A CREDIBILITY MODEL WITH RANDOM FLUCTUATIONS IN DELAY PROBABILITIES FOR THE PREDICTION OF IBNR CLAIMS.[*]

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ABSTRACT

We consider a general credibility model for the prediction of IBNR-claims which allows for random fluctuations in the underlying delay distribution. Such fluctuations always bring about decreasing credibility. It is shown that even negative credibility is achieved for more substantial fluctuations in the delay distribution. Special attention is paid to the mixed Poisson case for claim numbers including the discussion of parameter estimation.

KEYWORDS

Credibility theory; IBNR-reserve; delay distribution; Dirichlet distribution; linear sufficiency.

1. INTRODUCTION

The IBNR-problem is a classical problem extensively dealt with in actuarial literature. The models developed in this field are based on different sets of assumptions with regard to the (stochastic) nature of the loss reserving process. It is generally assumed in all these models that the development is to some extent stable. In a number of papers (BUHLMANN, SCHNIEPER and STRAUB, 1980; DE VYLDER, 1982; NORBERG, 1986 and WITTING, 1987a) it has been explicitly laid down in the model assumptions that there is a fixed delay distribution common to all occurrence years.

Taking the development of a fixed occurrence year into consideration the situation dealt with in these models can be illustrated as shown in Figure 1. The broken curves indicate a positive correlation between early and later development years. It should be pointed out that this behaviour is caused by stochastic effects relating solely to the occurrence year. Should it be of relevance to include additional stochastic effects like variations in the claims handling or recording process, then models that allow only for positive correlation may prove inadequate.

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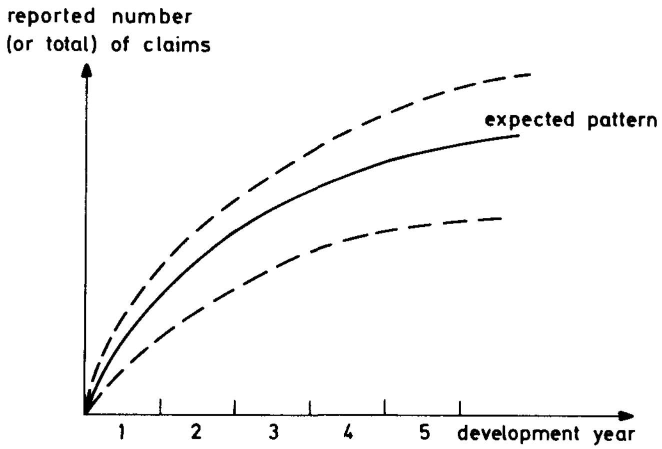

FIGURE 1. The expected pattern of development and two typical realizations of the development process (broken curves).

In such a case we might be faced with situations as illustrated in Figure 2. In this case a smaller number of claims observed in earlier development years would indicate a larger number of claims in later development years (as compared to the expected pattern), and vice versa.

The aim of the present paper is to construct a credibility model which also brings in the case of negative correlation. Whilst the model of WITTING (1987a) only allows for negative credibility (negative correlation) in some rare cases, the present authors feel that the natural way to introduce negative credibility is by means of random fluctuations in the delay distribution.

The idea of allowing for random fluctuations in the underlying delay distribution was mentioned by NORBERG (1986). However, in his quite general framework model this would not affect the moment structure. Also the Kalman filter approach, as advocated by DE JONG and ZEHNWIRTH (1983), can be said to take such fluctuations into account. De Jong and Zehnwirth state that "each year of origin gives rise to such a (delay) distribution and in any one (calendar) year we sample one component from each of an array of such distributions". In order to embed the loss reserving problem into the state-space framework they (need to) pick up a multiplicative decomposition, similar to the separation technique of DE VYLDER (1982). As a consequence, the genuine part of the delay distribution—describing the dynamics of development years—is kept deterministic and vanishes into the known design matrix. The present authors prefer to model the specific sources of variation rather than laying down a macro-model which, hopefully, is sufficiently flexible to reflect the main features of the process.

Our main purpose is to investigate how random fluctuations in delay probabilities affect the credibility estimator. Consequently, we shall not spend too great an effort on modelling the rest of the process but simply take a framework model which, in the case of fixed delay probabilities, contains a number of important IBNR-models developed in the actuarial literature as special cases.

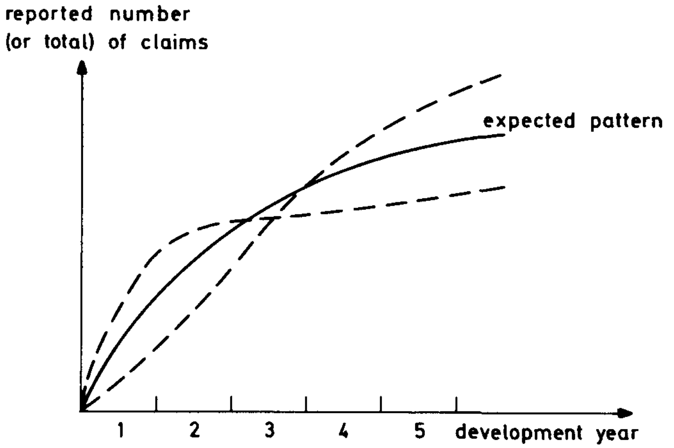

FIGURE 2. The expected pattern of development and two typical realizations of the development process (broken curves).

2. THE GENERAL MODEL

As usual in loss reserving we consider n occurrence years (calendar years) numbered consecutively by $j = 1, \ldots, n$ and, for each occurrence year, n development years $i = 1, \ldots, n$. That is, we assume all claims fully developed within n years.

For each occurrence year j we introduce a random vector $\boldsymbol{\pi}_j = ({}^1\pi_j, \ldots, {}^n\pi_j)$ where ${}^i\pi_j$ can be interpreted as either

(i) the probability that an individual claim incurred in occurrence year j is reported in the development year i, when only numbers of claims are dealt with,

or

(ii) the proportion of the final claim amount which is expected to be developed in development year i, when totals of claims are dealt with.

Note that by assumption $\sum_{i=1}^{n} {}^i\pi_j = 1$ for each j.

The claims statistics corresponding to occurrence year j are $X_j = ({}^1X_j, \ldots, {}^nX_j)$ where ${}^iX_j$ can denote either

(i) the number of claims incurred in occurrence year j and reported in development year i,

or

(ii) the part of the total claim amount incurred in occurrence year j and settled in development year i.

It is also convenient to introduce the accumulated quantities

$$X_{ij} = \sum_{m=1}^{i} {}^{m}X_j \qquad \text{and} \qquad X_j = X_{nj}.$$

The following assumptions define our basic framework model:

(A1) Quantities relating to different occurrence years are independent.

(A2) $\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_n$ are i.i.d. with first and second order moments

$$p_i = E\,{}^{i}\pi_j$$

and

$$c_{ik} = \operatorname{Cov}({}^{i}\pi_j, {}^{k}\pi_j); \qquad 1 \leqslant i, k \leqslant n.$$

(A3) For given delay distribution $\boldsymbol{\pi}_j$ the claims statistics X_j has a moment structure given by

$$E[{}^{i}X_j \mid \boldsymbol{\pi}_j] = {}^{i}\pi_j m_j;$$

$$\operatorname{Cov}({}^{i}X_j, {}^{k}X_j \mid \boldsymbol{\pi}_j) = \delta_{ik}\,{}^{i}f_j(\boldsymbol{\pi}_j) + {}^{i}\pi_j\,{}^{k}\pi_j d_j,$$

where $m_j \geqslant 0$ and d_j are known or unknown constants independent of the development year, ${}^{i}f_j(\cdot)$ is a given function and δ_{ik} the Kronecker symbol.

The choice of m_j, d_j and ${}^{i}f_j(\cdot)$ depends on further specification of the model assumptions. Two natural choices are indicated below.

The (conditional) moment structure displayed in (A3) comprises a number of important cases already treated in the literature when the underlying delay distribution is assumed to be fixed. Two such cases are:

(i) Let X_j denote the claim numbers. WITTING (1987a) assumes that the times of delay for single claims are i.i.d. random variables, independent of the total number of claims X_j. He derives the moment structure in (A3) with $m_j = EX_j$, $d_j = \operatorname{Var} X_j - EX_j$ and ${}^{i}f_j(\boldsymbol{\pi}_j) = {}^{i}\pi_j m_j$.

(ii) When X_j denotes the totals of claims, we can assume the multiplicative form $X_j = b(\theta_j)Y_j$, where $b(\theta_j)$ is a random "claims cost index" independent of $Y_j = ({}^{1}Y_j, \ldots, {}^{n}Y_j)$ (and $\boldsymbol{\pi}_j$). Furthermore, we can assume that

$$E({}^{i}Y_j \mid \boldsymbol{\pi}_j) = V_j\,{}^{i}\pi_j;$$

$$\operatorname{Cov}({}^{i}Y_j, {}^{k}Y_j \mid \boldsymbol{\pi}_j) = \delta_{ik} V_j r_i^2,$$

with V_j being a (known) measure of volume. Again we arrive at (A3), with

$$m_j = V_j E b(\theta_j),$$

$$d_j = V_j^2 \operatorname{Var} b(\theta_j)$$

and

$${}^{i}f_j(\boldsymbol{\pi}_j) = V_j r_i^2\, E(b(\theta_j)^2).$$

This (conditional) moment structure is a special case of that assumed in HACHEMEISTER's (1975) regression model. In an IBNR-context it is analysed by DE VYLDER (1982) in the special case where r_i^2 (and hence ${}^if_j(\cdot)$) is independent of i. NORBERG (1986) comments upon this additional assumption: "Although mathematically convenient, it is hardly appropriate as an *a priori* description of the IBNR process. A reasonable way of relaxing this assumption could be to replace r^2 by r_i^2. In order to limit the number of parameters, the r_i^2's could be taken as some simple parametric functions of i, e.g. $r_i^2 = \alpha + \beta i$."

Note that

(1) $$d_j = \mathrm{Cov}\left(\frac{{}^iX_j}{{}^i\pi_j}, \frac{{}^kX_j}{{}^k\pi_j}\,\middle|\, \boldsymbol{\pi}_j\right); \qquad i \neq k$$

in some sense measures the correlation originating from the stochastic effects of the occurrence year.

We write ${}^i\tau_j$ for the expected value $E[{}^if_j(\boldsymbol{\pi}_j)]$, and the unconditional moment structure is readily obtained from (A2) and (A3).

(2) $$E\,{}^iX_j = p_i m_j.$$
$$\mathrm{Cov}({}^iX_j, {}^kX_j) = \delta_{ik}\,{}^i\tau_j + c_{ik}(d_j + m_j^2) + p_i p_k d_j.$$

Besides the necessity of reducing the number of unknown parameters we are interested in getting an explicit formula for the credibility estimator of the outstanding claims. This is (of course) not possible in the general setting of (2). Therefore we add further specification of the assumption (A2):

(A2′) $\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_n$ are independent and identically Dirichlet distributed with parameters $\alpha_1, \ldots, \alpha_n$.

Then the delay distribution has the moment structure (DE GROOT, 1970, p. 49):

(3) $$p_i = E\,{}^i\pi_j = \alpha_i/\alpha,$$
$$c_{ik} = \mathrm{Cov}({}^i\pi_j, {}^k\pi_j) = \frac{1}{1+\alpha}(\delta_{ik}p_i - p_i p_k),$$

where $\alpha = \alpha_1 + \cdots + \alpha_n$ measures the correlation within the delay process.

By inserting (3) into the second equation of (2) we obtain the following unconditional covariance structure:

(4) $$\mathrm{Cov}({}^iX_j, {}^kX_j) = \delta_{ik}(p_i\Phi_j + {}^i\tau_j) + p_i p_k \Psi_j,$$

where

(5) $$\Phi_j = \frac{d_j + m_j^2}{1+\alpha},$$

and

(6) $$\Psi_j = \frac{\alpha d_j - m_j^2}{1+\alpha} = d_j - \Phi_j.$$

An interpretation of Ψ_j — analogous to the interpretation of d_j in (1) — follows from (4):

$$\text{(7)} \qquad \Psi_j = \mathrm{Cov}\left(\frac{{}^iX_j}{p_i}, \frac{{}^kX_j}{p_k}\right) = \frac{\alpha}{1+\alpha}\, d_j + \frac{1}{1+\alpha}\,(-m_j^2); \qquad i \neq k.$$

Whilst d_j measures (in a sense) the correlation for fixed delay distribution, Ψ_j additionally contains the effect of a random delay distribution. If follows from (7) that the correlation is decreased by the introduction of such fluctuations.

3. THE CREDIBILITY FORMULA

Our aim is to calculate the credibility estimator of the total outstanding claims $\mathrm{IBNR}(j) = X_j - X_{\tilde{n}j}$, where $\tilde{n} = n - j + 1$ denotes the latest observed development year corresponding to occurrence year j. As usual the observed data are assumed to be organized in a run-off triangle

$$\nabla X = \{{}^iX_j;\ j = 1, \ldots, n;\ i = 1, \ldots, \tilde{n}\}.$$

In regard of (A1) and comparing the covariance structure (4) with the one treated in JEWELL (1976) or WITTING (1987b), we find that

$$\text{(8)} \qquad \mathring{X}_{\tilde{n}j} = \sum_{i=1}^{\tilde{n}} \frac{p_i}{p_i\Phi_j + {}^i\tau_j}\, {}^iX_j$$

is a linearly sufficient statistic for calculating the credibility estimator. When dealing only with claim numbers (${}^i\tau_j = p_i m_j$) $\mathring{X}_{\tilde{n}j}$ reduces (up to a constant factor) to the cumulative number of claims $X_{\tilde{n}j}$.

Some simple calculations lead to the following credibility formula:

$$\text{(9)} \qquad \widehat{\mathrm{IBNR}}(j) = (1 - F(\tilde{n}))\left\{Z(j)\,\frac{\mathring{X}_{\tilde{n}j}}{G(j)} + (1 - Z(j))m_j\right\},$$

with credibility factor

$$\text{(10)} \qquad Z(j) = \frac{G(j)\Psi_j}{G(j)\Psi_j + 1},$$

$$\text{(11)} \qquad G(j) = \sum_{i=1}^{\tilde{n}} \frac{p_i^2}{p_i\Phi_j + {}^i\tau_j}$$

and

$$F(\tilde{n}) = \sum_{i=1}^{\tilde{n}} p_i\,.$$

Now, two limiting cases are of interest. First $\alpha \to \infty$ (where $\Phi_j \to 0$ and $\Psi_j \to d_j$)

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corresponding to the situation with fixed delay distribution, compare (3). Secondly, $\alpha \to 0$ (where $\Phi_j \to d_j + m_j^2$ and $\Psi_j \to -m_j^2$) which represents the case with minimal prior information about $\boldsymbol{\pi}_j$.

In our example (i) (claim numbers; ${}^i\tau_j = p_i m_j$) it holds for $\alpha \to \infty$ that

$$\mathring{X}_{\tilde{n}j} \to \frac{X_{\tilde{n}j}}{m_j}, \qquad G(j) \to \frac{F(\tilde{n})}{m_j} \qquad \text{and} \qquad Z(j) \to \frac{F(\tilde{n})d_j}{m_j + F(\tilde{n})d_j}.$$

It is encouraging to notice that formulas in this limiting case correspond to those derived in WITTING (1987a). Likewise, as $\alpha \to 0$ we find

$$\mathring{X}_{\tilde{n}j} \to \frac{X_{\tilde{n}j}}{d_j + m_j + m_j^2}, \qquad G(j) \to \frac{F(\tilde{n})}{d_j + m_j + m_j^2}$$

and

$$Z(j) \to \frac{-F(\tilde{n})m_j^2}{(1 - F(\tilde{n}))m_j^2 + d_j + m_j}.$$

It follows that a minimum of prior information about $\boldsymbol{\pi}_j$ always leads to negative credibility. This result reflects precisely the intuitive supposition that a high number of reported claims $X_{\tilde{n}j}$ could indicate that a major part of X_j has already been reported and, consequently, the outstanding number of claims IBNR(j) should be expected to be smaller.

In fact, it can easily be seen that the credibility weight $Z(j)$ in (10) is an increasing function of α. Hence, when uncertainty about the delay distribution is introduced, the credibility is reduced to $Z(j)$, where

$$Z(j) \in \left(\frac{-F(\tilde{n})m_j^2}{(1 - F(\tilde{n}))m_j^2 + d_j + m_j}, \frac{F(\tilde{n})d_j}{m_j + F(\tilde{n})d_j} \right).$$

The example (ii), totals of claims, can be dealt with analogously. Particularly, the formulas derived by DE VYLDER (1982) (${}^i\tau_j = \tau_j$) appear in the limiting case $\alpha \to \infty$.

Negative credibility is a consequence of negative correlation between observed and outstanding ${}^iX_j$'s. In the general case (not necessarily Dirichlet distributed delay probabilities) it holds for $i \leqslant \tilde{n}$ that (compare (2))

$$\begin{aligned} \text{Cov}({}^iX_j, \widehat{\text{IBNR}}(j)) &= \sum_{k = \tilde{n}+1}^{n} \text{Cov}({}^iX_j, {}^kX_j) \\ &= \left(\sum_{k = \tilde{n}+1}^{n} c_{ik} \right) (d_j + m_j^2) + p_i \left(\sum_{k = \tilde{n}+1}^{n} p_k \right) d_j. \end{aligned}$$

Since $(d_j + m_j^2) > 0$ it follows that random fluctuations in $\boldsymbol{\pi}_j$ causes a negative contribution to the credibility if

$$\sum_{k = \tilde{n}+1}^{n} c_{ik} < 0.$$

When $\boldsymbol{\pi}_j$ is Dirichlet distributed it holds that $c_{ik} < 0$ for all $i \neq k$ (compare (3)), whereas in the general case it can only be concluded that

$$\sum_{i=1}^{\tilde{n}}\left(\sum_{k=\tilde{n}+1}^{n} c_{ik}\right) = \mathrm{Cov}\left(\left(\sum_{i=1}^{\tilde{n}} {}^{i}\pi_j\right), \left(\sum_{k=\tilde{n}+1}^{n} {}^{k}\pi_j\right)\right)$$

$$= -\frac{1}{2}\left(\mathrm{Var}\left(\sum_{i=1}^{\tilde{n}} {}^{i}\pi_j\right) + \mathrm{Var}\left(\sum_{k=\tilde{n}+1}^{n} {}^{k}\pi_j\right)\right) < 0.$$

In particular, when predicting IBNR(j) on the basis of $X_{\tilde{n}j}$, random fluctuations always gives a negative contribution to the credibility.

4. THE MIXED POISSON CASE: PARAMETER ESTIMATION

In this section we only consider the case of claim numbers. The distribution of X_j is assumed to depend upon an unobservable structure variable θ_j representing the hidden risk characteristics of occurrence year j. We assume that, given θ_j, X_j is Poisson distributed with parameter $V_j\theta_j$, where V_j is a known measure of volume (e.g. the number of policies in occurrence year j).

Furthermore, as it is standard, $\theta_1, \ldots, \theta_n$ are taken to be i.i.d. with moments $\mu = E\theta_j$ and $w = \mathrm{Var}\ \theta_j$. Then it holds that $m_j = V_j\mu$, $d_j = V_j^2 w$, $\Phi_j = V_j^2\Phi$ and $\Psi_j = V_j^2\Psi$, with

$$\Phi = \frac{w+\mu^2}{1+\alpha} \quad \text{and} \quad \Psi = \frac{\alpha w - \mu^2}{1+\alpha}. \tag{12}$$

The insertion of these parameters into (10) and (11) leads to the following expression for the credibility weight:

$$Z(j) = \frac{F(\tilde{n})V_j\Psi}{F(\tilde{n})V_j\Psi + \mu + V_j\Phi}. \tag{13}$$

In the situation without random fluctuations in the delay distribution the credibility weight is

$$Z(j) = \frac{F(\tilde{n})V_j w}{\mu + F(\tilde{n})V_j w}$$

(WITTING, 1987a, formula (9)).

Comparing, the credibility in (13) is decreased by the replacements $w \to \Psi$ and $\mu \to \mu + V_j\Phi$. In fact, this results in negative credibility if $\alpha < 1/\gamma^2$, where $\gamma = \sqrt{w}/\mu$ is the coefficient of variation of the underlying structure distribution.

A set of estimators for the unknown parameters p_i, μ, Φ and Ψ can be based on the equations

$$E\,{}^{i}X_j = V_j\mu p_i, \tag{14.a}$$

$$E({}^{i}X_j\,{}^{k}X_j) = \delta_{ik}p_i(V_j^2\Phi + V_j\mu) + p_ip_kV_j^2(\Psi+\mu^2), \tag{14.b}$$

$$1 \leqslant i \leqslant k \leqslant n-j+1.$$

Based on (14.a) the following estimators suggest themselves (least squares

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estimators using natural weights):

(15) $$\widehat{p_i\mu} = \left(\sum_{j=1}^{n-i+1} V_j\right)^{-1} \sum_{j=1}^{n-i+1} {}^iX_j; \qquad i = 1, \ldots, n,$$

(15) $$\hat{\mu} = \sum_{i=1}^{n} \widehat{p_i\mu},$$

(16) $$\hat{p}_i = \widehat{p_i\mu}/\hat{\mu}; \qquad i = 1, \ldots, n.$$

Weighted least squares estimators for Ψ and Φ, based on (14.b), can be obtained by minimizing the quadratic form

$$Q(\Phi, \Psi) = \sum_{j=1}^{n} \sum_{i,k} w_j(i,k)\{{}^iX_j {}^kX_j - \delta_{ik} p_i(V_j\mu + V_j^2\Phi) - p_i p_k V_j^2(\Psi + \mu^2)\}^2$$

the sums ranging over all $j = 1, \ldots, n$ and $1 \leqslant i \leqslant k \leqslant n - j + 1$.

It is straightforward to verify (formulas (A.4), (A.5) in the Appendix) that these estimators can be written in the form

(17) $$\hat{\Psi} = \frac{A_1B_3 - A_2B_1}{B_2B_3 - B_1^2} - \mu^2,$$

(18) $$\hat{\Phi} = \frac{A_2B_2 - A_1B_1}{B_2B_3 - B_1^2},$$

where

(19) $$A_1 = \sum_{j,i,k} w_j(i,k) p_i p_k {}^{ik}T_j,$$

(20) $$A_2 = \sum_{j,i} w_j(i,i) p_i {}^{ii}T_j,$$

(21) $$B_1 = \sum_{j,i} w_j(i,i) p_i^3,$$

(22) $$B_2 = \sum_{j,i,k} w_j(i,k) p_i^2 p_k^2,$$

(23) $$B_3 = \sum_{j,i} w_j(i,i) p_i^2,$$

(the sums ranging over all $j = 1, \ldots, n$ and $1 \leqslant i \leqslant k \leqslant n - j + 1$), and

(24) $${}^{ik}T_j = ({}^iX_j {}^kX_j - \delta_{ik} p_i V_j \mu)/V_j^2.$$

The corresponding least squares estimator for α is obtained by use of (12) ((A.6) in the Appendix)

$$\hat{\alpha} = \frac{A_1B_3 - A_2B_1}{A_2B_2 - A_1B_1}.$$

It is also shown in the Appendix that a set of "natural" weights are

(25) $$w_j(i,i) = V_j^3[4V_j^2 p_i^3 \mu^2 + 6V_j p_i^2 \mu + p_i]^{-1},$$

(26) $$w_j(i,k) = V_j^2[p_i p_k + V_j p_i p_k(p_i + p_k)\mu]^{-1}; \qquad i \neq k.$$

Finally, genuine estimators are gained by inserting the estimators $\hat{\mu}$ and $\hat{p}_i$ ((15) and (16)) into (19)–(26).

As the calculations of these estimators are very time consuming the authors would also suggest a (half-) Bayesian approach: Assume that the value of α can be preassigned. We get from (14) the 2nd factorial moment of $X_{\tilde{n}j}$

$$EX_{\tilde{n}j}^{(2)} = \Phi V_j^2 F(\tilde{n})(1 + F(\tilde{n})\alpha),$$

which immediately leads to the estimator

$$\hat{\Phi} = \frac{\sum_{j=1}^{n} X_{\tilde{n}j}^{(2)}}{\sum_{j=1}^{n} \hat{F}(\tilde{n}) V_j^2 (1 + \hat{F}(\tilde{n})\alpha)}.$$

In fixing α subjectively it might be helpful to interpret $1/(1+\alpha)$ as the ratio of $\mathrm{Var}\,{}^i\pi_j$ and the variance of a 0–1-variable with mean p_i (compare (3)). A corresponding interpretation is shown by FERGUSON (1973) (see also ZEHNWIRTH, 1977) to be valid for the continuous case, that is, for the so called Dirichlet process introduced by FERGUSON (1973).

5. CONCLUSION

It is the authors' intention to present a general credibility model for the prediction of outstanding claims. The model comprises a fairly broad category of situations, namely the cases of claim numbers and total of claims, making allowance for a random delay distribution. As in any other framework model the old conflict between realism and simplicity arises (see NORBERG (1986) for an extensive discussion). Even though it may be difficult to obtain reasonable parameter estimates in the framework model we believe that the result derived within this model will shed some light on the possible effects of different sources of fluctuations in IBNR-problems.

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APPENDIX

Motivated by (14.b) we estimate Φ and Ψ by minimizing the quadratic form

$$Q(\Phi, \Psi) = \sum_{j=1}^{n} \sum_{i,k} w_j(i,k) \{ {}^iX_j\, {}^kX_j - \delta_{ik} p_i (V_j \mu + V_j^2 \Phi) - p_i p_k V_j^2 (\Psi + \mu^2) \}^2,$$

the sum ranging over all $j = 1, \ldots, n$ and $1 \leqslant i \leqslant k \leqslant n - j + 1$. By introducing

$${}^{ik}T_j = ({}^iX_j\, {}^kX_j - \delta_{ik} p_i V_j \mu) / V_j^2,$$

and by redefining the weights, we may write

$$Q(\Phi, \Psi) = \sum_{j,i,k} w_j(i,k)\{{}^{ik}T_j - \delta_{ik} p_i \Phi - p_i p_k (\Psi + \mu^2)\}^2. \tag{A.1}$$

It is straightforward to check that (A.1) is minimized by $\hat{\Phi}, \hat{\Psi}$, satisfying

$$\hat{\Phi} = \frac{A_2 - (\hat{\Psi} + \mu^2) B_1}{B_3}, \tag{A.2}$$

$$\hat{\Psi} = \frac{A_1 - \hat{\Phi} B_1 - \mu^2 B_2}{B_2}, \tag{A.3}$$

where

$$A_1 = \sum_{j,i,k} w_j(i,k) p_i p_k \, {}^{ik}T_j,$$

$$A_2 = \sum_{j,i} w_j(i,i) p_i \, {}^{ii}T_j,$$

$$B_1 = \sum_{j,i} w_j(i,i) p_i^3,$$

$$B_2 = \sum_{j,i,k} w_j(i,k) p_i^2 p_k^2,$$

$$B_3 = \sum_{j,i} w_j(i,i) p_i^2,$$

and that equations (A.2), (A.3) are solved for

$$\hat{\Phi} = \frac{A_2 B_2 - A_1 B_1}{B_2 B_3 - B_1^2}, \tag{A.4}$$

$$\hat{\Psi} = \frac{A_1 B_3 - A_2 B_1}{B_2 B_3 - B_1^2} - \mu^2. \tag{A.5}$$

Since

$$\Phi = \frac{w + \mu^2}{1 + \alpha},$$

and $\Phi + \Psi = w$ (compare (12)), we find that

$$\alpha = \frac{\Psi + \mu^2}{\Phi},$$

and the corresponding least squares estimator for α is

$$\hat{\alpha} = \frac{A_1 B_3 - A_2 B_1}{A_2 B_2 - A_1 B_1}. \tag{A.6}$$

The optimal weights in (A.1) cannot be derived for the general case. When both π_j and θ_j are degenerate it holds that ${}^iX_j$; $i = 1, \ldots, \tilde{n}$ are independent and Poisson distributed with parameters $p_i V_j \mu$. If $Z \sim \text{Poisson}(\lambda)$, then $EZ^{(m)} = \lambda^m$, and we find the 2nd and 4th order non-central moments to be

$$EZ^2 = \lambda^2 + \lambda$$

and

$$EZ^4 = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda,$$

respectively. From the definition of ${}^{ik}T_j$ we then get

$$\text{Var}\,{}^{ii}T_j = [4V_j^3 p_i^3 \mu^3 + 6V_j^2 p_i^2 \mu^2 + V_j p_i \mu] / V_j^4,$$

$$\text{Var}\,{}^{ik}T_j = [V_j^2 p_i p_k \mu^2 + V_j^3 p_i p_k (p_i + p_k) \mu^3] / V_j^4; \qquad i \neq k.$$

A natural choice of weights would therefore be

$$w_j(i,i) = V_j^3 [4V_j^2 p_i^3 \mu^2 + 6V_j p_i^2 \mu + p_i]^{-1},$$

(A.7)

$$w_j(i,k) = V_j^2 [p_i p_k + V_j p_i p_k (p_i + p_k) \mu]^{-1}; \qquad i \neq k.$$

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