularity conditions on the solution to the stochastic HJB equation (3.9), we have the following result for the reinsurer’s optimization problem.

Proposition 3.2. *Suppose that $(\Phi^L(t, x), \Psi^L(t, x))$ is a solution to the stochastic HJB equation (3.9) such that the following regularity conditions are satisfied*

- i. $\{\Phi^L(t, X(t)) | t \in [0, T]\}$ is uniformly integrable;
- ii. $\Phi_x^L(\cdot, X(\cdot)) \in \mathcal{L}_{\mathbb{F}, \mathbb{P}}^{2,loc}(0, T; \mathbb{R}^2)$ and $\Psi^L(\cdot, X(\cdot)) \in \mathcal{L}_{\mathbb{F}, \mathbb{P}}^{2,loc}(0, T; \mathbb{R})$.

If the supremum in the drift of (3.9) is achieved at $p^(\cdot) \in \mathcal{A}_L$, then $V^L(t, x) = \Phi^L(t, x)$ and $p^*(\cdot)$ are the value function and the optimal strategy of Problem (2.12), respectively.*

Proof. The proposition can be proved in a similar manner to Proposition 3.1. We omit the proof here. ■

Even when the risk model is given by Grandell’s original diffusion approximation model with constant coefficients, in general, we have to resort to stochastic HJB equations to solve the Stackelberg game (2.11)–(2.12). The difficulty is that for general utility functions the optimal strategy of any one of the players may depend on its own state process and even its opponent’s state and control processes in an anticipating manner, which may render the coefficients of the other player’s state process non-Markovian. This issue is encountered in the linear-quadratic case with deterministic coefficients (see Yong (2002)). Thereby, deterministic HJB equations are not applicable once utility functions are general no matter whether the model coefficients are random or not, and the discussions in this section are helpful in this situation. The interested reader may refer to Appendix B for an example in which optimal strategies are anticipating and deterministic HJB equations are not applicable.

4. SOLUTION TO EXPONENTIAL UTILITY CASE WITH RANDOM PARAMETERS

As surpluses may take negative values, the most widely used utility function in the literature on optimal reinsurance problems is the exponential utility function. In this section, we consider the case that the reinsurer’s and insurer’s preferences are modeled by exponential utility functions:

$$U_L(x_L) := -\frac{1}{\gamma_L} \exp(-\gamma_L x_L), \quad U_F(x_F) := -\frac{1}{\gamma_F} \exp(-\gamma_F x_F), \quad (4.1)$$

where $\gamma_L > 0$ and $\gamma_F > 0$ are CARA coefficients. With this special structure of utility functions, we are able to solve the game problem with the aid of BSDEs. More specifically, we are able to verify the regularity conditions imposed in Propositions 3.1–3.2 and express the value functions and optimal strategies of the game by unique solutions to two quadratic BSDEs. In the finance and insurance literature, the BSDE approach is a useful technique to solve utility maximization problems with random parameters (see Hu *et al.* (2005), Shen and Wei (2016), and the references therein).

The specification of exponential utilities gives us an explicit terminal condition of the value function in the insurer’s problem (2.11). Thus, we consider an *ansatz* for the value function:

$$\phi^F(t, x) = -\frac{1}{\gamma_F} \exp[-\gamma_F(h^F(t)x_F + Y^F(t))], \quad (4.2)$$

where $h^F(t)$ is a deterministic, continuously differentiable function such that $h^F(T) = 1$, and $Y^F(t)$ is an \mathbb{F} -adapted process and satisfies a BSDE:

$$dY^F(t) = -f^F(t, Y^F(t), Z^F(t))dt - Z^F(t)dW(t), \quad t \in [0, T], \quad Y^F(T) = 0. \quad (4.3)$$

Indeed, $Y^F(\cdot)$ is the first component of the solution to the BSDE (4.3), and $f^F(t, Y^F(t), Z^F(t))$ is the driver of the BSDE (4.3). A pair of processes $(Y^F(\cdot), Z^F(\cdot))$ is said a solution to the BSDE (4.3) if $Y^F(\cdot)$ is continuous \mathbb{F} -adapted and $Z^F(\cdot)$ is \mathbb{F} -adapted, (4.3) holds \mathbb{P} -a.s., and the following condition is satisfied:

$$\int_0^T |f^F(s, Y^F(s), Z^F(s))|ds + \int_0^T |Z^F(s)|^2 ds < \infty, \quad \mathbb{P}\text{-a.s.} \quad (4.4)$$

In the next proposition, we determine $h^F(t)$ and $f^F(t, Y^F(t), Z^F(t))$.

Proposition 4.1. *In the ansatz (4.2), if*

- i. $\{\phi^F(t, X(t)) | t \in [0, T]\}$ is uniformly integrable,
- ii. the function $h^F(t)$ is given by

$$h^F(t) = e^{\int_t^T \rho_F(s) ds}, \quad (4.5)$$

iii. the BSDE (4.3) has a solution $(Y^F(\cdot), Z^F(\cdot))$ and the driver of the BSDE (4.3) is specified by

$$\begin{aligned}
 f^F(t, Z^F(t)) &= -\frac{\gamma_F}{2} \left[\operatorname{dist}_{q \in Q} \left(h^F(t)\sigma(t)q, Z^F(t) + \frac{p(t) - a(t)}{\gamma_F\sigma(t)} \right) \right]^2 \\
 &\quad + \frac{p(t) - a(t)}{\sigma(t)} Z^F(t) \\
 &\quad + \frac{1}{2\gamma_F} \left(\frac{p(t) - a(t)}{\sigma(t)} \right)^2 + h^F(t)[(1 + \theta(t))a(t) - p(t)], \quad (4.6)
 \end{aligned}$$

where

$$\begin{aligned}
 &\operatorname{dist}_{q \in Q} \left(h^F(t)\sigma(t)q, Z^F(t) + \frac{p(t) - a(t)}{\gamma_F\sigma(t)} \right) \\
 &:= \inf_{q \in Q} \left\{ \left| h^F(t)\sigma(t)q - \left(Z^F(t) + \frac{p(t) - a(t)}{\gamma_F\sigma(t)} \right) \right| \right\}, \quad (4.7)
 \end{aligned}$$

then $\{\phi^F(t, X(t)) | t \in [0, T]\}$ is a supermartingale for any $(p(\cdot), q(\cdot)) \in \mathcal{A}_L \times \mathcal{A}_F$ and a martingale when

$$q^*(t) = \alpha^*(t, p(t)) = \operatorname{arg\,min}_{q \in Q} \left[q - \frac{1}{h^F(t)\sigma(t)} \left(Z^F(t) + \frac{p(t) - a(t)}{\gamma_F\sigma(t)} \right) \right]^2 \quad (4.8)$$

is taken for any $p(\cdot) \in \mathcal{A}_L$.

Proof. By Itô’s formula, we derive

$$\begin{aligned}
 &d\phi^F(t, X(t)) \\
 &= \left\{ \exp \left[-\gamma_F(h^F(t)X_F(t) + Y^F(t)) \right] X_F(t) [h_t^F(t) + \rho_F(t)h^F(t)] + \Pi(t) \right\} dt \\
 &\quad + \exp \left[-\gamma_F(h^F(t)X_F(t) + Y^F(t)) \right] [h^F(t)\sigma(t)q(t) - Z^F(t)] dW(t), \quad (4.9)
 \end{aligned}$$

where

$$\begin{aligned}
 \Pi(t) &:= \exp \left[-\gamma_F(h^F(t)X_F(t) + Y^F(t)) \right] \\
 &\quad \times \left\{ h^F(t) [\theta(t)a(t) - (1 - q(t))(p(t) - a(t))] \right. \\
 &\quad - f^F(t, Z^F(t)) - \frac{1}{2} \gamma_F (h^F(t))^2 \sigma^2(t) q^2(t) + \gamma_F h^F(t) \sigma(t) Z^F(t) q(t) \\
 &\quad \left. - \frac{1}{2} \gamma_F |Z^F(t)|^2 \right\}. \quad (4.10)
 \end{aligned}$$

It will turn out that f^F does not depend on Y^F (see (4.12)). So we have written $f^F(t, Z^F(t)) := f^F(t, Y^F(t), Z^F(t))$.

If we set

$$h_t^F(t) + \rho_F(t)h^F(t) = 0, \tag{4.11}$$

whose explicit solution is clearly given by (4.5), and

$$\begin{aligned} & f^F(t, Z^F(t)) \\ &= \sup_{q \in Q} \left\{ h^F(t)[\theta(t)a(t) - (1 - q)(p(t) - a(t))] - \frac{1}{2}\gamma_F(h^F(t))^2\sigma^2(t)q^2 \right. \\ & \quad \left. + \gamma_F h^F(t)Z^F(t)\sigma(t)q - \frac{1}{2}\gamma_F|Z^F(t)|^2 \right\} \\ &= -\frac{\gamma_F}{2} \inf_{q \in Q} \left[h^F(t)\sigma(t)q - \left(Z^F(t) + \frac{p(t) - a(t)}{\gamma_F\sigma(t)} \right) \right]^2 + \frac{p(t) - a(t)}{\sigma(t)} Z^F(t) \\ & \quad + \frac{1}{2\gamma_F} \left(\frac{p(t) - a(t)}{\sigma(t)} \right)^2 + h^F(t)[(1 + \theta(t))a(t) - p(t)], \end{aligned} \tag{4.12}$$

then the drift of (4.9) is non-positive for any $q(\cdot) \in \mathcal{A}_F$. In fact, the infimum in the second equality of (4.12) can be rewritten as the distance function in (4.6).

By the definition of the solution $(Y^F(\cdot), Z^F(\cdot))$ to the BSDE (4.3) (refer to (4.4)), we see that for any $q(\cdot) \in \mathcal{A}_F$, the process $\int_0^\cdot [h^F(s)\sigma(s)q(s) - Z^F(s)]dW(s)$ is an (\mathbb{F}, \mathbb{P}) -local martingale, and $Y_F(\cdot)$ is a continuous \mathbb{F} -adapted process. By Definition 2.1, for any $(p(\cdot), q(\cdot)) \in \mathcal{A}_L \times \mathcal{A}_F$, the insurer’s surplus process $X_F(\cdot)$ is continuous, \mathbb{F} -adapted and square integrable (in the sense of $\mathcal{S}_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R})$). Therefore, the process $\exp[-\gamma_F(h^F(\cdot)X_F(\cdot) + Y^F(\cdot))]$ is locally bounded (refer to Page 140 of Revuz and Yor (2005)). So, the stochastic integral with respect to the Brownian motion in (4.9), i.e.,

$$\int_0^\cdot \exp[-\gamma_F(h^F(s)X_F(s) + Y^F(s))][h^F(s)\sigma(s)q(s) - Z^F(s)]dW(s),$$

is also a local martingale (again refer to Page 140 of Revuz and Yor (2005)). Moreover, as $\{\phi^F(t, X(t))|t \in [0, T]\}$ is uniformly integrable and $\Pi(\cdot)$ is non-positive, we can use similar localization techniques as in the proof of Proposition 3.1 to derive that

$$\mathbb{E}[\phi^F(s, X(s))|\mathcal{F}_t] = \phi^F(t, X(t)) + \mathbb{E}\left[\int_t^s \Pi(v)dv|\mathcal{F}_t\right], \quad 0 \leq t \leq s \leq T. \tag{4.13}$$

Observing that $\Pi(\cdot)$ is a non-positive process for any $q(\cdot) \in \mathcal{A}_F$ and becomes zero when $\alpha^*(\cdot, p(\cdot))$ is taken, we immediately obtain the desired results. ■

Remark 4.1. Since the optimal reinsurance strategy $q^*(t)$ depends on $c(t)$ via $a(t)$ and $\theta(t)$ as well as other random parameters, it is random in general. This is legitimate in practice. For instance, if an insurance company has foreign business and receives premium in a foreign currency, converting the premium denominated in the foreign currency to one denominated in the domestic currency may pose foreign exchange risk to the insurer’s decision making process. This would then render $c(t)$ and $q^*(t)$ random.

Now that we have obtained the explicit structure of the driver of the BSDE (4.3), we can discuss the solvability of this BSDE.

Proposition 4.2. *The BSDE (4.3) admits a unique solution $(Y^F(\cdot), Z^F(\cdot)) \in \mathcal{S}_{\mathbb{F}, \mathbb{P}}^\infty(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}, \mathbb{P}}^{2, BMO}(0, T; \mathbb{R})$.*

Proof. Note that if the minimum in (4.8) is attainable at an interior point of Q , then $f^F(t, z)$ is a linear function of z ; otherwise, if the minimum in (4.8) is achievable at the boundary of Q , then $f^F(t, z)$ is quadratic in z . In both cases, from the boundedness of coefficients, we can find a $K > 0$ such that the following quadratic growth condition is satisfied:

$$|f^F(t, z)| \leq K(1 + |z|^2). \tag{4.14}$$

Applying the triangle inequality to the distance defined by (4.7), we obtain that for any $z_1, z_2 \in \mathbb{R}$,

$$|f^F(t, z_1) - f^F(t, z_2)| \leq K(1 + |z_1| + |z_2|)|z_1 - z_2|. \tag{4.15}$$

Then, the desired result follows from Kobylanski (2000) and Morlais (2009). ■

The following verification result can be given by using the property of the solution to the BSDE (4.3).

Proposition 4.3. *The value function and the optimal strategy of Problem (2.11) are given by*

$$V^F(t, x; p(\cdot)) = -\frac{1}{\gamma_F} \exp \left[-\gamma_F(h^F(t)x_F + Y^F(t)) \right], \tag{4.16}$$

and

$$q^*(t) = \alpha^*(t, p(t)) = \arg \min_{q \in Q} \left[q - \frac{1}{h^F(t)\sigma(t)} \left(Z^F(t) + \frac{p(t) - a(t)}{\gamma_F\sigma(t)} \right) \right]^2. \tag{4.17}$$

Proof. The proof can be conducted by using the Davis–Varaiya martingale optimality principle. Essentially speaking, we need to show that the regularity condition in Proposition 4.1 is satisfied.

Note that

$$\begin{aligned}
 h^F(t)X^F(t) &= e^{\int_0^t \rho_F(s)ds} x_{0F} + \int_0^t h^F(s)[\theta(s)a(s) - (1 - q(s))(p(s) - a(s))]ds \\
 &\quad + \int_0^t h^F(s)q(s)\sigma(s)dW(s).
 \end{aligned}
 \tag{4.18}$$

By the boundedness of $h^F(t)$, $Y^F(t)$ and other model paramters, we have that for any $m \geq 1$ and $q(\cdot) \in \mathcal{A}_F$,

$$\begin{aligned}
 &\mathbb{E}[|\phi^F(t, X(t))|^m] \\
 &\leq K\mathbb{E}\left[e^{-m\gamma_F h^F(t)X^F(t)}\right] \\
 &= K\mathbb{E}\left[e^{-m\gamma_F \left\{e^{\int_0^t \rho_F(s)ds} x_{0F} + \int_0^t h^F(s)[\theta a - (1-q)(p-a)]ds\right\}} \cdot e^{-m\gamma_F \int_0^t h^F(s)q(s)\sigma(s)dW(s)}\right] \\
 &\leq K\mathbb{E}\left[e^{-\frac{1}{2}m^2\gamma_F^2 \int_0^t [h^F(s)q(s)\sigma(s)]^2 ds - m\gamma_F \int_0^t h^F(s)q(s)\sigma(s)dW(s)}\right] \\
 &= K < \infty, \quad \forall t \in [0, T].
 \end{aligned}
 \tag{4.19}$$

Therefore, choosing $m > 1$ and taking supremum in the above inequality give

$$\sup_{t \in [0, T]} \mathbb{E}[|\phi^F(t, X(t))|^m] < \infty.
 \tag{4.20}$$

This guarantees that $\{\phi^F(t, X(t)) | t \in [0, T]\}$ is uniformly integrable. By Proposition 4.1, we see that $\phi^F(t, X(t))$ is a supermartingale for any $q(\cdot) \in \mathcal{A}_F$, and is a martingale when $q^*(\cdot)$ defined by (4.17) is taken. Using the martingale optimality principle, we affirm that $V^F(t, x; p(\cdot)) = \phi^F(t, x)$ is the value function of Problem (2.11) and the optimal reinsurance strategy is given by $q^*(\cdot)$. ■

Remark 4.2. As mentioned in Section 2, the insurer’s optimal strategy (4.8) is expressed by the random map $\alpha^* : [0, T] \times \Omega \times \mathcal{A}_L \rightarrow \mathcal{A}_F$. Whichever reinsurance premium strategy $p(\cdot) \in \mathcal{A}_L$ the reinsurer decides to execute, the insurer, as the follower of the Stackelberg game, will apply the optimal strategy $\alpha^*(\cdot, p(\cdot))$ accordingly.

Next, we study the reinsurer’s optimization problem (2.12). Similarly, we try the following *ansatz* for the value function of the reinsurer’s problem:

$$\phi^L(t, x) = -\frac{1}{\gamma_L} \exp\left[-\gamma_L(h^L(t)x_L + Y^L(t))\right].
 \tag{4.21}$$

As in the insurer’s problem, the solution of Problem (2.12) can be expressed by the solution to a BSDE. First of all, we discuss the solvability of the BSDE

related to the reinsurer’s problem:

$$dY^L(t) = -f^L(t, Z^L(t))dt - Z^L(t)dW(t), \quad t \in [0, T], \quad Y^L(T) = 0, \quad (4.22)$$

where

$$\begin{aligned} f^L(t, z) = \sup_{p \in P(t)} & \left\{ -\frac{1}{2}\gamma_L(h^L(t))^2\sigma^2(t)(1 - \alpha^*(t, p))^2 \right. \\ & + \gamma_L h^L(t)\sigma(t)(1 - \alpha^*(t, p))z \\ & \left. + h^L(t)(1 - \alpha^*(t, p))(p - a(t)) \right\} - \frac{1}{2}\gamma_L z^2, \end{aligned} \quad (4.23)$$

and

$$h^L(t) = e^{\int_t^T \rho_L(s)ds}.$$

Proposition 4.4. *The BSDE (4.22) has a unique solution $(Y^L(\cdot), Z^L(\cdot)) \in \mathcal{S}_{\mathbb{F}, \mathbb{P}}^\infty(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}, \mathbb{P}}^{2, BMO}(0, T; \mathbb{R})$.*

Proof. By the boundedness of $h^L(t)$, $\alpha^*(t, p)$ and other model parameters, we see that the driver $f^L(t, z)$ has a quadratic growth in z , i.e.,

$$|f^L(t, z)| \leq K(1 + |z|^2). \quad (4.24)$$

If the minimum in (4.8) is achieved at the boundary of Q , i.e., $\alpha^*(t, p) = 0$ or 1 , it is obvious that for any $z_1, z_2 \in \mathbb{R}$,

$$|f^L(t, z_1) - f^L(t, z_2)| \leq K(1 + |z_1| + |z_2|)|z_1 - z_2|. \quad (4.25)$$

It follows from Kobylanski (2000) and Morlais (2009) that the BSDE admits a unique solution $(Y^L(\cdot), Z^L(\cdot)) \in \mathcal{S}_{\mathbb{F}, \mathbb{P}}^\infty(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}, \mathbb{P}}^{2, BMO}(0, T; \mathbb{R})$.

Otherwise, if the minimum in (4.8) is achieved at an interior point of Q , then

$$\begin{aligned} f^L(t, z) = \sup_{p \in P(t)} & \left\{ -\frac{1}{2}\gamma_L(h^L(t))^2\sigma^2(t) \left[1 - \frac{Z^F(t) + \frac{p-a(t)}{\gamma_F\sigma(t)}}{h^F(t)\sigma(t)} \right]^2 \right. \\ & + \gamma_L h^L(t)\sigma(t) \left[1 - \frac{Z^F(t) + \frac{p-a(t)}{\gamma_F\sigma(t)}}{h^F(t)\sigma(t)} \right] z \\ & \left. + h^L(t) \left[1 - \frac{Z^F(t) + \frac{p-a(t)}{\gamma_F\sigma(t)}}{h^F(t)\sigma(t)} \right] (p - a(t)) \right\} - \frac{1}{2}\gamma_L z^2 \end{aligned}$$

$$\begin{aligned}
 &= -h^L(t)\sigma^2(t) \left[\frac{1}{2}\gamma_L h^L(t) + \gamma_F h^F(t) \right] \\
 &\quad \times \left\{ \operatorname{dist}_{p \in P(t)} \left[\frac{p}{\gamma_F h^F(t)\sigma^2(t)}, 1 + \frac{a(t) - \gamma_F \sigma(t) Z^F(t)}{\gamma^F h^F(t)\sigma^2(t)} \right. \right. \\
 &\quad \left. \left. - \frac{\gamma_F h^F(t)\sigma(t) + \gamma_L h^L(t)z - \gamma_F Z^F(t)}{h^L(t)\sigma(t)[\gamma_L h^L(t) + 2\gamma_F h^F(t)]} \right] \right\}^2 \\
 &\quad + \frac{[\gamma_F h^F(t)\sigma(t) + \gamma_L h^L(t)z - \gamma_F Z^F(t)]^2}{2h^L(t)[\gamma_L h^L(t) + 2\gamma_F h^F(t)]} - \frac{1}{2}\gamma_L z^2, \tag{4.26}
 \end{aligned}$$

where $\operatorname{dist}_{p \in P(t)} [\cdot, \cdot]$ is the distance defined similarly as (4.7). Again, thanks to the triangle inequality of the distance, we have that for any $z_1, z_2 \in \mathbb{R}$,

$$|f^L(t, z_1) - f^L(t, z_2)| \leq C(1 + |Z^F(t)| + |z_1| + |z_2|)|z_1 - z_2|. \tag{4.27}$$

Noting that $Z^F(\cdot) \in \mathcal{H}_{\mathbb{F}, \mathbb{P}}^{2, \text{BMO}}(0, T; \mathbb{R})$, we can show as Kobylanski (2000) and Morlais (2009) that the BSDE (4.22) admits a unique solution $(Y^L(\cdot), Z^L(\cdot)) \in \mathcal{S}_{\mathbb{F}, \mathbb{P}}^\infty(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}, \mathbb{P}}^{2, \text{BMO}}(0, T; \mathbb{R})$. ■

Proposition 4.5. *The value function of Problem (2.12) is given by*

$$V^L(t, x) = -\frac{1}{\gamma_L} \exp \left[-\gamma_L(h^L(t)x_L + Y^L(t)) \right], \tag{4.28}$$

and the optimal strategy of Problem (2.12) is given by

$$\begin{aligned}
 p^*(t) = \operatorname{arg\,max}_{p \in P(t)} &\left[-\frac{1}{2}\gamma_L(h^L(t))^2\sigma^2(t)(1 - \alpha^*(t, p))^2 \right. \\
 &+ \gamma_L h^L(t)\sigma(t)(1 - \alpha^*(t, p))Z^L(t) \\
 &\left. + h^L(t)(1 - \alpha^*(t, p))(p - a(t)) \right]. \tag{4.29}
 \end{aligned}$$

Proof. Applying Itô’s formula, we derive

$$\begin{aligned}
 &d\phi^L(t, X(t)) \\
 &= \left\{ \exp \left[-\gamma_L(h^L(t)X_L(t) + Y^L(t)) \right] X_L(t) [h_i^L(t) + \rho_L(t)h^L(t)] + \Xi(t) \right\} dt \\
 &\quad + \exp \left[-\gamma_L(h^L(t)X_L(t) + Y^L(t)) \right] [h^L(t)(1 - \alpha^*(t, p(t)))\sigma(t) - Z^L(t)] dW(t), \tag{4.30}
 \end{aligned}$$

where

$$\begin{aligned} \Xi(t) := & \exp \left[-\gamma_L(h^L(t)X_L(t) + Y^L(t)) \right] \\ & \times \left\{ h^L(t)(1 - \alpha^*(t, p(t)))(p(t) - a(t)) - f^L(t, Z^L(t)) \right. \\ & - \frac{1}{2}\gamma_L(h^L(t))^2\sigma^2(t)(1 - \alpha^*(t, p(t)))^2 \\ & \left. + \gamma_L h^L(t)\sigma(t)Z^L(t)(1 - \alpha^*(t, p(t))) - \frac{1}{2}\gamma_L|Z^L(t)|^2 \right\}. \end{aligned} \tag{4.31}$$

In equation (4.30), the drift is non-positive, while the stochastic integral is a local martingale. The remaining arguments can resemble those in the proofs of Propositions 4.1 and 4.3, thereby are omitted here. ■

As a by-product, in the next proposition we find unique solutions to the stochastic HJB equations (3.1) and (3.9).

Proposition 4.6. *The stochastic HJB equations (3.1) and (3.9) admit unique solutions*

$$\phi^F(t, x) = -\frac{1}{\gamma_F} \exp \left[-\gamma_F(h^F(t)x_F + Y^F(t)) \right], \tag{4.32}$$

$$\psi^F(t, x) = -\exp \left[-\gamma_F(h^F(t)x_F + Y^F(t)) \right] Z^F(t), \tag{4.33}$$

and

$$\phi^L(t, x) = -\frac{1}{\gamma_L} \exp \left[-\gamma_L(h^L(t)x_L + Y^L(t)) \right], \tag{4.34}$$

$$\psi^L(t, x) = -\exp \left[-\gamma_L(h^L(t)x_L + Y^L(t)) \right] Z^L(t), \tag{4.35}$$

where $(Y^F(\cdot), Z^F(\cdot))$ and $(Y^L(\cdot), Z^L(\cdot))$ are the unique solutions to the BSDEs (4.3) and (4.22), respectively.

Proof. Denote by

$$\Lambda(t) := e^{-\frac{1}{2}m^2\gamma_F^2 \int_0^t [h^F(s)q(s)\sigma(s)]^2 ds - m\gamma_F \int_0^t h^F(s)q(s)\sigma(s) dW(s)}. \tag{4.36}$$

Clearly, $\Lambda(\cdot)$ is a Radon–Nikodym derivative and satisfies

$$d\Lambda(t) = -m\gamma_F h^F(t)q(t)\sigma(t)\Lambda(t)dW(t), \quad \Lambda(0) = 1. \tag{4.37}$$

This is a SDE satisfying the Lipchitz and linear growth conditions. So, the SDE (4.37) has a unique solution such that $\Lambda(\cdot) \in S_{\mathbb{F}, \mathbb{P}}^1(0, T; \mathbb{R})$. As the derivation in

(4.19), we deduce that for any $m > 1$ and $q(\cdot) \in \mathcal{A}_F$,

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |\phi^F(t, X(t))|^m \right] \\ & \leq K \mathbb{E} \left[\sup_{t \in [0, T]} e^{-\frac{1}{2}m^2\gamma_F^2 \int_0^t [h^F(s)q(s)\sigma(s)]^2 ds - m\gamma_F \int_0^t h^F(s)q(s)\sigma(s)dW(s)} \right] \\ & = K \mathbb{E} \left[\sup_{t \in [0, T]} |\Lambda(t)| \right] < \infty. \end{aligned} \tag{4.38}$$

This immediately implies that $\{\phi^F(t, X(t)) | t \in [0, T]\}$ is uniformly integrable.

On the other hand, it follows from the definition of $\Pi(t)$ in the proof of Proposition 4.1 that

$$|\Pi(t)| \leq K|\phi^F(t, X(t))|(1 + |Z^F(t)|^2). \tag{4.39}$$

Then, we derive

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \Pi(s) ds \right)^2 \right] \\ & \leq K \mathbb{E} \left[\left(\int_0^T |\phi^F(s, X(s))|(1 + |Z^F(s)|^2) ds \right)^2 \right] \\ & \leq K \mathbb{E} \left[\sup_{s \in [0, T]} |\phi^F(s, X(s))|^2 \left(1 + \int_0^T |Z^F(s)|^2 ds \right)^2 \right] \\ & \leq K \left\{ 1 + \mathbb{E} \left[\sup_{s \in [0, T]} |\phi^F(s, X(s))|^4 \right] + \mathbb{E} \left[\left(\int_0^T |Z^F(s)|^2 ds \right)^4 \right] \right\} \\ & \leq K \left\{ 1 + \mathbb{E} \left[\sup_{s \in [0, T]} |\phi^F(s, X(s))|^4 \right] + \|Z^F(\cdot)\|_{\mathcal{H}_{\text{BMO}}^2}^8 \right\} < \infty. \end{aligned} \tag{4.40}$$

Here, we have used the energy inequality for BMO martingales to proceed from the second last line to the last one. Setting $m = 2$ in (4.38) and combining with the square integrability of $\Pi(\cdot)$, we can easily obtain that $\phi^F(\cdot, X(\cdot)) - \int_0^\cdot \Pi(s) ds$ is a square-integrable martingale. Thus, the martingale representation theorem guarantees the existence of a unique square-integrable process $\Theta^F(\cdot) := \{\Theta^F(t) | t \in [0, T]\}$ such that

$$\phi^F(t, X(t)) - \int_0^t \Pi(s) ds = \phi^F(0, x_0) + \int_0^t \Theta^F(s) dW(s). \tag{4.41}$$

Recalling (4.9), we can rewrite it in the integral form as follows:

$$\begin{aligned} \phi^F(t, X(t)) - \phi^F(0, x_0) &= \int_0^t \Pi(s)ds + \int_0^t \Theta^F(s, X(s))dW(s) \\ &= \int_0^t \Theta^F(s, X(s))dW(s) - \int_0^t (-\Pi(s))ds, \end{aligned} \tag{4.42}$$

where

$$\Theta^F(s, X(s)) := \exp \left[-\gamma_F(h^F(s)X_F(s) + Y^F(s)) \right] [h^F(s)\sigma(s)q(s) - Z^F(s)].$$

By (4.10), we know that $\Pi(t) \leq 0$, for all $t \in [0, T]$, thereby $\{\int_0^t (-\Pi(s))ds | t \in [0, T]\}$ is an increasing process with zero being its initial value. Moreover, it can be shown as (4.40) that

$$\begin{aligned} &\mathbb{E} \left[\int_0^T |\Theta^F(s, X(s))|^2 ds \right] \\ &\leq K \mathbb{E} \left[\int_0^T |\phi^F(s, X(s))|^2 (1 + |Z^F(s)|^2) ds \right] \\ &\leq K \left\{ 1 + \mathbb{E} \left[\sup_{s \in [0, T]} |\phi^F(s, X(s))|^4 \right] + \mathbb{E} \left[\left(\int_0^T |Z^F(s)|^2 ds \right)^2 \right] \right\} \\ &\leq K \left\{ 1 + \mathbb{E} \left[\sup_{s \in [0, T]} |\phi^F(s, X(s))|^4 \right] + \|Z^F(\cdot)\|_{\mathcal{H}_{\text{BMO}}^2}^4 \right\} < \infty. \end{aligned} \tag{4.43}$$

That is, $\{\Theta^F(t, X(t)) | t \in [0, T]\}$ is a square-integrable process. Using the existence and uniqueness of $\Theta^F(\cdot)$ in the martingale representation (4.41) and comparing (4.41) with (4.42), we obtain that $\Theta^F(t) = \Theta^F(t, X(t))$ is uniquely determined. Moreover, as $X(\cdot)$ and $Y^F(\cdot)$ are càdlàg processes, $\phi^F(\cdot, X(\cdot))$ is also a càdlàg process. Combining this observation with the uniform integrability of $\{\phi^F(t, X(t)) | t \in [0, T]\}$, we know that the supermartingale $\{\phi^F(t, X(t)) - \phi^F(0, x_0) | t \in [0, T]\}$ is of Class (D). It then follows from the Doob–Meyer decomposition theorem that the increasing process $\{\int_0^t (-\Pi(s))ds | t \in [0, T]\}$ is unique (up to indistinguishability) in the Doob–Meyer decomposition (4.42).

Because $\phi^F(t, x)$ defined in (4.32) is the value function of the insurer’s problem, it is the first component of the solution to the stochastic HJB equation (3.1), i.e., $\Phi^F(t, x) = \phi^F(t, x)$. Then, we apply Itô–Kunita’s formula to $\Phi^F(t, X(t))$

and get

$$\begin{aligned}
 &\phi^F(t, X(t)) - \phi^F(0, x_0) \\
 &= \Phi^F(t, X(t)) - \Phi^F(0, x_0) \\
 &= \int_0^t [\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi^F(s, X(s))]dW(s) - \int_0^t (-\Pi'(s))ds,
 \end{aligned}
 \tag{4.44}$$

where

$$\begin{aligned}
 \Pi'(s) := & - \sup_{q(\cdot) \in \mathcal{A}_F} [H(s, X(s), p(s), q(s), \Phi_x^F(s, X(s)), \Phi_{xx}^F(s, X(s)), \Psi_x^F(s, X(s)))] \\
 & + H(s, X(s), p(s), q(s), \Phi_x^F(s, X(s)), \Phi_{xx}^F(s, X(s)), \Psi_x^F(s, X(s))).
 \end{aligned}
 \tag{4.45}$$

Note that $\{\int_0^t (-\Pi'(s))ds | t \in [0, T]\}$ is also an increasing process with zero as the initial value and (4.44) is also a Doob–Meyer decomposition of $\{\phi^F(t, X(t)) - \phi^F(0, x_0) | t \in [0, T]\}$. Thanks to the uniqueness of the increasing process in the Doob–Meyer decomposition (4.42), we obtain that $\int_0^t \Pi'(s)ds = \int_0^t \Pi(s)ds, dt \otimes d\mathbb{P}$ -a.e.. So, replacing $\int_0^t \Pi'(s)ds$ by $\int_0^t \Pi(s)ds$ in equation (4.44) leads to

$$\begin{aligned}
 &\phi^F(t, X(t)) - \phi^F(0, x_0) - \int_0^t \Pi(s)ds \\
 &= \int_0^t [\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi^F(s, X(s))]dW(s).
 \end{aligned}
 \tag{4.46}$$

By using the Burkholder–Davis–Gundy (BDG) inequality, we deduce

$$\begin{aligned}
 &k \cdot \mathbb{E} \left[\int_0^T |\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi^F(s, X(s))|^2 ds \right] \\
 &\leq \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t [\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi^F(s, X(s))]dW(s) \right|^2 \right],
 \end{aligned}
 \tag{4.47}$$

where k is a universal positive constant in the BDG inequality. On the other hand, it follows from the inequality of arithmetic and geometric means, (4.38)

and (4.40) that

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left| \phi^F(t, X(t)) - \phi^F(0, x_0) - \int_0^t \Pi(s) ds \right|^2 \right] \\ & \leq K \left\{ 1 + \mathbb{E} \left[\sup_{t \in [0, T]} |\phi^F(t, X(t))|^2 \right] + \mathbb{E} \left[\left(\int_0^T \Pi(s) ds \right)^2 \right] \right\} < \infty, \end{aligned} \tag{4.48}$$

where $K := 3(1 \vee \|\phi^F(0, x_0)\|_\infty^2)$ is a positive finite constant, which is guaranteed by equation (4.32) and the boundedness of $Y^F(\cdot)$. Here, $\|\cdot\|_\infty$ denotes the essential supremum under \mathbb{P} . Note that the right-hand side of (4.47) and the left-hand side of (4.48) are identical (see equation (4.46)). Thus, combining (4.47) with (4.48) leads to

$$\begin{aligned} & \mathbb{E} \left[\int_0^T |\Phi_x^F(s, X(s))^\top D(s, q(s)) + \Psi^F(s, X(s))|^2 ds \right] \\ & \leq \frac{K}{k} \left\{ 1 + \mathbb{E} \left[\sup_{t \in [0, T]} |\phi^F(t, X(t))|^2 \right] + \mathbb{E} \left[\left(\int_0^T \Pi(s) ds \right)^2 \right] \right\} < \infty, \end{aligned} \tag{4.49}$$

that is, $\Phi_x^F(\cdot, X(\cdot))^\top D(\cdot, q(\cdot)) + \Psi^F(\cdot, X(\cdot)) \in \mathcal{L}_{\mathbb{P}, \mathbb{P}}^2(0, T; \mathbb{R})$. It again follows from the uniqueness of $\Theta^F(\cdot)$ in the martingale representation (4.41) that

$$\Theta^F(t) = \Theta^F(t, X(t)) = \Phi_x^F(t, X(t))^\top D(t, q(t)) + \Psi^F(t, X(t)), \quad dt \otimes d\mathbb{P}\text{-a.e.} \tag{4.50}$$

Obviously,

$$\begin{aligned} \Phi_x^F(t, X(t))^\top D(t, q(t)) &= \phi_x^F(t, X(t))^\top D(t, q(t)) \\ &= \exp \left[-\gamma_F(h^F(t) X_F(t) + Y^F(t)) \right] h^F(t) \sigma(t) q(t), \end{aligned} \tag{4.51}$$

and this ensures that

$$\begin{aligned} \Psi^F(t, X(t)) &= \Theta^F(t, X(t)) - \Phi_x^F(t, X(t))^\top D(t, q(t)) \\ &= -\exp \left[-\gamma_F(h^F(t) X_F(t) + Y^F(t)) \right] Z^F(t) = \psi^F(t, X(t)), \end{aligned} \tag{4.52}$$

is uniquely determined. Making substitution of $\Phi^F(t, X(t))$ and $\Psi^F(t, X(t))$ in (4.45), we can further confirm that $\Pi'(t) = \Pi(t)$. Consequently, $(\Phi^F(t, x), \Psi^F(t, x)) = (\phi^F(t, x), \psi^F(t, x))$ constitutes a unique solution pair to the stochastic HJB equation (3.1).

The existence and uniqueness of the solution to the stochastic HJB equation (3.9) can be proved similarly, and we do not repeat the proof here. ■

Remark 4.3. As the explicit solutions to the stochastic HJB equations (3.1) and (3.9) have been obtained in Proposition 4.6, an alternative approach to verifying the optimality is to use Propositions 3.1 and 3.2 directly. In fact, we only need to check the regularity conditions in these two propositions. The explicit expressions of $(\Phi^F(t, x), \Psi^F(t, x))$ and $(\Phi^L(t, x), \Psi^L(t, x))$ and the properties of the solutions to BSDEs (4.3) and (4.22) allow us to confirm the regularity conditions easily.

5. SOLUTION TO EXPONENTIAL UTILITY CASE WITH CONSTANT COEFFICIENTS

In this section, we assume that all model coefficients are deterministic and are given by positive constants. More specifically, we assume that for any $(t, \omega) \in [0, T] \times \Omega$,

$$a(t) = a > 0, \quad \sigma(t) = \sigma > 0, \quad \theta(t) = \theta > 0, \\ \eta(t) = \eta > 0, \quad \rho_F(t) = \rho_F > 0, \quad \rho_L(t) = \rho_L > 0.$$

In this case, the risk model reduces to the original diffusion approximation model proposed by Grandell (1990). Now, the lower and upper bounds of the the reinsurance premium are also given by positive constants $c := (1 + \theta)a$ and $g := (1 + \eta)a$, respectively. So, the respective domains for the reinsurance strategy $q(\cdot)$ and the reinsurance premium strategy $p(\cdot)$ become $Q = [0, 1]$ and $P = [c, g]$.

Let us first introduce an auxiliary admissible set of reinsurance premium strategies.

Definition 5.1. *The auxiliary admissible set $\bar{\mathcal{A}}_L$ is the space of all deterministic reinsurance premium strategies such that $p(\cdot) \in \mathcal{A}_L$.*

To simplify our presentation, we denote

$$M(t) := \frac{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 1}{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 2}, \quad N^\theta(t) := \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}}, \quad N^\eta(t) := \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}},$$

and

$$O_1 := \{t : \frac{p(t)-a}{\sigma^2} \leq \gamma_F e^{\rho_F(T-t)}\}, \quad O_2 := \{t : \frac{p(t)-a}{\sigma^2} > \gamma_F e^{\rho_F(T-t)}\}, \quad \bar{O}_1 := \{t : M(t) < N^\theta(t)\}, \\ \bar{O}_2 := \{t : N^\theta(t) \leq M(t) < N^\eta(t)\}, \quad \bar{O}_3 := \{t : N^\eta(t) \leq M(t)\}, \quad \bar{O}_4 := \{t : N^\theta(t) \geq 1\}.$$

Before solving the game problem, we present closed-form solutions to the BSDEs (4.3) and (4.22) in the next proposition, which also serves to validate

that it is adequate to search for the optimal reinsurance premium strategy within $\bar{\mathcal{A}}_L$.

Proposition 5.1. *Suppose that the model coefficients are constant. Then the unique solutions to the BSDEs (4.3) and (4.22) are given by*

$$\begin{aligned}
 Y^F(t) = & \int_t^T \left\{ \left[-e^{\rho_F(T-s)}((p(s) - a) - \theta a) + \frac{1}{2} \frac{(p(s) - a)^2}{\gamma_F \sigma^2} \right] I_{O_1}(s) \right. \\
 & \left. + \left[\theta a e^{\rho_F(T-s)} - \frac{1}{2} \gamma_F \sigma^2 e^{2\rho_F(T-s)} \right] I_{O_2}(s) \right\} ds, \tag{5.1}
 \end{aligned}$$

$$\begin{aligned}
 Y^L(t) = & \int_t^T \left\{ \left[-\frac{1}{2} \gamma_L \sigma^2 e^{2\rho_L(T-s)} \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right)^2 \right. \right. \\
 & \left. \left. + e^{\rho_L(T-s)} \theta a \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right) \right] I_{\bar{O}_1}(s) \right. \\
 & \left. + \frac{\sigma^2}{2\gamma_L} \left[\frac{1}{(\gamma_F e^{\rho_F(T-s)})^2} + \frac{2}{\gamma_L \gamma_F e^{(\rho_L + \rho_F)(T-s)}} \right] I_{\bar{O}_2}(s) \right. \\
 & \left. + \left[-\frac{1}{2} \gamma_L \sigma^2 e^{2\rho_L(T-s)} \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right)^2 \right. \right. \\
 & \left. \left. + e^{\rho_L(T-s)} \eta a \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right) \right] I_{\bar{O}_3}(s) \right\} ds, \tag{5.2}
 \end{aligned}$$

and

$$Z^F(t) = 0, \quad Z^L(t) = 0, \quad \forall t \in [0, T], \tag{5.3}$$

where I_E is the indicator function of E , for $E := O_1, O_2, \bar{O}_1, \bar{O}_2, \bar{O}_3$.

Proof. Note that it has been shown in Propositions 4.2 and 4.4 that the BSDEs (4.3) and (4.22) have unique solutions in the solution space $\mathcal{S}_{\mathbb{F}, \mathbb{P}}^\infty(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}, \mathbb{P}}^{2, \text{BMO}}(0, T; \mathbb{R})$. Therefore, once we find an explicit solution pair to each of (4.3) and (4.22) in the solution space, the proof is completed.

Since the drivers of the two BSDEs have only deterministic coefficients, we conjecture that $Z^F(t) = 0$ and $Z^L(t) = 0$. Then, the drivers become

$$f^F(t, 0) = \max_{q \in \mathcal{Q}} \left\{ h^F(t) [\theta a - (1 - q)(p(t) - a)] - \frac{1}{2} \gamma_F (h^F(t))^2 \sigma^2 q^2 \right\}, \tag{5.4}$$

and

$$f^L(t, 0) = \max_{p \in P} \left\{ -\frac{1}{2} \gamma_L (h^L(t))^2 \sigma^2 (1 - \alpha^*(t, p))^2 + h^L(t) (1 - \alpha^*(t, p)) (p - a) \right\}, \tag{5.5}$$

where

$$\alpha^*(t, p) = \arg \max_{q \in Q} \left\{ h^F(t) [\theta a - (1 - q)(p - a)] - \frac{1}{2} \gamma_F (h^F(t))^2 \sigma^2 q^2 \right\}. \tag{5.6}$$

In fact, we can express $\alpha^*(t, p)$ more explicitly as

$$\alpha^*(t, p) = \frac{1}{\gamma_F e^{\rho_F(T-t)}} \frac{p - a}{\sigma^2} \cdot I_{O_1}(t) + 1 \cdot I_{O_2}(t). \tag{5.7}$$

Thus, $f^F(t, 0)$ can be further simplified as

$$f^F(t, 0) = - \left[e^{\rho_F(T-t)} ((p(t) - a) - \theta a) - \frac{1}{2} \frac{(p(t) - a)^2}{\gamma_F \sigma^2} \right] I_{O_1}(t) - \left[-\theta a e^{\rho_F(T-t)} + \frac{1}{2} \gamma_F \sigma^2 e^{2\rho_F(T-t)} \right] I_{O_2}(t). \tag{5.8}$$

On the other hand, substituting (5.6) into (5.5) gives

$$p^*(t) = \arg \max_{p \in P} \left\{ -\frac{1}{2} \gamma_L (h^L(t))^2 \sigma^2 (1 - \alpha^*(t, p))^2 + h^L(t) (1 - \alpha^*(t, p)) (p - a) \right\} = (1 + \theta)a \cdot I_{\tilde{O}_1}(t) + \left[\gamma_F \sigma^2 e^{\rho_F(T-t)} \frac{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 1}{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 2} + a \right] \cdot I_{\tilde{O}_2}(t) + (1 + \eta)a \cdot I_{\tilde{O}_3}(t) \tag{5.9}$$

and

$$f^L(t, 0) = \left[-\frac{1}{2} \gamma_L \sigma^2 e^{2\rho_L(T-t)} \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}} \right)^2 + e^{\rho_L(T-t)} \theta a \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}} \right) \right] I_{\tilde{O}_1}(t) + \frac{\sigma^2}{2\gamma_L} \left[\frac{1}{(\gamma_F e^{\rho_F(T-t)})^2} + \frac{2}{\gamma_L \gamma_F e^{(\rho_L + \rho_F)(T-t)}} \right] I_{\tilde{O}_2}(t)$$

$$\begin{aligned}
 & + \left[-\frac{1}{2} \gamma_L \sigma^2 e^{2\rho_L(T-t)} \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}} \right)^2 \right. \\
 & \left. + e^{\rho_L(T-t)} \eta a \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}} \right) \right] I_{\bar{D}_3}(t). \tag{5.10}
 \end{aligned}$$

Clearly, the first components $Y^F(\cdot)$ and $Y^L(\cdot)$ are given by in (5.1) and (5.2), respectively. Obviously, both $(Y^F(\cdot), Z^F(\cdot))$ and $(Y^L(\cdot), Z^L(\cdot))$ defined by (5.1)–(5.3) are in $S_{\mathbb{F}, \mathbb{P}}^\infty(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}, \mathbb{P}}^{2, \text{BMO}}(0, T; \mathbb{R})$. Therefore, we can conclude that (5.1)–(5.3) constitute unique solutions to the BSDEs (4.3) and (4.22). ■

Having obtained explicit unique solutions to (4.3) and (4.22), we are ready to give the solution to the Stackelberg game in the following proposition:

Proposition 5.2. *i. If $M(t) < N^\theta(t)$ and $N^\theta(t) \geq 1$, then the optimal strategies are given by any reinsurance premium strategy $p(\cdot) \in \mathcal{A}_L$ and*

$$q^*(t) = 1;$$

ii. If $M(t) < N^\theta(t) < 1$, then the optimal strategies are given by

$$p^*(t) = (1 + \theta)a, \tag{5.11}$$

and

$$q^*(t) = \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}}; \tag{5.12}$$

iii. If $N^\theta(t) \leq M(t) < N^\eta(t)$, the optimal strategies are given by

$$p^*(t) = \gamma_F \sigma^2 e^{\rho_F(T-t)} \frac{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 1}{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 2} + a, \tag{5.13}$$

and

$$q^*(t) = \frac{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 1}{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 2}; \tag{5.14}$$

iv. If $N^\eta(t) \leq M(t)$, then the optimal strategies are given by

$$p^*(t) = (1 + \eta)a, \tag{5.15}$$

and

$$q^*(t) = \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}}. \tag{5.16}$$

The value functions are given by

$$\begin{aligned}
 V^L(t, x) = & -\frac{1}{\gamma_L} \exp \left\{ -\gamma_L x_L e^{\rho_L(T-t)} \right. \\
 & + \int_t^T \left[\frac{1}{2} \gamma_L^2 \sigma^2 e^{2\rho_L(T-s)} \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right)^2 - \gamma_L e^{\rho_L(T-s)} \theta a \right. \\
 & \quad \times \left. \left. \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right) \right] I_{\bar{\delta}_1 \cap \bar{\delta}_4^c}(s) ds \right. \\
 & - \int_t^T \frac{\sigma^2}{2} \left[\frac{1}{(\gamma_F e^{\rho_F(T-s)})^2} + \frac{1}{\gamma_L \gamma_F e^{(\rho_L + \rho_F)(T-s)}} \right] I_{\bar{\delta}_2}(s) ds \\
 & + \int_t^T \left[\frac{1}{2} \gamma_L^2 \sigma^2 e^{2\rho_L(T-s)} \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right)^2 \right. \\
 & \quad \left. - \gamma_L e^{\rho_L(T-s)} \eta a \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right) \right] I_{\bar{\delta}_3}(s) ds \left. \right\}, \tag{5.17}
 \end{aligned}$$

and

$$\begin{aligned}
 V^F(t, x) = & -\frac{1}{\gamma_F} \exp \left\{ -\gamma_F x_F e^{\rho_F(T-t)} \right. \\
 & + \int_t^T \left[-\gamma_F \theta a e^{\rho_F(T-s)} + \frac{1}{2} \gamma_F^2 \sigma^2 e^{2\rho_F(T-s)} \right] I_{\bar{\delta}_4}(s) ds \\
 & - \int_t^T \frac{\theta^2 a^2}{2\sigma^2} I_{\bar{\delta}_1 \cap \bar{\delta}_4^c}(s) ds \\
 & + \int_t^T \gamma_F^2 \sigma^2 e^{2\rho_F(T-s)} \left[M(s) - N^\theta(s) - \frac{1}{2} M^2(s) \right] I_{\bar{\delta}_2}(s) ds \\
 & + \int_t^T \left[\gamma_F e^{\rho_F(T-s)} (\eta - \theta) a - \frac{\eta^2 a^2}{2\sigma^2} \right] I_{\bar{\delta}_3}(s) ds \left. \right\}. \tag{5.18}
 \end{aligned}$$

Proof. First, we can rewrite the maximizers (5.6) and (5.9) in the proof of the previous proportion as follows:

$$\alpha^*(t, p) = \frac{1}{\gamma_F h^F(t)} \frac{p - a}{\sigma^2} \wedge 1, \tag{5.19}$$

and

$$p^*(t) = \left[\gamma_F \sigma^2 h^F(t) \frac{\frac{\gamma_L h^L(t)}{\gamma_F h^F(t)} + 1}{\frac{\gamma_L h^L(t)}{\gamma_F h^F(t)} + 2} + a \right] \vee (1 + \theta) a \wedge (1 + \eta) a. \tag{5.20}$$

Then,

$$q^*(t) = \alpha^*(t, p^*(t)) = \left\{ \frac{\frac{\gamma_L h^L(t)}{\gamma_F h^F(t)} + 1}{\frac{\gamma_L h^L(t)}{\gamma_F h^F(t)} + 2} \vee \frac{\theta a}{\gamma_F \sigma^2 h^F(t)} \wedge \frac{\eta a}{\gamma_F \sigma^2 h^F(t)} \right\} \wedge 1. \tag{5.21}$$

Case 1^o is a special case. Indeed, if $M(t) < N^\theta(t)$, then

$$\gamma_F \sigma^2 e^{\rho_F(T-t)} \frac{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 1}{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 2} + a < (1 + \theta)a. \tag{5.22}$$

It looks that we should take $p^*(t) = (1 + \theta)a$ as the optimal reinsurance premium strategy, and

$$q^*(t) = \alpha^*(t, p^*(t)) = \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}} \wedge 1, \tag{5.23}$$

as the optimal reinsurance strategy. However, the condition $N^\theta(t) \geq 1$ implies that $q^*(t) = 1$, and thus, the reinsurer’s strategy no longer enters into the game problem. Particularly, the reinsurer’s surplus is now $X_L(t) = x_{0L} e^{\rho_L t}$. Therefore, any reinsurance premium $p(\cdot) \in \mathcal{A}_L$ is indifferent and hence optimal.

It remains to consider Cases 2^o–4^o, which correspond to $I_{\tilde{\mathcal{O}}_1 \cap \tilde{\mathcal{O}}_4^c}(t) = 1$, $I_{\tilde{\mathcal{O}}_2}(t) = 1$ and $I_{\tilde{\mathcal{O}}_3}(t) = 1$, respectively. Thus, for Cases 2^o–4^o (i.e., given $N^\theta(t) < 1$), equations (5.20) and (5.21) can be expressed by the indicator functions as

$$p^*(t) = (1 + \theta)a \cdot I_{\tilde{\mathcal{O}}_1 \cap \tilde{\mathcal{O}}_4^c}(t) + \left[\gamma_F \sigma^2 h^F(t) \frac{\frac{\gamma_L h^L(t)}{\gamma_F h^F(t)} + 1}{\frac{\gamma_L h^L(t)}{\gamma_F h^F(t)} + 2} + a \right] \cdot I_{\tilde{\mathcal{O}}_2}(t) + (1 + \eta)a \cdot I_{\tilde{\mathcal{O}}_3}(t) \tag{5.24}$$

and

$$q^*(t) = \alpha^*(t, p^*(t)) = \frac{\theta a}{\gamma_F \sigma^2 h^F(t)} \cdot I_{\tilde{\mathcal{O}}_1 \cap \tilde{\mathcal{O}}_4^c}(t) + \frac{\frac{\gamma_L h^L(t)}{\gamma_F h^F(t)} + 1}{\frac{\gamma_L h^L(t)}{\gamma_F h^F(t)} + 2} \cdot I_{\tilde{\mathcal{O}}_2}(t) + \frac{\eta a}{\gamma_F \sigma^2 h^F(t)} \cdot I_{\tilde{\mathcal{O}}_3}(t). \tag{5.25}$$

Obviously, when $I_{\tilde{\mathcal{O}}_1 \cap \tilde{\mathcal{O}}_4^c}(t) = 1$ (resp., $I_{\tilde{\mathcal{O}}_2}(t) = 1$ or $I_{\tilde{\mathcal{O}}_3}(t) = 1$), equations (5.24)–(5.25) reduce to the optimal strategies given in Case 2^o (resp., Case 3^o or 4^o).

To derive the value functions, we note

$$Y^F(t) = \int_t^T f^F(s, 0) ds, \quad Y^L(t) = \int_t^T f^L(s, 0) ds, \tag{5.26}$$

and plug them into (4.16) and (4.28)

$$\begin{aligned}
 V^F(t, x; p^*(\cdot)) = & -\frac{1}{\gamma_F} \exp \left\{ -\gamma_F x_F e^{\rho_F(T-t)} \right. \\
 & - \gamma_F \int_t^T \left[h^F(s) [\theta a - (1 - \alpha^*(s, p^*(s)))(p^*(s) - a)] \right. \\
 & \left. \left. - \frac{1}{2} \gamma_F (h^F(s))^2 \sigma^2 (\alpha^*(s, p^*(s)))^2 \right] ds \right\}, \tag{5.27}
 \end{aligned}$$

and

$$\begin{aligned}
 V^L(t, x) = & -\frac{1}{\gamma_L} \exp \left\{ -\gamma_L x_L e^{\rho_L(T-t)} \right. \\
 & - \gamma_L \int_t^T \left[-\frac{1}{2} \gamma_L (h^L(s))^2 \sigma^2 (1 - \alpha^*(s, p^*(s)))^2 \right. \\
 & \left. \left. + h^L(s) (1 - \alpha^*(s, p^*(s)))(p^*(s) - a) \right] ds \right\}. \tag{5.28}
 \end{aligned}$$

Finally, substituting the values of $p^*(t)$ and $q^*(t) = \alpha^*(t, p^*(t))$ in Cases 1^o–4^o into the above two expressions and by some algebraic calculation, we can see that the above two expressions match (5.18) and (5.17). The proof is completed. ■

Remark 5.1. When the Stackelberg equilibrium is achieved in the interior case (i.e., Case (iii)), the optimal reinsurance premium follows the variance premium principle. Indeed, for every one unit of risk, the total instantaneous reinsurance premium associated with the ceded proportion $100(1 - q^*(t))\%$ can be written as

$$(1 - q^*(t))p^*(t) = (1 - q^*(t))a + [\gamma_L e^{\rho_L(T-t)} + \gamma_F e^{\rho_F(T-t)}](1 - q^*(t))^2 \sigma^2, \tag{5.29}$$

where the first term accounts for the mean component, and the second for the variance component. Note that the premium principle for reinsurance considered in this paper is general. The Stackelberg game framework provides theoretical support that the variance premium principle is an ideal candidate among all possible premium principles when the proportional reinsurance is applied (see Chen et al. (2016) and the references therein). In fact, this finding has a duality relation of the well-known result in the literature, that is, given the variance premium principle, the proportional reinsurance is optimal among all types of reinsurance treaties.

It can be seen that the variance component weights heavier if

1. the insurer and the reinsurer are more risk averse;
2. the insurer and the reinsurer can earn higher risk-free rates of returns.

It is easy to understand the first point. The insurer and the reinsurer are averse to the uncertainty of the insurance claim, measured by σ . To explain the second point, we note that in nature the risk-free return is a return with certainty. Thus, the risk-free return has a crowding-out effect to the insurance risk. The increase of the risk-free return rates would make the uncertain insurance business less favorable, resulting in a higher cost of making reinsurance arrangement.

Remark 5.2. We observe that the maximizers in (5.4) and (5.5), i.e.,

$$\alpha^*(t, p) = \arg \max_{q \in Q} \left\{ h^F(t) [\theta a - (1 - q)(p - a)] - \frac{1}{2} \gamma_F \sigma^2 (h^F(t))^2 q^2 \right\}, \tag{5.30}$$

and

$$p^*(t) = \arg \max_{p \in P} \left\{ h^L(t) (1 - \alpha^*(t, p))(p - a) - \frac{1}{2} \gamma_L \sigma^2 (h^L(t))^2 (1 - \alpha^*(t, p))^2 \right\}, \tag{5.31}$$

are visibly deterministic functions. Thus, the optimal reinsurance premium strategy $p^*(\cdot)$ must be in $\bar{\mathcal{A}}_L$.

In fact, the game problem can be also discussed by the stochastic HJB equation approach. In particular, the second components of the unique solutions to the stochastic HJB equations vanish, i.e.,

$$\Psi^F(t, x) = - \exp \left[- \gamma_F (h^F(t) x_F + Y^F(t)) \right] Z^F(t) = 0, \tag{5.32}$$

and

$$\Psi^L(t, x) = - \exp \left[- \gamma_L (h^L(t) x_L + Y^L(t)) \right] Z^L(t) = 0. \tag{5.33}$$

Thus, the stochastic HJB equations (3.1) and (3.9) reduce to the HJB PDEs (A.8) and (A.18) presented in Appendix A. This also implies that the optimal strategies live in $\bar{\mathcal{A}}_L \times \mathcal{A}_F$. In both the BSDE and stochastic HJB equation approaches, the admissible set \mathcal{A}_L is in a very general form. Even in this general form, it is found that the optimal reinsurance strategy $p^*(\cdot)$ is deterministic. This validates that when model coefficients are constant/deterministic and utilities are given by exponential functions, the optimal strategies in $\bar{\mathcal{A}}_L \times \mathcal{A}_F$ are also optimal in $\mathcal{A}_L \times \mathcal{A}_F$ for the game problem and vice versa. In Appendix A, we will apply the HJB PDE approach to discuss the game problem in the admissible set $\bar{\mathcal{A}}_L \times \mathcal{A}_F$, which is equivalent to that in the original admissible set $\mathcal{A}_L \times \mathcal{A}_F$.

6. NUMERICAL EXAMPLES

In this section, we illustrate our results in the preceding section with several numerical examples, where the insurer and the reinsurer have exponential utilities

and all the model coefficients are constant. To this end, we fix the following parameter values as our benchmark:

$$t = 0, \quad T = 3, \quad a = 10, \quad \sigma = 3, \quad \theta = 0.35, \quad \eta = 0.45, \\ x_F = 10, \quad x_L = 30, \quad \rho_F = 0.1, \quad \rho_L = 0.1, \quad \gamma_F = 0.5, \quad \gamma_L = 0.5.$$

As the reinsurer is the leader of the game and possesses the dominate power, the wealth of the reinsurer is chosen to be much bigger than that of the insurer. And, a is more than three times of σ so that the probability of ruin is very small. In each of the following figures, we vary the value of one parameter and examine the sensitivity of optimal solutions with respect to the change of that parameter. The purpose of our numerical examples is mainly to show the respective roles of the insurer and the reinsurer in achieving the optimal reinsurance agreement. For this reason, we only present the effects of the insurer's and reinsurer's subjective and characteristic parameters, including ρ_F , ρ_L , γ_F and γ_L , on their own and counterparties' optimal strategies and value functions. Though the sensitivity analyses of other parameters may be also interesting, they are of less value to understand the interactive roles of the insurer and the reinsurer in the Stackelberg game for optimal reinsurance.

In Figures 1 and 2, we show how the parameters γ_F , γ_L , ρ_F and ρ_L affect the optimal strategies p^* and q^* , and use four different lines to depict the trends of p^* and q^* in Cases (i)–(iv). These four cases are numerical illustrations of Section 5, and exactly correspond to Cases (i)–(iv) in Proposition 5.2. First, we show the effects of the parameters γ_F and γ_L on the optimal premium strategy p^* and the optimal reinsurance strategy q^* at $t = 0$. Since γ_F represents the insurer's risk aversion parameter, a larger γ_F corresponds to an insurer that is more risk-averse. As γ_F grows, the insurer tends to reduce its risk exposure by decreasing the retention level and increasing the ceding level of insurance risk. This can be seen clearly in Figure 1(b). Moreover, we can observe from Figure 1(b) that the insurer is willing to withhold all insurance risk when γ_F is below 0.3 approximately. On the other hand, Figure 1(a) reveals that at this circumstance the reinsurer is unable to attract the insurer to transfer the risk even if the reinsurance premium is reduced to its lower limit, that is, the cheap reinsurance is applied; the insurer is indifferent about the reinsurance premium and will not pursue any reinsurance protection. In this case, the Stackelberg equilibrium cannot be achieved. Once γ_F is raised above 0.3 approximately, the insurer cannot fully afford the insurance risk and is opt to cede the partially unaffordable risk to the reinsurer. By observing the insurer's optimal strategy, the reinsurer would know that the insurer will cede more insurance risk through reinsurance when the insurer becomes more risk averse. Therefore, when γ_F is slightly above 0.3, the reinsurer will choose a cheap reinsurance (see Case (ii) in Figure 1(a)) to attract the insurer to cede insurance risk. Doing so is obviously optimal from the reinsurer's point of view, since the reinsurer is not bearing any risk and so is keen to undertake the ceded risk at this stage. As γ_F increases further, the proportion of the transferred insurance risk grows higher and the

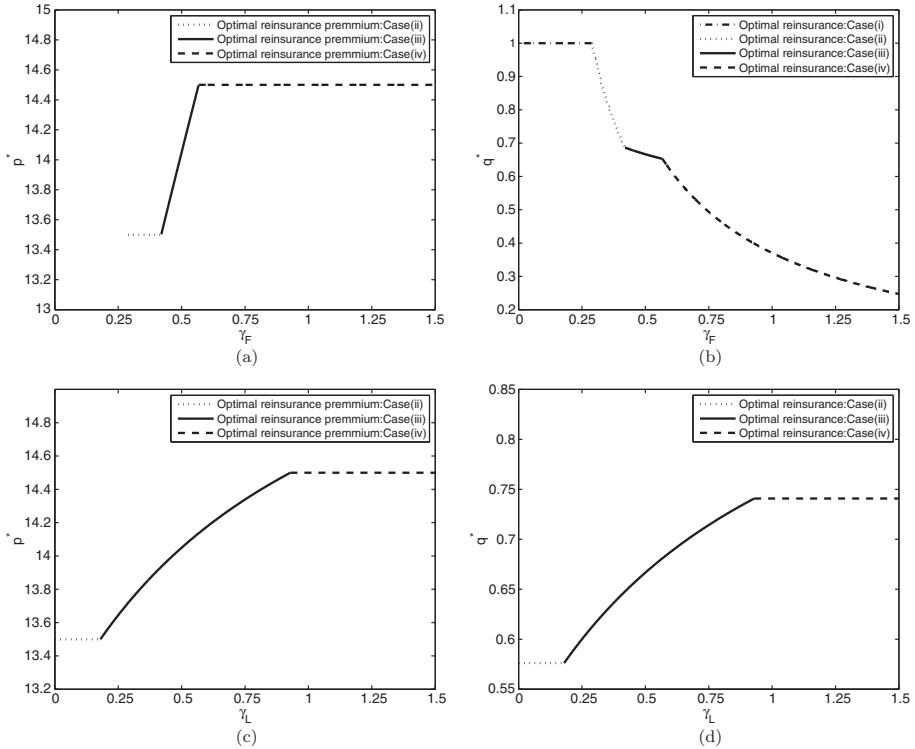


FIGURE 1: Effects of γ_F and γ_L on the optimal strategy (p^*, q^*) .

reinsurer, taking into account its own risk-taking capacity, will increase the reinsurance premium accordingly at the appropriate time (refer to the change points in Figure 1(a)). The increased reinsurance premium discourages the insurer to transfer risk. In fact, after the reinsurance premium starts to increase from the cheap one, the growth rate of the transferred risk proportion is slowed down, as shown from Case (ii) to Case (iii) in Figure 1(b). When the premium reaches the upper bound, the reinsurer is no longer permitted to increase it. After this change point, the growth rate of the insurer's ceded risk is fueled up, and as the insurer becomes even more risk averse, the proportion of retained risk dives to 0. Figure 1(c) and (d) depicts the responses of p^* and q^* to the reinsurer's risk aversion parameter γ_L . An upward trend can be observed in the optimal reinsurance premium with respect to the increment of risk aversion parameter γ_L . Clearly, the upper and lower limits make the reinsurance premium like one segment of staircases. The growing reinsurance premium drives the insurer to raise the retained proportion of risk because its degree of risk aversion is invariant to the change of γ_L . Moreover, we can see that the trend of the optimal reinsurance strategy q^* with respect to γ_L copies that of the optimal reinsurance premium p^* . This is in accordance with our intuition. As the reinsurer becomes more risk

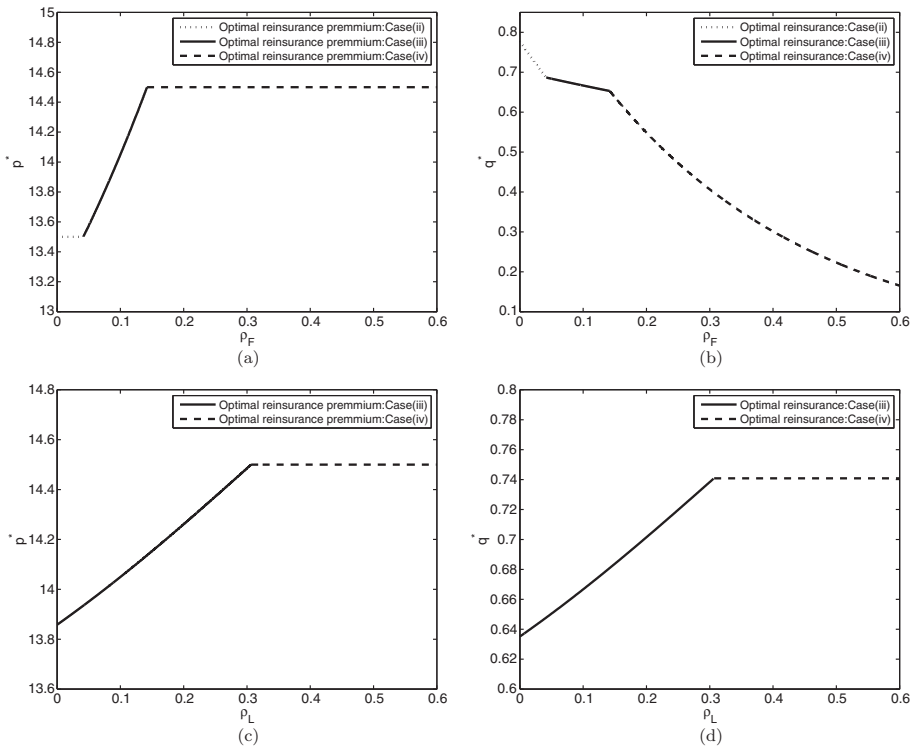


FIGURE 2: Effects of ρ_F and ρ_L on the optimal strategy (p^*, q^*) .

averse, it is less willing to undertake the ceded risk and thereby increases the reinsurance premium. The increased reinsurance premium makes it more costly for the insurer to transfer the same proportion of risk. To offset the increased cost of managing insurance risk, the insurance company will certainly retain more risk and thereby transfer less risk. This is consistent with Figure 1(d) since the optimal strategy q^* represents the insurer’s retained proportion of risk, while $1 - q^*$ is the insurer’s ceded proportion of risk. It is interesting to note that the increases of both γ_F and γ_L lead to hiking the reinsurance premium; however, the former makes the insurer cede more insurance risk, while the latter forces the insurer to retain more. This implies that due to their different roles, the insurer’s and the reinsurer’s risk aversion parameters have different impacts on the game.

In Figure 2(a)–(d), we show the impacts of risk-free interest rates ρ_F and ρ_L on p^* and q^* . First of all, we analyze the insurer’s response to the increase of its own risk-free interest rate ρ_F (refer to Figure 2(b)). On the one hand, when ρ_F becomes larger, from the insurer’s perspective, not only the credit interest rate (for positive surplus) but also the debit one (for negative surplus) increases. The latter corresponds to a higher borrowing cost. If the aggregate claim, i.e., $C(t)$, is sufficiently large, the surplus may become negative, which is more likely when

a higher proportion of risk is retained. In this circumstance, the insurer has to borrow money to mitigate insolvency; however, the higher cost would force the insurer to retain less insurance risk to avoid excessive borrowing. Although a positive surplus would also bring a higher return as credit interest, this cannot compensate the potential increased borrowing cost associated with a negative surplus since the insurer is risk averse. Therefore, the net effect for the insurer is decreasing risk retention to fight against the adverse situation with the potential negative surplus. In this regard, Figure 2(b) is consistent with the literature on optimal reinsurance from the insurer's perspective only (see, e.g., Figure 1(c) in Zeng *et al.* (2013) and Figure 1 in Li *et al.* (2017)). On the other hand, rather than a form of consumption, buying reinsurance protection/retaining insurance risk is more like a form of risky investment. This is because the risk retention strategy enters into not only the drift (return) of the wealth, but also the diffusion (risk) of the wealth; it plays essentially the same role as an investment strategy, which rebalances the return and risk of the wealth simultaneously. As all the coefficients are constant, the optimal strategy q^* is pretty much like Merton's myopic portfolio strategy under an exponential utility. Moreover, since investing in the risk-free asset is an alternative to retaining insurance risk, i.e., "risky investment", and provides a more attractive return, the insurer will naturally shift investment from the risky insurance business to the risk-free asset. Since the insurer is inclined to reduce its own risk retention by ceding more risk, the reinsurer, as the leader of the game and with a fixed risk appetite, will naturally increase the reinsurance premium (refer to Figure 2(a)). This is the only action the reinsurer can take to discourage the insurer to transfer more proportion of risk. In fact, by charging a higher reinsurance premium, the reinsurer manages to slow the decreasing rate of the insurer's retained risk; however, once the reinsurance premium hits its upper limit, the insurer accelerates transferring risk to the reinsurer. It is clear that the insurer's activity is not only affected by its own risk-free rate, but also is led by the reinsurer. The economic rationale underneath the insurer's and the reinsurer's decisions is two-fold: (1) from the insurer's perspective, investment return with certainty would be preferred over uncertain investment, i.e., assuming insurance risk, unless the uncertain investment compensates the insurer with a much higher return rate; (2) from the reinsurer's perspective, adjusting the price of reinsurance (i.e., reinsurance premium) is a measure to restrain the insurer's demand for reinsurance. These implications are in accordance with our economic/financial intuition: (1) investors always prefer a certain dollar to an uncertain one; (2) the law of demand states that conditional on all else being equal (after ρ_F has been changed), as the price of a good increases, quantity demanded decreases. Figure 2(c) reveals that when the reinsurer can earn a higher rate of risk-free return at ρ_L , it tends to reduce its exposure to potential insurance risk by raising the reinsurance premium. As shown in Figure 2(d), the insurer, as the follower of the game, is then forced to retain more risk. It is worthwhile to mention that not all cases of Proposition 5.2 are present in Figures 1 and 2. This is because which case is in effect is determined by a combination of several parameters. Therefore, with the value of

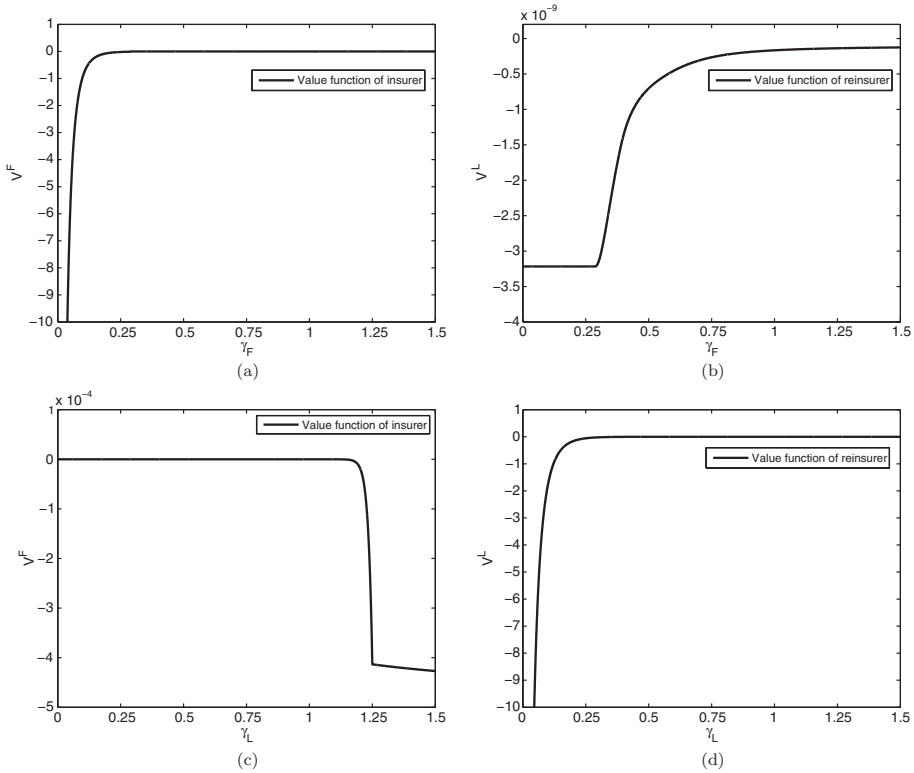


FIGURE 3: Effects of γ_F and γ_L on the value function.

only one parameter changed and those of other parameters fixed, some cases of optimal strategies are absent in these figures.

In Figure 3(a)–(d), we report the effects of the risk aversion parameters γ_F and γ_L on the value functions V^F and V^L . It can be seen from Figure 3(a) and (d) that the value functions V^F and V^L are concave in γ_F and γ_L , respectively, and they tend to zero as γ_F and γ_L increase. These properties are in line with and, indeed, are inherited from those of exponential utilities since for any positive initial surpluses, U_L and U_F are concave and increasing in γ_F and γ_L , respectively. However, as shown by Figure 3(b) and (c), the value functions are irregular to the changes on the parameters of the other parties, and remains constant (thereby not strictly monotone) at the beginning. The reason may lie in the constraints imposed on the insurer’s and reinsurer’s strategies, which result in the change points in the strategies (see Figure 1(a)–(d)). This reveals that the counterparties’ features have more complicated effects on the game between the insurer and the reinsurer. Ignoring the impact of the counterparties’ risk aversion on the game may misguide the insurer and the reinsurer to execute non-optimal reinsurance and reinsurance premium strategies.

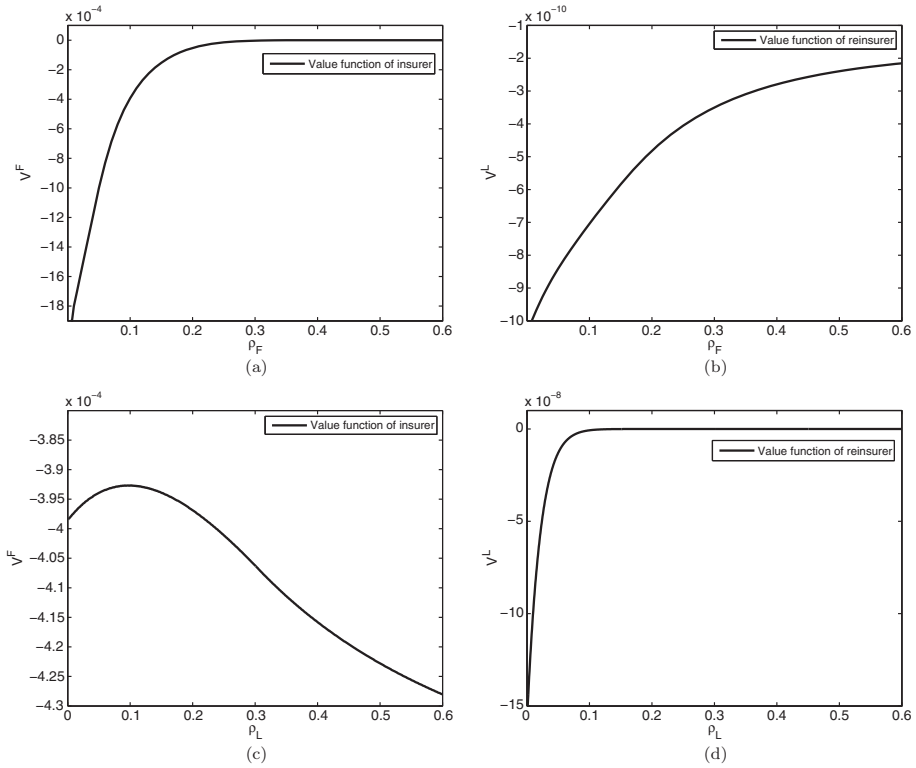


FIGURE 4: Effects of ρ_F and ρ_L on the value function.

The effects of the parameters ρ_F and ρ_L on the value functions are described in Figure 4(a)–(d). The higher rates of risk-free returns are more favorable for both the insurer and the reinsurer. Increasing ρ_F and ρ_L enhances their risk-free investment returns and encourages them to take less risks, which reduces the uncertainty of their strategies. This further elevates their own value functions and makes them tend to zero. This phenomenon is consistent with the properties of exponential utility functions. Indeed, decreasing the uncertainty of strategies is similar to increasing certainty equivalents for the insurer and the reinsurer. Therefore, in consideration of a higher ρ_F (resp., ρ_L), the insurer (resp., the reinsurer) would choose a less uncertain strategy and act as if with a greater initial surplus, which results in a higher level at the value function. On the other hand, the effects of changes in ρ_F and ρ_L are of different patterns on the counterparties' value functions. When ρ_F is increased, the reinsurer can achieve a higher value in V_L ; however, when ρ_L becomes bigger, the insurer's value function V_F increases only for a short period and eventually decreases to a lower level. This observation is interesting and reflects the different roles of the insurer and the reinsurer. Knowing that more risk will be transferred from the insurer when ρ_F becomes bigger, the reinsurer will increase the reinsurance premium to manage

its own risk exposure (see Figure 2(a)–(b)). This in turn would force the insurer to cede a lower proportion of risk to the reinsurer. Guaranteed by the reinsurer’s leadership role, such a series of interactions results in a higher value function for the reinsurer. On the contrary, when ρ_L is increased, the reinsurer tends to take less risk by charging a higher reinsurance premium, which compels the insurer to retain a greater proportion of risk (refer to Figure 2(c)–(d)). As a follower of the game, the insurer has to accept the situation of a decreased value function; unlike the reinsurer, the insurer has no dominant power in the game to improve its own value function when its opponent’s risk-free return rate increases.

Admittedly, the value functions are very sensitive to the game players’ own parameter values. This is particularly apparent in Figure 3. However, the effects of the opponents’ parameter values on the value functions are not negligible. First, when we evaluate whether the value function is sensitive to the change of one parameter, it would be better to compare the relative change rather than the absolute change. The rationale behind this is simple: a \$100 change is definitely different to a person with \$100 and that with \$1,000,000. So, the relative change is a more legitimate measure for sensitivity analysis. Indeed, at least three reasons drive us to apply the relative change to measure sensitivity. First, as the exponential utility is negative, zero is an upper limit for the value function, which makes it less meaningful to compare absolute changes of the value function around zero. Second, in Figure 3(a) and (d) the extreme sensitive case is caused by the “ $\frac{1}{\gamma_F}$ ” and “ $\frac{1}{\gamma_L}$ ” terms in front of the exponential functions. Note that when these two terms are dropped, $-\exp(-\gamma_F x_F)$ and $-\exp(-\gamma_L x_L)$ are still utility functions, and indeed this form of utility functions is directly defined as exponential utility in some classical textbooks for financial economics (e.g., p.154 in Cochrane (2005)). Third, the magnitude of surpluses x_F and x_L may veil the sensitivity of the value functions to the changes of parameter values. In our numerical examples, the large values of x_F and x_L make the value functions fluctuate within very small (absolute) ranges below zero. Now for Figure 3(b) and (c), we define the relative change by $|\frac{\max_{\gamma_i} V_j(x_j; \gamma_i) - \min_{\gamma_i} V_j(x_j; \gamma_i)}{\min_{\gamma_i} V_j(x_j; \gamma_i)}|$, for $i = L, F$ and $j = L, F$. Here, writing the value functions as $V_j(x_j; \gamma_j)$ highlights their dependence on the risk aversion parameters. Using the relative change measures can help us overcome difficulties in sensitivity analysis due to the aforementioned reasons. Particularly, it can be seen from Proposition 5.2 that the relative change measures can cancel the terms “ $\frac{1}{\gamma_F}$ ”, “ $\frac{1}{\gamma_L}$ ”, “ $-\gamma_F x_F e^{\rho_F(T-t)}$ ” and “ $-\gamma_L x_L e^{\rho_L(T-t)}$ ”. It can be calculated that when γ_F changes from 0.25 to 0.5 in Figure 3(b), the relative change in the value function V_L is 78.11%; when γ_L changes from 1.21 to 1.46 in Figure 3(b), the relative change in the value function V_F is 92.51%. This implies that in both cases, the relative changes are not negligible. Moreover, comparing Figure 4(c) and (d), we find that even for the absolute changes, the value function of one player could be more sensitive to its opponent’s parameters than its own’s. In addition, the sensitivity of the value function to one player’s own parameters seems less important than that to its opponent’s. The reason is that the player should have a better knowledge of its

own parameters, but more likely misestimate its opponent's parameters. Thus, it is of greater importance to look at the sensitivity of one player's value function to its opponent's parameters, which reveals how much adverse impact that may cause due to the misestimation of its opponent's parameters.

7. CONCLUSION

We consider a stochastic Stackelberg differential game between an insurer and a reinsurer. This provides us a new paradigm to understand how the insurer and the reinsurer reach an agreement on reinsurance policies, which are mutually beneficial to both parties. The agreement is achieved when both the insurer's and the reinsurer's expected utilities are maximized in the sense of the Stackelberg equilibrium. Various directions are worth further exploring. For instance, the objectives of the insurer and the reinsurer can be changed to minimizing ruin probabilities or risk measures and optimizing mean-variance criteria. In our companion paper, Chen and Shen (2017), we will present time consistent solutions to the Stackelberg game between insurers and reinsurers under a mean-variance criterion. It is also interesting to consider the Stackelberg game between multiple insurers and one reinsurer, where the insurers are allowed to compete with each other.

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APPENDIX A. HJB EQUATION APPROACH FOR EXPONENTIAL UTILITY CASE WITH CONSTANT COEFFICIENTS

In this appendix, we introduce the HJB equation approach to solve the game problem (2.11) and (2.12) in the same setting of Section 5, that is, when the insurer's and reinsurer's preferences are modeled by exponential utilities and all model coefficients are constant. This approach is more elementary and caters to readers who are less familiar with BSDEs.

As we mentioned at the end of Section 3, even when all model coefficients are deterministic, Bellman's dynamic programming principle may fail to work and stochastic HJB equations would be unavoidably needed to solve the problem for general utility functions. One may refer to Appendix B for some discussions on a class of power utilities. Fortunately, the special structure of exponential utilities allows for the use of Bellman's dynamic programming principle and HJB PDEs to tackle the Stackelberg game in the case with constant/deterministic coefficients.

Before introducing the HJB equations related to the game problem, we show some properties on the value functions of the game problem.

Proposition A.1. *The value functions of the insurer’s and reinsurer’s problems (2.11) and (2.12), i.e., $V^F(t, x; p(\cdot))$ and $V^L(t, x)$, are increasing and concave in x , for any $t \in [0, T]$.*

Proof. We first show the increasing and concave properties for $V^F(t, x; p(\cdot))$. First of all, we fix a reinsurance premium strategy $p(\cdot) \in \mathcal{A}_L$. Let $x_1 \in (\mathbb{R}^+)^2$ and $x_2 \in (\mathbb{R}^+)^2$ be two possible values of the wealth state at time t . Denote by $X_F^{(x; p(\cdot), q(\cdot))}(T)$ the terminal wealth of the insurer associated with $X(t) = x \in (\mathbb{R}^+)^2$ and the pair of strategies $p(\cdot) \in \mathcal{A}_L$ and $q(\cdot) \in \mathcal{A}_F$.

If $0 < x_1 \leq x_2$, then it is clear that $X_F^{(x_1; p(\cdot), q(\cdot))}(T) \leq X_F^{(x_2; p(\cdot), q(\cdot))}(T)$. Thus, by the increasing property of the exponential utility function, we have

$$\begin{aligned} V^F(t, x_1; p(\cdot)) &= \sup_{q(\cdot) \in \mathcal{A}_F} \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp(-\gamma_F X_F^{(x_1; p(\cdot), q(\cdot))}(T)) \right] \\ &\leq \sup_{q(\cdot) \in \mathcal{A}_F} \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp(-\gamma_F X_F^{(x_2; p(\cdot), q(\cdot))}(T)) \right] = V^F(t, x_2; p(\cdot)). \end{aligned} \tag{A.1}$$

If $x_1 \in (\mathbb{R}^+)^2$ and $x_2 \in (\mathbb{R}^+)^2$, then $\bar{x} := \lambda x_1 + (1 - \lambda)x_2$, where $\lambda \in [0, 1]$, is also in $(\mathbb{R}^+)^2$. Denote by

$$q_i^*(\cdot) = \alpha_i^*(\cdot, p(\cdot)) := \arg \max_{q(\cdot) \in \mathcal{A}_F} \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp(-\gamma_F X_F^{(x_i; p(\cdot), q(\cdot))}(T)) \right], \tag{A.2}$$

i.e., the optimal reinsurance strategy associated with x_i , for $i = 1, 2$. Clearly, $\lambda q_1^*(t) + (1 - \lambda)q_2^*(t) \in [0, 1]$, for any $t \in [0, T]$; associated with $X(t) = \bar{x}$ and $(p(\cdot), \lambda q_1^*(\cdot) + (1 - \lambda)q_2^*(\cdot))$, the state equation (2.8) has a unique strong solution $X_F^{(\bar{x}; p(\cdot), \lambda q_1^*(\cdot) + (1 - \lambda)q_2^*(\cdot))}(\cdot)$, that is,

$$X_F^{(\bar{x}; p(\cdot), \lambda q_1^*(\cdot) + (1 - \lambda)q_2^*(\cdot))}(\cdot) = \lambda X_F^{(x_1; p(\cdot), q_1^*(\cdot))}(\cdot) + (1 - \lambda) X_F^{(x_2; p(\cdot), q_2^*(\cdot))}(\cdot). \tag{A.3}$$

From Definition 2.1, we see $\lambda q_1^*(\cdot) + (1 - \lambda)q_2^*(\cdot) \in \mathcal{A}_L$. Thus, by the concavity of the exponential utility function and the definition of supremum, we derive

$$\begin{aligned} &\lambda V^F(t, x_1; p(\cdot)) + (1 - \lambda) V^F(t, x_2; p(\cdot)) \\ &= \lambda \sup_{q(\cdot) \in \mathcal{A}_F} \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp(-\gamma_F X_F^{(x_1; p(\cdot), q(\cdot))}(T)) \right] \\ &\quad + (1 - \lambda) \sup_{q(\cdot) \in \mathcal{A}_F} \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp(-\gamma_F X_F^{(x_2; p(\cdot), q(\cdot))}(T)) \right] \\ &= \lambda \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp(-\gamma_F X_F^{(x_1; p(\cdot), q_1^*(\cdot))}(T)) \right] + (1 - \lambda) \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp(-\gamma_F X_F^{(x_2; p(\cdot), q_2^*(\cdot))}(T)) \right] \\ &\leq \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp(-\gamma_F X_F^{(\bar{x}; p(\cdot), \lambda q_1^*(\cdot) + (1 - \lambda)q_2^*(\cdot))}(T)) \right] \\ &\leq \sup_{q(\cdot) \in \mathcal{A}_F} \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp(-\gamma_F X_F^{(\bar{x}; p(\cdot), q(\cdot))}(T)) \right] = V^F(t, \bar{x}; p(\cdot)). \end{aligned} \tag{A.4}$$

Next, we show the increasing and concave properties for $V^L(t, x)$. To this end, we claim that the insurer’s optimal strategy is independent of the initial wealth, i.e.,

$$q_1^*(\cdot) = \alpha_1^*(\cdot, p(\cdot)) = \alpha_2^*(\cdot, p(\cdot)) = q_2^*(\cdot). \tag{A.5}$$

In fact, we know

$$\begin{aligned} X_F^{(x_i; p(\cdot), q(\cdot))}(T) &= x_{iF} e^{\rho_F(T-t)} + \int_t^T e^{\rho_F(T-s)} [\theta a - (1 - q(s))(p(s) - a)] ds \\ &\quad + \int_t^T e^{\rho_F(u)(T-s)} q(s) \sigma dW(s), \end{aligned} \tag{A.6}$$

where x_{iF} denotes the wealth of the insurer at time t , i.e., $X_F(t) = x_{iF}$. Thus, we can separate x_{iF} from the insurer’s optimization problem in the following way

$$\begin{aligned} &\sup_{q(\cdot) \in \mathcal{A}_F} \mathbb{E}_t \left[-\frac{1}{\gamma_F} \exp \left(-\gamma_F X_F^{(x_i; p(\cdot), q(\cdot))}(T) \right) \right] \\ &= -\frac{1}{\gamma_F} \exp \left(-\gamma_F x_{iF} e^{\rho_F(T-t)} \right) \\ &\quad \times \sup_{q(\cdot) \in \mathcal{A}_F} \mathbb{E}_t \left[\exp \left\{ -\gamma_F \int_t^T e^{\rho_F(T-s)} [\theta a - (1 - q(s))(p(s) - a)] ds \right\} \right. \\ &\quad \left. \times \exp \left\{ -\gamma_F \int_t^T e^{\rho_F(T-s)} q(s) \sigma dW(s) \right\} \right]. \end{aligned} \tag{A.7}$$

Clearly, this shows that the relation (A.5) holds. Therefore, the arguments to prove the increasing and concavity properties of $V^F(t, x; p(\cdot))$ also work for $V^L(t, x)$. Thus, we do not repeat them here. ■

It has been shown in Section 5 (refer to the proof of Proposition 5.1) that the optimal solution to the game under $\bar{\mathcal{A}}_L \times \mathcal{A}_F$ is the same as that under $\mathcal{A}_L \times \mathcal{A}_F$. Thus, it suffices to solve the game over $\bar{\mathcal{A}}_L \times \mathcal{A}_F$. As a consequence, we discuss the insurer’s problem (2.11) by the following HJB equation:

$$\begin{cases} \max_{q \in [0,1]} \left\{ v_t^F(t, x) + v_x^F(t, x) [\rho_F x_F + \theta a - (1 - q)(p(t) - a)] + \frac{1}{2} v_{xx}^F(t, x) \sigma^2 q^2 \right\} \\ = 0, \quad p(\cdot) \in \bar{\mathcal{A}}_L, \\ v^F(T, x) = -\frac{1}{\gamma_F} e^{-\gamma_F x_F}. \end{cases} \tag{A.8}$$

From the form of (A.8), it is clear that if $p(\cdot)$ is random, the HJB equation collapses immediately. This is the reason that we should focus on the auxiliary admissible set $\bar{\mathcal{A}}_L$, when we would apply the HJB equation approach to solve the problem.

Proposition A.2. *For any $p(\cdot) \in \bar{\mathcal{A}}_L$, the solution to the HJB equation (A.8) is given by*

$$v^F(t, x) = -\frac{1}{\gamma_F} \exp \left(-\gamma_F x_F e^{\rho_F(T-t)} \right) g^F(t), \tag{A.9}$$

where

$$g^F(t) = \exp \left\{ \int_t^T \left[\gamma_F e^{\rho_F(T-s)} ((p(s) - a) - \theta a) - \frac{1}{2} \frac{(p(s) - a)^2}{\sigma^2} \right] I_{O_1}(s) ds + \int_t^T \left[-\gamma_F \theta a e^{\rho_F(T-s)} + \frac{1}{2} \gamma_F^2 \sigma^2 e^{2\rho_F(T-s)} \right] I_{O_2}(s) ds \right\}. \tag{A.10}$$

The maximizer in the HJB equation (A.8) is given by

$$\alpha^*(t, p(t)) = \frac{1}{\gamma_F e^{\rho_F(T-t)}} \frac{p(t) - a}{\sigma^2} \wedge 1. \tag{A.11}$$

Proof. Inspired by the increasing and concavity properties of the value function, we assume initially that $v_x^F > 0$ and $v_{xx}^F < 0$, which will be verified at the end of the proof.

If the maximum in the HJB equation (A.8) is achieved at an interior point of $[0, 1]$, then by the first-order condition, we obtain

$$\alpha^o(t, p(t)) = -\frac{v_x^F}{v_{xx}^F} \frac{p(t) - a}{\sigma^2}. \tag{A.12}$$

By the assumption that $v_x^F > 0$ and $v_{xx}^F < 0$ and the fact that $p(t) \geq (1 + \theta)a > a$, we know that $\alpha^o(t, p(t)) > 0$ is automatically satisfied, for any $t \in [0, T]$ and $p(\cdot) \in \tilde{\mathcal{A}}_L$. Taking into account the upper bound on admissible reinsurance strategies, we can express the optimal reinsurance strategy as

$$\alpha^*(t, p(t)) = -\frac{v_x^F}{v_{xx}^F} \frac{p(t) - a}{\sigma^2} \wedge 1. \tag{A.13}$$

We try the following *ansatz*

$$v^F(t, x) = -\frac{1}{\gamma_F} \exp(-\gamma_F x_F e^{\rho_F(T-t)}) g^F(t), \tag{A.14}$$

where the terminal condition $g^F(T) = 1$ is satisfied. Substituting this into equation (A.13) gives

$$\alpha^*(t, p(t)) = \frac{1}{\gamma_F e^{\rho_F(T-t)}} \frac{p(t) - a}{\sigma^2} \wedge 1. \tag{A.15}$$

Plugging (A.14) and (A.15) to the HJB equation (A.8), we obtain that if $\alpha^*(t, p(t)) = \alpha^o(t, p(t))$, then $g^F(t)$ satisfies

$$g_t^F(t) + g^F(t) \left[\gamma_F ((p(t) - a) - \theta a) e^{\rho_F(T-t)} - \frac{1}{2} \frac{(p(t) - a)^2}{\sigma^2} \right] = 0; \tag{A.16}$$

if $\alpha^*(t, p(t)) = 1$, then $g^F(t)$ satisfies

$$g_t^F(t) + g^F(t) \left[-\gamma_F \theta a e^{\rho_F(T-t)} + \frac{1}{2} \gamma_F^2 \sigma^2 e^{2\rho_F(T-t)} \right] = 0. \tag{A.17}$$

Clearly, the solution to the above two cases is given by (A.10). Since the admissible strategy $p(\cdot)$ is constrained in a bounded set, $g^F(t)$ is a strictly positive function. It holds that

$$v_x^F(t, x) = e^{\rho_F(T-t)} \exp(-\gamma_F x_F e^{\rho_F(T-t)}) g^F(t) > 0,$$

and

$$v_{xx}^F(t, x) = -\gamma_F e^{2\rho_F(T-t)} \exp(-\gamma_F x_F e^{\rho_F(T-t)}) g^F(t) < 0,$$

which affirm the desired results. ■

To solve the reinsurer problem (2.12), it remains to solve the following HJB equation:

$$\begin{cases} \max_{p \in [c, g]} \left\{ v_t^L(t, x) + v_x^L(t, x) [\rho_L x_L + (1 - \alpha^*(t, p))(p - a)] + \frac{1}{2} v_{xx}^L(t, x) \sigma^2 (1 - \alpha^*(t, p))^2 \right\} \\ = 0, \\ v^L(T, x) = -\frac{1}{\gamma_L} e^{-\gamma_L x_L}. \end{cases} \tag{A.18}$$

In view of Proposition A.2, it is obvious that if $\alpha^*(t, p) = 1$, the game problem is trivial.

Proposition A.3. *When $\alpha^*(t, p) = 1$, the value of p has no impact on the maximum function in the HJB equation (A.18), and the solution to the HJB equation (A.18) is given by*

$$v^L(t, x) = -\frac{1}{\gamma_L} \exp(-\gamma_L x_L e^{\rho_L(T-t)}). \tag{A.19}$$

When $\alpha^*(t, p) = \frac{1}{\gamma_F e^{\rho_F(T-t)}} \frac{p-a}{\sigma^2}$, the maximizer in the HJB equation (A.18) achieves the maximum at

$$p^*(t) = \left[\gamma_F \sigma^2 e^{\rho_F(T-t)} \frac{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 1}{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 2} + a \right] \vee (1 + \theta)a \wedge (1 + \eta)a; \tag{A.20}$$

and the solution to the HJB Equation (A.18) is given by

$$v^L(t, x) = -\frac{1}{\gamma_L} \exp(-\gamma_L x_L e^{\rho_L(T-t)}) g^L(t), \tag{A.21}$$

where

$$\begin{aligned}
 g^L(t) = \exp \left\{ \int_t^T \left[\frac{1}{2} \gamma_L^2 \sigma^2 e^{2\rho_L(T-s)} \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right)^2 \right. \right. \\
 - \gamma_L e^{\rho_L(T-s)} \theta a \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right) \Big] I_{\bar{O}_1}(s) ds \\
 - \int_t^T \frac{\sigma^2}{2} \left[\frac{1}{(\gamma_F e^{\rho_F(T-s)})^2} + \frac{1}{\gamma_L \gamma_F e^{(\rho_L + \rho_F)(T-s)}} \right] I_{\bar{O}_2}(s) ds \\
 + \int_t^T \left[\frac{1}{2} \gamma_L^2 \sigma^2 e^{2\rho_L(T-s)} \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right)^2 \right. \\
 \left. \left. - \gamma_L e^{\rho_L(T-s)} \eta a \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-s)}} \right) \right] I_{\bar{O}_3}(s) ds \right\},
 \end{aligned}$$

and $I_{\bar{O}_i}$ is the indicator function of \bar{O}_i , for $i = 1, 2, 3$.

Proof. When $\alpha^*(t, p) = 1$, the HJB equation (A.18) reduces to an ordinary differential equation (ODE), which is irrelevant to the value of p and obviously has the solution given by (A.19).

When $\alpha^*(t, p) = \alpha^o(t, p)$, we denote by

$$\begin{aligned}
 \Gamma(p; t, x) &:= v_t^L(t, x) + v_x^L(t, x) [\rho_L x_L + (1 - \alpha^o(t, p))(p - a)] + \frac{1}{2} v_{xx}^L(t, x) \sigma^2 (1 - \alpha^o(t, p))^2 \\
 &= v_t^L(t, x) + v_x^L(t, x) \rho_L x_L + \frac{1}{2} v_{xx}^L(t, x) \sigma^2 + \left[v_x^L(t, x) - \frac{v_{xx}^L(t, x)}{\gamma_F e^{\rho_F(T-t)}} \right] (p - a) \\
 &\quad - \frac{1}{\gamma_F e^{\rho_F(T-t)}} \left[v_x^L(t, x) - \frac{v_{xx}^L(t, x)}{2\gamma_F e^{\rho_F(T-t)}} \right] \frac{(p - a)^2}{\sigma^2}. \tag{A.22}
 \end{aligned}$$

As in the proof of Proposition A.2, we assume initially that $v_x^L(t, x) > 0$ and $v_{xx}^L(t, x) < 0$. Indeed, this assumption can be verified easily once we get the explicit expression of $v^L(t, x)$. So, (A.22) is a quadratic function in p , and the coefficient of p^2 is negative.

Then the first-order condition shows that the critical point of $\Gamma(p; t, x)$ in p is

$$p^o(t) = \gamma_F \sigma^2 e^{\rho_F(T-t)} \frac{\frac{v_x^L}{\gamma_F e^{\rho_F(T-t)}} - v_x^L}{\frac{v_{xx}^L}{\gamma_F e^{\rho_F(T-t)}} - 2v_x^L} + a. \tag{A.23}$$

We try the following *ansatz*:

$$v^L(t, x) = -\frac{1}{\gamma_L} \exp(-\gamma_L x_L e^{\rho_L(T-t)}) g^L(t). \tag{A.24}$$

Making substitution of (A.24) into (A.23) gives

$$p^o(t) = \gamma_F \sigma^2 e^{\rho_F(T-t)} \frac{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 1}{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 2} + a. \tag{A.25}$$

Noting that $p(t)$ is constrained in $[c, g] = [(1 + \theta)a, (1 + \eta)a]$, we can express the optimal reinsurance premium strategy by

$$p^*(t) = \gamma_F \sigma^2 e^{\rho_F(T-t)} \frac{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 1}{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 2} \vee \theta a \wedge \eta a + a. \tag{A.26}$$

In what follows, we consider three cases depending on whether the optimal reinsurance premium strategy is achieved at the lower bound, the interior point or the upper bound.

If $M(t) < N^\theta(t)$, then the optimal reinsurance premium is achieved at the lower bound, i.e.,

$$p^*(t) = (1 + \theta)a. \tag{A.27}$$

We input (A.27) and the *ansatz* (A.24) into the HJB equation (A.18) and obtain an ODE for $g^L(t)$:

$$g_t^L(t) + g^L(t) \left[\frac{1}{2} \gamma_L^2 \sigma^2 e^{2\rho_L(T-t)} \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}} \right)^2 - \gamma_L e^{\rho_L(T-t)} \theta a \left(1 - \frac{\theta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}} \right) \right] = 0. \tag{A.28}$$

If $N^\theta(t) \leq M(t) < N^\eta(t)$, then the optimal reinsurance premium is given by the interior point:

$$p^*(t) = \gamma_F \sigma^2 e^{\rho_F(T-t)} \frac{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 1}{\frac{\gamma_L e^{\rho_L(T-t)}}{\gamma_F e^{\rho_F(T-t)}} + 2} + a. \tag{A.29}$$

Substituting (A.29) and the *ansatz* (A.24) into the HJB equation (A.18), we obtain that $g^L(t)$ satisfies the following ODE:

$$g_t^L(t) + g^L(t) \left[-\frac{\sigma^2}{2} \frac{1}{\frac{1}{(\gamma_F e^{\rho_F(T-t)})^2} + \frac{2}{\gamma_L \gamma_F e^{(\rho_L + \rho_F)(T-t)}}} \right] = 0. \tag{A.30}$$

If $N^\eta(t) \leq M(t)$, then the optimal reinsurance premium is achieved at the upper bound:

$$p^*(t) = (1 + \eta)a. \tag{A.31}$$

A substitution of (A.31) and the *ansatz* (A.24) into the HJB equation (A.18) gives an ODE for $g^L(t)$:

$$g_t^L(t) + g^L(t) \left[\frac{1}{2} \gamma_L^2 \sigma^2 e^{2\rho_L(T-t)} \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}} \right)^2 - \gamma_L e^{\rho_L(T-t)} \eta a \left(1 - \frac{\eta a}{\gamma_F \sigma^2 e^{\rho_F(T-t)}} \right) \right] = 0. \tag{A.32}$$

Combining the above three cases leads to the explicit representations for $g^L(t)$ and $v^L(t, x)$. Some simple calculation can validate $v_x^L(t, x) > 0$ and $v_{xx}^L(t, x) < 0$. This completes the proof. ■

Remark A.1. It is worthwhile to mention that the concept of viscosity solutions is not needed in our paper. Even if the control constraints are present in the insurer’s problem (2.11) and reinsurer’s problem (2.12), thanks to the special structure of exponential utility, it can be shown as Chen et al. (2016) that the value functions $v^F(\cdot, \cdot)$ and $v^L(\cdot, \cdot)$ are sufficiently smooth so that they are classical solutions to the HJB equations (A.8) and (A.18).

Combining the results in Propositions A.2 and A.3, we can summarize the optimal strategy $(p^*(\cdot), q^*(\cdot))$ of the Stackelberg game as Cases 1^a–4^a in Proposition 5.2, and represent the value functions $V^L(t, x)$ and $V^F(t, x) = V^F(t, x; p^*(\cdot))$ as (5.17)–(5.18) therein. Since Proposition 5.2 is very lengthy, we choose not to repeat it here. Instead, we only provide a verification theorem for the optimality of $(p^*(\cdot), q^*(\cdot))$. The verification theorem serves to confirm that the maximizers of HJB equations (A.8) and (A.18) are optimal strategies of the game problem.

Proposition A.4 (Verification theorem). *The strategy $(p^*(\cdot), q^*(\cdot))$ defined in Proposition 5.2 achieves optimality in $\mathcal{A}_L \times \mathcal{A}_F$.*

Proof. First, it is clear that the strategies $p^*(\cdot)$ and $q^*(\cdot)$ obtained in Proposition 5.2 satisfy Conditions (i)–(ii) in Definition 2.1, i.e., they are \mathbb{F} -adapted and bounded within $P = [c, g]$ and $Q = [0, 1]$, respectively. This, together with the assumption of constant parameters, guarantees that the state process (2.8) has a unique solution in $S_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R}^2)$. Thus, $(p^*(\cdot), q^*(\cdot))$ is an admissible pair of strategies, i.e., $(p^*(\cdot), q^*(\cdot)) \in \mathcal{A}_L \times \mathcal{A}_F$.

Next, we show the optimality of $(p^*(\cdot), q^*(\cdot))$ and first concentrate on the auxiliary admissible set $\bar{\mathcal{A}}_L \times \mathcal{A}_F$. Note that $v^F(t, x)$ does not depend on x_L (see (A.9)). With a little abuse of notation, we suppress x_L in $v^F(t, x)$ and denote by $v^F(t, x_F)$ as the solution to the HJB equation (A.8). For any $(p(\cdot), q(\cdot)) \in \bar{\mathcal{A}}_L \times \mathcal{A}_F$ and $\vartheta \in [t, T]$, we apply Itô’s formula to $v^F(t, X_F(t))$ and deduce

$$\begin{aligned} v^F(\vartheta, X_F(\vartheta)) &= v^F(t, X_F(t)) + \int_t^\vartheta \left\{ v_s^F(s, X_F(s)) \right. \\ &\quad \left. + v_{x_F}^F(s, X_F(s))[\rho_F X_F(s) + \theta a - (1 - q(s))(p(s) - a)] \right. \\ &\quad \left. + \frac{1}{2} v_{x_F x_F}^F(s, X_F(s)) \sigma^2 q^2(s) \right\} ds + \int_t^\vartheta v_{x_F}^F(s, X_F(s)) q(s) \sigma dW(s). \end{aligned} \tag{A.33}$$

Since the function $v^F(t, x_F)$ is the solution of the HJB equation (A.8), taking expectation on both sides of (A.33) conditional on $X_F(t) = x_F$, we have

$$\mathbb{E}_{t, x_F} [v^F(\vartheta, X_F(\vartheta))] \leq v^F(t, x_F) + \mathbb{E}_{t, x_F} \left[\int_t^\vartheta v_{x_F}^F(s, X_F(s)) q(s) \sigma dW(s) \right]. \tag{A.34}$$

From the wealth process (2.6), we have

$$X_F(s) = e^{\rho_F s} \left\{ x_{0F} + \int_0^s e^{-\rho_F u} [\theta a - (1 - q(u))(p(u) - a)] du + \int_0^s e^{-\rho_F u} q(u) \sigma dW(u) \right\}. \tag{A.35}$$

Denote by

$$C_1(s) := \exp \left\{ -2\gamma_F e^{\rho_F T} \cdot \int_0^s e^{-\rho_F u} q(u) \sigma dW(u) \right\},$$

$$C_2(s) := \exp \left\{ -2\gamma_F e^{\rho_F T} \cdot \left[x_{0F} + \int_0^s e^{-\rho_F u} [\theta a - (1 - q(u))(p(u) - a)] du \right] \right\} \cdot (g^F(s))^2,$$

and

$$C_3(s) := \exp \left\{ \gamma_F^2 e^{2\rho_F T} \cdot \int_0^s e^{-2\rho_F u} q^2(u) \sigma^2 du \right\}.$$

Note that $(v_x^F(s, X_F(s)))^2 = C_1(s)C_2(s)$, and for any $(p(\cdot), q(\cdot)) \in \bar{\mathcal{A}}_L \times \mathcal{A}_F$, both $C_2(s)$ and $C_3(s)$ are positive and bounded on $[0, T]$. Furthermore, it is clear

$$\Lambda(s) := (C_3(s))^{-\frac{1}{2}} (C_1(s))^{\frac{1}{2}}$$

$$= \exp \left\{ -\frac{1}{2} \gamma_F^2 e^{2\rho_F T} \cdot \int_0^s e^{-2\rho_F u} q^2(u) \sigma^2 du - \gamma_F e^{\rho_F T} \cdot \int_0^s e^{-\rho_F u} q(u) \sigma dW(u) \right\},$$

$s \in [0, T],$

is a stochastic exponential martingale, satisfying $\Lambda(\cdot) \in \mathcal{S}_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R})$.

Let K_2 and K_3 be upper bounds of $C_2(s)$ and $C_3(s)$ on $[0, T]$, respectively. Then, we can derive

$$\begin{aligned} \mathbb{E} \left[\int_0^T [v_x^F(s, X_F(s))q(s)\sigma]^2 ds \right] &\leq \sigma^2 \mathbb{E} \left[\int_0^T C_1(s)C_2(s) ds \right] \\ &\leq K_2 \sigma^2 \cdot \mathbb{E} \left[\int_0^T C_1(s) ds \right] \\ &\leq K_2 \sigma^2 \cdot \mathbb{E} \left[\sup_{s \in [0, T]} \left\{ C_3(s) | (C_3(s))^{-\frac{1}{2}} (C_1(s))^{\frac{1}{2}} |^2 \right\} \right] \\ &\leq K_2 K_3 \sigma^2 \cdot \mathbb{E} \left[\sup_{s \in [0, T]} |\Lambda(s)|^2 \right] < \infty, \end{aligned}$$

which implies that $\int_0^\cdot v_x^F(s, X_F(s))q(s)\sigma dW(s)$ is an (\mathbb{F}, \mathbb{P}) -martingale.

Therefore, the conditional expectation on the right-hand side of the inequality (A.34) vanishes, and setting $\vartheta = T$ in (A.34) yields

$$\mathbb{E}_{t, X_F} \left[-\frac{1}{\gamma_F} \exp \left(-\gamma_F X_F^{(p(\cdot), q(\cdot))}(T) \right) \right] \leq v^F(t, X_F), \quad \forall (p(\cdot), q(\cdot)) \in \bar{\mathcal{A}}_L \times \mathcal{A}_F.$$

Obviously, when the maximizer in equation (A.8) is taken, i.e., $q(\cdot) = \alpha^*(\cdot, p(\cdot))$, the above inequality becomes an equality. That is,

$$\mathbb{E}_{t, X_F} \left[-\frac{1}{\gamma_F} \exp \left(-\gamma_F X_F^{(p(\cdot), \alpha^*(\cdot, p(\cdot)))}(T) \right) \right] = v^F(t, X_F), \quad \forall p(\cdot) \in \bar{\mathcal{A}}_L. \quad (\text{A.36})$$

Following similar derivations, we can obtain

$$\mathbb{E}_{t,x_L} \left[-\frac{1}{\gamma_L} \exp \left(-\gamma_L X_L^{(p(\cdot), \alpha^*(\cdot, p(\cdot)))}(T) \right) \right] \leq v^L(t, x_L), \quad \forall p(\cdot) \in \bar{\mathcal{A}}_L,$$

and

$$\mathbb{E}_{t,x_L} \left[-\frac{1}{\gamma_L} \exp \left(-\gamma_L X_L^{(p^*(\cdot), \alpha^*(\cdot, p^*(\cdot)))}(T) \right) \right] = v^L(t, x_L). \tag{A.37}$$

This confirms that $(p^*(\cdot), \alpha^*(\cdot, p^*(\cdot)))$ is optimal in $\bar{\mathcal{A}}_L \times \mathcal{A}_F$. Combining the result obtained in Section 5 (see Remark 5.2), we can therefore conclude that the strategy $(p^*(\cdot), q^*(\cdot)) = (p^*(\cdot), \alpha^*(\cdot, p^*(\cdot)))$ defined in Proposition 5.2 achieves optimality in $\mathcal{A}_L \times \mathcal{A}_F$. ■

APPENDIX B. DISCUSSIONS ON POWER UTILITY CASE WITH CONSTANT COEFFICIENTS

In Appendix A, we demonstrate that the HJB equation approach can be applied to solve the game problem in the exponential utility case with constant coefficients. However, for a general utility, the assumption of constant/deterministic coefficients does not guarantee that optimal strategies are Markovian, which is a sufficient condition that the method of HJB PDEs can be used.

In this appendix, we consider a class of power utility functions:

$$U_F(x_F) = \frac{x_F^{\gamma_F}}{\gamma_F}, \quad U_L(x_L) = \frac{x_L^{\gamma_L}}{\gamma_L}, \tag{B.1}$$

where $\gamma_i := \frac{1}{s_i}$ with s_i being any odd number greater than 1 and $i = L, F$. Furthermore, we assume that all model coefficients are constant as Section 5. This class of power utility functions has the domain of \mathbb{R} , and allows for the utility of negative surplus. To simplify our analysis, we drop the constraints in the original Stackelbeg game problem (2.11) and (2.12). In what follows, we show that in this case both the insurer’s and reinsurer’s optimal strategies are non-Markovian and anticipating.

For the insurer’s problem (2.11), it can be shown as Section 4 in the paper (also refer to Hu *et al.* (2005) and Shen and Wei (2016)) that the optimal strategy and the value function are given by

$$\alpha^*(t, p(t)) = \frac{p(t) - a}{(1 - \gamma_F)\sigma^2} (X_F(t) + Y_1^F(t)) - \frac{Z_1^F(t)}{\sigma} + \frac{Z_2^F(t)}{\sigma} \frac{X_F(t) + Y_1^F(t)}{Y_2^F(t)} \tag{B.2}$$

and

$$V_F(t, x_F) = \frac{(x_F + Y_1^F(t))^{\gamma_F}}{\gamma_F} (Y_2^F(t))^{1-\gamma_F}, \tag{B.3}$$

where $(Y_1^F(\cdot), Z_1^F(\cdot))$ and $(Y_2^F(\cdot), Z_2^F(\cdot))$ are unique solutions to the following BSDEs

$$\begin{cases} dY_1^F(t) = - \left[\rho_F Y_1^F(t) + \frac{p(t) - a}{\sigma} Z_1^F(t) + (1 + \theta)a - p(t) \right] dt + Z_1^F(t) dW(t), \\ Y_1^F(T) = 0, \end{cases} \tag{B.4}$$

and

$$\begin{cases} dY_2^F(t) = - \left\{ \frac{\gamma_F}{1 - \gamma_F} \left[\rho_F + \frac{(p(t) - a)^2}{2(1 - \gamma_F)\sigma^2} \right] Y_2^F(t) + \frac{\gamma_F(p(t) - a)}{(1 - \gamma_F)\sigma} Z_2^F(t) \right\} dt \\ \quad + Z_2^F(t) dW(t), \\ Y_2^F(T) = 1. \end{cases} \tag{B.5}$$

The BSDEs (B.4) and (B.5) can be rewritten in the following integral form:

$$Y_1^F(t) = \int_t^T \left[\rho_F Y_1^F(s) + \frac{p(s) - a}{\sigma} Z_1^F(s) + (1 + \theta)a - p(s) \right] ds - \int_t^T Z_1^F(s) dW(s),$$

and

$$\begin{aligned} Y_2^F(t) = & 1 + \int_t^T \left\{ \frac{\gamma_F}{1 - \gamma_F} \left[\rho_F + \frac{(p(s) - a)^2}{2(1 - \gamma_F)\sigma^2} \right] Y_2^F(s) \right. \\ & \left. + \frac{\gamma_F(p(s) - a)}{(1 - \gamma_F)\sigma} Z_2^F(s) \right\} ds \\ & - \int_t^T Z_2^F(s) dW(s). \end{aligned}$$

Thus, the first components of solutions $Y_1^F(\cdot)$ and $Y_2^F(\cdot)$ at time t , i.e., $Y_1^F(t)$ and $Y_2^F(t)$, depend on $\{p(s) | s \in [t, T]\}$, and so does $\alpha^*(t, p(t))$. This implies that $Y_1^F(\cdot)$, $Y_2^F(\cdot)$ and hence $\alpha^*(\cdot, p(\cdot))$ are anticipating.

By substituting the map $\alpha^*(\cdot, p(\cdot))$ into the state equation (2.8) of $X(\cdot) = (X_L(\cdot), X_F(\cdot))^\top$, we can see that the state equation depends on $Y^F(\cdot) = (Y_1^F(\cdot), Y_2^F(\cdot))^\top$ and $Z^F(\cdot) = (Z_1^F(\cdot), Z_2^F(\cdot))^\top$, i.e., the solutions to the BSDEs (B.4) and (B.5). Therefore, the reinsurer's problem (2.12) becomes a stochastic optimal control problem with state processes $X(\cdot)$ and $(Y^F(\cdot), Z^F(\cdot))$, which are governed by a forward-backward stochastic differential equation (FBSDE):

$$\begin{cases} dX(t) = [AX(t) + \widehat{B}(X(t), Y^F(t), Z^F(t), p(t))]dt + \widehat{D}(X(t), Y^F(t), Z^F(t), p(t))dW(t), \\ dY^F(t) = -[G(p(t))Y^F(t) + L(p(t))Z^F(t) + K(p(t))]dt + Z^F(t)dW(t), \\ X(0) = (x_{0L}, x_{0F})^\top, \quad Y^F(T) = (0, 1)^\top, \end{cases} \tag{B.6}$$

where

$$A := \begin{pmatrix} \rho_L & 0 \\ 0 & \rho_F \end{pmatrix}, \quad G(p(t)) := \begin{pmatrix} \rho_F & 0 \\ 0 & \frac{\gamma_F}{1 - \gamma_F} \left[\rho_F + \frac{(p(t) - a)^2}{2(1 - \gamma_F)\sigma^2} \right] \end{pmatrix},$$

$$\widehat{B}(X(t), Y^F(t), Z^F(t), p(t)) := \begin{pmatrix} \left[1 - \frac{p(t)-a}{(1-\gamma_F)\sigma^2} (X_F(t) + Y_1^F(t)) + \frac{Z_1^F(t)}{\sigma} - \frac{Z_2^F(t)}{\sigma} \frac{X_F(t)+Y_1^F(t)}{Y_2^F(t)} \right] (p(t) - a) \\ \theta a - \left[1 - \frac{p(t)-a}{(1-\gamma_F)\sigma^2} (X_F(t) + Y_1^F(t)) + \frac{Z_1^F(t)}{\sigma} - \frac{Z_2^F(t)}{\sigma} \frac{X_F(t)+Y_1^F(t)}{Y_2^F(t)} \right] (p(t) - a) \end{pmatrix},$$

$$L(p(t)) := \begin{pmatrix} \frac{p(t)-a}{\sigma} & 0 \\ 0 & \frac{\gamma_F(p(t)-a)}{(1-\gamma_F)\sigma} \end{pmatrix}, \quad K(p(t)) := \begin{pmatrix} (1 + \theta)a - p(t) & \\ & 0 \end{pmatrix},$$

$$\widehat{D}(X(t), Y^F(t), Z^F(t), p(t)) := \begin{pmatrix} \left[1 - \frac{p(t)-a}{(1-\gamma_F)\sigma^2} (X_F(t) + Y_1^F(t)) + \frac{Z_1^F(t)}{\sigma} - \frac{Z_2^F(t)}{\sigma} \frac{X_F(t)+Y_1^F(t)}{Y_2^F(t)} \right] \sigma \\ \left[\frac{p(t)-a}{(1-\gamma_F)\sigma^2} (X_F(t) + Y_1^F(t)) - \frac{Z_1^F(t)}{\sigma} + \frac{Z_2^F(t)}{\sigma} \frac{X_F(t)+Y_1^F(t)}{Y_2^F(t)} \right] \sigma \end{pmatrix}.$$

Since the cost functional of the reinsurer’s problem is also of a power form, we conjecture that an optimal strategy $p^*(\cdot)$ would rely on the state processes $X(\cdot)$ and $(Y^F(\cdot), Z^F(\cdot))$, and $p^*(\cdot)$ could also be anticipating. Indeed, it will turn out that associated with $p^*(\cdot)$, the FBSDE (B.6) is fully coupled.

To solve the reinsurer’s problem (2.12), we apply the stochastic maximum principle for forward-backward control systems (see Øksendal and Sulem (2009)). To this end, we define a Hamiltonian $\widehat{\mathcal{H}} : [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\widehat{\mathcal{H}}(t, x, y^F, z^F, p, \psi, \varphi, \phi) := [G(p)y^F + L(p)z^F + K(p)]^\top \psi + [Ax + \widehat{B}(x, y^F, z^F, p)]^\top \varphi + [\widehat{D}(x, y^F, z^F, p)]^\top \phi. \tag{B.7}$$

Then, the adjoint equation to the reinsurer’s control problem is given by a new FBSDE:

$$\begin{cases} d\psi(t) = \widehat{\mathcal{H}}_{y^F}(t, X(t), Y^F(t), Z^F(t), p(t), \psi(t), \varphi(t), \phi(t))dt \\ \quad + \widehat{\mathcal{H}}_{z^F}(t, X(t), Y^F(t), Z^F(t), p(t), \psi(t), \varphi(t), \phi(t))dW(t), \\ d\varphi(t) = -\widehat{\mathcal{H}}_x(t, X(t), Y^F(t), Z^F(t), p(t), \psi(t), \varphi(t), \phi(t))dt + \phi(t)dW(t), \\ \psi(0) = (0, 0)^\top, \quad \varphi(T) = ((X_L(T))^{\gamma_L-1}, 0)^\top, \end{cases} \tag{B.8}$$

where

$$\widehat{\mathcal{H}}_a(t, x, y^F, z^F, p, \psi, \varphi, \phi) = \frac{\partial \widehat{\mathcal{H}}}{\partial a}(t, x, y^F, z^F, p, \psi, \varphi, \phi), \quad \text{for } a := x, y^F, z^F, \tag{B.9}$$

and the solution to the FBSDE (B.8), i.e., $(\psi(\cdot), \varphi(\cdot), \phi(\cdot))$ is called the adjoint process, where $\psi(\cdot) := (\psi_1(\cdot), \psi_2(\cdot))^\top$, $\varphi(\cdot) := (\varphi_1(\cdot), \varphi_2(\cdot))^\top$ and $\phi(\cdot) := (\phi_1(\cdot), \phi_2(\cdot))^\top$.

By the first-order condition, the reinsurer’s optimal strategy is determined by

$$\frac{\partial \mathcal{H}}{\partial p}(t, X(t), Y^F(t), Z^F(t), p, \psi(t), \varphi(t), \phi(t)) = 0, \tag{B.10}$$

which yields

$$\begin{aligned}
 p^*(t) = & \left[\frac{\gamma_F Y_2^F(t)}{(1 - \gamma_F)^2 \sigma^2} \psi_2(t) - \frac{2(X_F(t) + Y_1^F(t))}{(1 - \gamma_F)\sigma^2} (\phi_1(t) - \phi_2(t)) \right]^{-1} \\
 & \times \left[\left(1 - \frac{Z_1^F(t)}{\sigma} \right) \psi_1(t) - \frac{\gamma_F Z_2^F(t)}{(1 - \gamma_F)\sigma} \psi_2(t) + \frac{X_F(t) + Y_1^F(t)}{(1 - \gamma_F)\sigma} (\phi_1(t) - \phi_2(t)) \right. \\
 & \left. - \left(1 + \frac{Z_1^F(t)}{\sigma} - \frac{Z_2^F(t)}{\sigma} \frac{X_F(t) + Y_1^F(t)}{Y_2^F(t)} \right) (\phi_1(t) - \phi_2(t)) \right] + a. \tag{B.11}
 \end{aligned}$$

Associated with $p^*(\cdot)$, the adjoint equation (B.8) is also a fully coupled FBSDE, and its solution $(\psi(\cdot), \varphi(\cdot), \phi(\cdot))$ depends on the path of $X(\cdot)$. Therefore, $(\psi(\cdot), \varphi(\cdot), \phi(\cdot))$ and $p^*(\cdot)$ are anticipating, which is consistent with our conjecture. It should be mentioned (B.6) and (B.8) are highly nonlinear FBSDEs, the solvability of which exceeds the scope of this paper. As our space is limited, we prefer not to go deeper in our discussions and plan to re-investigate the power utility case in our future research.

Remark B.1. In the exponential utility case (see Sections 4 and 5), it is shown that the optimal strategies $p^*(\cdot)$ and $q^*(\cdot)$ are independent of the surplus $X(\cdot)$, though they still depend on solutions to some BSDEs. For this reason, once coefficients are assumed to be constant in Section 5, the related BSDEs therein reduce to backward ODEs. Indeed, the optimal strategies are Markovian. Thus, both the BSDE approach and the HJB equation approach work for the exponential utility case with constant coefficients (see Appendix A).

However, for a general utility, such as the power utility case considered here, even if all coefficients are constant, optimal strategies may depend on the path of the surplus process. Then, the BSDE component, (i.e., (B.4)–(B.5)) of the state equation for the reinsurer’s problem does not degenerate to an ODE, thereby the solutions to (B.4)–(B.5) and the adjoint equation (B.8) are non-Markovian and anticipating, and so are optimal strategies $(p^*(\cdot), q^*(\cdot))$. This makes the HJB equation not applicable.

Remark B.1. In general, a sufficient condition that both the methods of stochastic HJB equations and HJB PDEs would work to solve the game problem is that the reinsurer’s (leader’s) optimal strategy is deterministic. In this case, we can restrict our attention on $p(\cdot) \in \bar{\mathcal{A}}_L$ and thereby Markov controls for the insurer’s optimization problem (2.11). Therefore, the HJB equation approach can be applied. When all coefficients are constant, the exponential utility case is an example such that the reinsurer’s optimal strategy is deterministic. In fact, when we only consider deterministic controls for the reinsurer’s optimization problem (2.12), the method of HJB PDEs is applicable to the game problem (2.11) and (2.12) even if the insurer’s and reinsurer’s preferences are modeled by general utility functions.