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WITTEN DEFORMATION FOR NONCOMPACT MANIFOLDS WITH BOUNDED GEOMETRY

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Abstract Motivated by the Landau–Ginzburg model, we study the Witten deformation on a noncompact manifold with bounded geometry, together with some tameness condition on the growth of the Morse function f near infinity. We prove that the cohomology of the Witten deformation d_{Tf} acting on the complex of smooth L^2 forms is isomorphic to the cohomology of the Thom–Smale complex of f as well as the relative cohomology of a certain pair (M,U) for sufficiently large T. We establish an Agmon estimate for eigenforms of the Witten Laplacian which plays an essential role in identifying these cohomologies via Witten's instanton complex, defined in terms of eigenspaces of the Witten Laplacian for small eigenvalues. As an application, we obtain the strong Morse inequalities in this setting.

Keywords: Witten deformation, Thom–Smale complex, Agmon estimate

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1. Introduction

1.1. Overview

In an extremely influential paper [21], Witten introduced a deformation of the de Rham complex by considering the new differential $d_f = d + df$, where d is the usual exterior derivative on forms and f is a Morse function. Setting

$$d_{Tf} := d + Tdf,$$

Witten observed that when $T > 0$ is large enough, the eigenfunctions of the small eigenvalues for the corresponding deformed Hodge Laplacian, the so-called Witten Laplacian, concentrate at the critical points of f. As a result, Witten deformation builds a direct bridge between the Betti numbers and the Morse indices of the critical points of f.

Witten deformation on closed manifolds has produced a whole range of beautiful applications, from Demailly's holomorphic Morse inequalities [5] to the proof of the Ray–Singer conjecture and its generalisation by Bismut and Zhang [2] to the instigation of the development of Floer homology theory.

Although the Witten deformation on noncompact manifolds is much less studied and understood, there has been interesting work in the direction. In [6] the cohomology of an affine algebraic variety is related to that of the Witten complex of $\mathbb{C}^m$ (see also [10] for further development).

This paper is motivated by the study of Landau–Ginzburg models (compare [14]), which according to Witten [22] are simply different phases of Calabi–Yau manifolds, and hence equivalent (in a certain sense) to Calabi–Yau manifolds. Suppose there is a nontrivial holomorphic function W (the superpotential) on a noncompact Kähler manifold M^n ($n = \dim_{\mathbb{C}} M$); then one considers the Witten deformation of the $\bar{\partial}$ operator:

$$\bar{\partial}_W = \bar{\partial} - \frac{i}{2}\partial W,$$

as its cohomology describes the quantum ground states of the Landau–Ginzburg model (M, W). If W is also a Morse function with k critical points, then complex Morse theoretic consideration leads to the expectation that

$$H^l_{\bar{\partial}_W}(M) = \begin{cases} \mathbb{C}^k & \text{if } l = n, \\ 0 & \text{otherwise.} \end{cases}$$

For the mathematical study of Landau–Ginzburg models and their significant applications, we point out the important work of [9]. On the other hand, in [17] the L^2 Hodge theory is used to give a dGBV algebra and Frobenius manifold structure for the Landau–Ginzburg models.

In this paper, we consider the more general case of Riemannian manifolds: we explore the relations between the Thom–Smale complex for a Morse function f on a noncompact manifold M and the deformed de Rham complex with respect to f. The first difficulty one encounters here is the presence of a continuous spectrum on a noncompact manifold; for that, one has to impose certain tameness conditions, consisting of the bounded geometry requirement for the manifold and growth conditions for the function. The notion of strong tameness is introduced in [7] in the Kähler setting, which guarantees the discreteness of the spectrum for the Witten Laplacian. Here we introduce a slightly weaker notion which allows a continuous spectrum but only outside a large interval starting from 0.

It is important to note – and this is another new phenomenon in the noncompact case – that the Thom–Smale complex may not be a complex in general. Namely, the square of its boundary operator need not be zero, since M is noncompact. However, we prove that with the tameness condition, it is.

The crucial technical part of our work is the Agmon estimate for eigenforms of the Witten Laplacian, which is essential in extending the usual analysis from the compact setting to the noncompact case. The Agmon estimate was discovered by S. Agmon in his study of N-body Schrödinger operators in the Euclidean setting and has found many important applications. The exponential decay of the eigenfunction is expressed in terms of the so-called Agmon distance (compare [1]). We make essential use of this Agmon estimate to carry out the isomorphism between the Witten instanton complex defined in terms of eigenspaces corresponding to the small eigenvalues and the Thom–Smale complex defined in terms of the critical point data of the function. We remark that the Agmon estimate near the critical points also plays an important role in the compact case [13]. The novelty here is that we make essential use of the exponential decay at spatial infinity provided by the Agmon estimate.

As an application of our results on noncompact manifolds, we obtain corresponding results for manifolds with boundaries which generalise recent work of [16].

In the rest of the introduction we give precise statements of our main results after setting up our notations. In subsequent work we will develop the local index theory and the Ray–Singer torsion for the Witten deformation in the noncompact setting.

1.2. Notations and basic setup

Let (M,g) be an n-dimensional noncompact connected complete Riemannian manifold with metric g. (M,g) is said to have bounded geometry if the following conditions hold:

1. The injectivity radius r_0 of M is positive.
2. $|\nabla^m R| \leq C_m$, where $\nabla^m R$ is the mth covariant derivative of the curvature tensor and C_m is a constant depending only on m.

On such a manifold, the Sobolev constant is uniformly bounded (see, e.g., [3]). Now let $f : M \to \mathbb{R}$ be a smooth function. In [7], the notion of strong tameness for the triple (M,g,f) is introduced.

Definition 1.1. *The triple (M,g,f) is said to be strongly tame if (M,g) has bounded geometry and*

$$\limsup_{p\to\infty} \frac{\left|\nabla^2 f\right|(p)}{\left|\nabla f\right|^2(p)} = 0$$

and

$$\lim_{p\to\infty} |\nabla f| \to \infty,$$

where $\nabla f, \nabla^2 f$ are the gradient and Hessian of f, respectively.

Remark 1.2. *Fix $p_0 \in M$ and let d be the distance function induced by g. Here $p \to \infty$ simply means that $d(p,p_0) \to \infty$.*

In this paper we only need the following weaker condition:

Definition 1.3. *The triple (M,g,f) is said to be well tame if (M,g) has bounded geometry and*

$$\limsup_{p\to\infty} \frac{\left|\nabla^2 f\right|(p)}{\left|\nabla f\right|^2(p)} < \infty$$

and

$$\liminf_{p\to\infty} |\nabla f| > 0.$$

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As usual, the metric g induces a canonical metric (still denote it by g) on $\Lambda^*(M)$, which then defines an inner product $(\cdot,\cdot)_{L^2}$ on $\Omega_c^*(M)$:

$$(\phi,\psi)_{L^2} = \int_M (\phi,\psi)_g dvol, \quad \phi,\psi \in \Omega_c^*(M).$$

Let $L^2\Lambda^*(M)$ be the completion of $\Omega_c^*(M)$ with respect to $\|\cdot\|_{L^2}$, and for simplicity we denote $L^2(M) := L^2\Lambda^0(M)$.

For any $T \geq 0$, let $d_{Tf} := d + Tdf\wedge : \Omega^*(M) \to \Omega^{*+1}(M)$ be the so-called Witten deformation of the de Rham operator d. It is an unbounded operator on $L^2\Lambda^*(M)$ with domain $\Omega_c^*(M)$. Also, d_{Tf} has a formal adjoint operator δ_{Tf}, with $\mathrm{Dom}\,(\delta_{Tf}) = \Omega_c^*(M)$, such that

$$(d_{Tf}\phi,\psi)_{L^2} = (\phi,\delta_{Tf}\psi)_{L^2}, \quad \phi,\psi \in \Omega_c^*(M).$$

Set $\Delta_{H,Tf} = (d_{Tf} + \delta_{Tf})^2$, and we denote the Friedrichs extension of $\Delta_{H,Tf}$ by $\Box_{Tf}$. As we will see (Theorem 2.1), if (M,g,f) is well tame, then $\Delta_{H,Tf}$ is essentially self-adjoint (and hence $\Box_{Tf}$ is the unique self-adjoint extension). In Appendix Appendix A (also see Theorem 2.3), we will prove the Hodge–Kodaira decomposition when (M,g,f) is well tame and T is large enough:

$$L^2\Lambda^*(M) = \ker\Box_{Tf} \oplus \mathrm{Im}\bar{d}_{Tf} \oplus \mathrm{Im}\bar{\delta}_{Tf}, \tag{1.1}$$

where $\bar{d}_{Tf}$ and $\bar{\delta}_{Tf}$ are the minimal extensions of d_{Tf} and δ_{Tf}, respectively.

Setting $\Omega^*_{(2)}(M) := L^2\Lambda^*(M) \cap \Omega^*(M)$, we have a chain complex (of unbounded operators)

$$\cdots \xrightarrow{d_{Tf}} \Omega^*_{(2)}(M) \xrightarrow{d_{Tf}} \Omega^{*+1}_{(2)}(M) \xrightarrow{d_{Tf}} \cdots.$$

Let $H^*_{(2)}(M,d_{Tf})$ denote the cohomology of this complex. In Appendix Appendix A we will show that $H^*_{(2)}(M,d_{Tf}) \cong \ker\Box_{Tf}$, provided (M,g,f) is well tame and T is large enough.

Finally we note the following well-known fact (compare [21, 23]):

Proposition 1.4. *The Hodge Laplacian $\Delta_{H,Tf}$ has the following local expression:*

$$\Delta_{H,Tf} = \Delta + T\nabla^2_{e_i,e_j} f\left[e^i \wedge, \iota_{e_j}\right] + T^2 |\nabla f|^2. \tag{1.2}$$

Here $\{e_i\}$ is a local frame on TM, $\{e^i\}$ is the dual frame on T^*M and Δ is the usual Hodge Laplacian.

1.3. Main results

In this subsection, we assume that (M,g) has bounded geometry and f is a Morse function with finite many critical points. Clearly this will be the case if (M,g,f) is well tame and f is Morse.

As we mentioned, the main technical result here is the Agmon estimate for the eigenforms of the Witten Laplacian:

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Theorem 1.1. *Let (M,g,f) be well tame and $\omega \in \mathrm{Dom}(\square_{Tf})$ be an eigenform of $\square_{Tf}$ whose eigenvalue is uniformly bounded in T. Then*

$$|\omega(p)| \leq CT^{(n+2)/2}\exp(-a\rho_T(p))\|\omega\|_{L^2},$$

for any $a \in (0,1)$ (provided T is sufficiently large and C is a constant depending on the dimension n, the function f, the curvature bound, the injectivity radius lower bound r_0 and a; for the precise choice of T and C, see the end of Section 3). The definition of the Agmon distance $\rho_T(p)$ will be given in Section 3.

The proof of the Agmon estimate, given in Section 7, is to carry out the idea of [1] in this more general setting.

Set $b_i(T) = \dim H^i_{(2)}(M,d_{Tf})$. If x is a critical point of f, denote by $n_f(x)$ the Morse index of f at x. Let m_i be the number of critical points of f with Morse index i. Then the strong Morse inequalities hold.

Theorem 1.2. *If (M,g,f) is well tame, then we have the following strong Morse inequality:*

$$(-1)^k\sum_{i=0}^{k}(-1)^i b_i(T) \leq (-1)^k\sum_{i=0}^{k}(-1)^i m_i, \quad \forall k \leq n,$$

provided T is large enough. And the equality holds for $k=n$.

In general, $b_i(T)$ may be very sensitive to T. However, we have the following result regarding the indepedence of $b_i(T)$ in T. Assume that the Morse function f satisfies the Smale transversality condition. Let $\left(C^*(W^u),\tilde{\partial}'\right)$ be the Thom–Smale complex given by f. It is important to note that in general, since M is noncompact, it could happen that $\left(\tilde{\partial}'\right)^2 \neq 0$. Also let $c>0$ be big enough, $U_c = \{p \in M : f(p) < -c\}$ and $(\Omega^*(M,U_c),d)$ be the relative de Rham complex.

Theorem 1.3. *If (M,g,f) is well tame, then $\left(\tilde{\partial}'\right)^2 = 0$, and therefore the cohomology $H^*\left(C^\bullet(W^u),\tilde{\partial}'\right)$ is well defined. Moreover, there exists $T_0 \geq 0$ such that $H^*_{(2)}(M,d_{Tf})$ is isomorphic to $H^*\left(C^\bullet(W^u),\tilde{\partial}'\right)$ for all $T > T_0$. In addition, $H^*\left(C^\bullet(W^u),\tilde{\partial}'\right)$, and hence $H^*_{(2)}(M,d_{Tf})$, is isomorphic to the relative de Rham cohomology $H^*_{dR}(M,U_c)$.*

Remark 1.5. *When (M,g,f) is strongly tame, $T_0 = 0$.*

By Theorem 1.3, we can refine our result of Theorem 1.2:

Corollary 1.4. *If (M,g,f) is well tame, then $b_i(T)$ is independent of T when T is big enough. When (M,g,f) is strongly tame, $b_i(T)$ is independent of $T>0$.*

Remark 1.6. *Assume that M is oriented and let $*$ be the Hodge star operator. Then $*\square_{Tf} = \square_{-Tf}*$. Hence we have the following Poincaré duality:*

$$H^k(M,d_{Tf}) \cong H^{n-k}(M,d_{-Tf}).$$

As an another application of Theorem 1.3, we study the Morse cohomology for compact manifolds with boundary.

Let M be a compact, oriented manifold of dimension n with boundary ∂M. Let N_i $(i \in \Lambda)$ be the connected components of ∂M. We fix a collar neighbourhood $(0,1] \times N_i \subset M$, and let r be the standard coordinate on the $(0,1]$ factor.

Definition 1.7. *A smooth function f on M is called a transversal Morse function if it satisfies the following conditions:*

1. *$f|_{M \setminus \partial M}$ is a Morse function on the manifold $M \setminus \partial M$.*
2. *$f|_{\partial M}$ is a Morse function on the manifold ∂M.*
3. *For any point x on the collar neighbourhood, $-\frac{\partial f}{\partial r}\Big|_x \neq 0$.*

For a transversal Morse function f on M, since $-\frac{\partial f}{\partial r}$ is continuous on any connected components of ∂M, we can call N_i positive (with respect to f) if $-\frac{\partial f}{\partial r}\Big|_{N_i} > 0$ and negative if $-\frac{\partial f}{\partial r}\Big|_{N_i} < 0$.

Let N^+ be the union of all positive boundaries and N^- the union of all negative boundaries. Suppose we have a partition of positive boundaries $N^+ = N_1^+ \sqcup N_2^+$ and a partition of negative boundaries $N^- = N_1^- \sqcup N_2^-$. Now we denote

- by $\mathrm{Crit}^{\circ,k}(f)$ the set of internal critical points of f with Morse index k, $m_k = \left|\mathrm{Crit}^{\circ,k}(f)\right|$;
- by $\mathrm{Crit}^{+,k}_{N_j^+}(f)$ $(j = 1,2)$ the set of critical points of $f|_{N_j^+}$ on the positive boundary N_j^+ with Morse index $k-1$, $n_{k,N_j^+} = \left|\mathrm{Crit}^{+,k}_{N_j^+}(f)\right|$;
- by $\mathrm{Crit}^{-,k}_{N_j^-}(f)$ $(j = 1,2)$ the set of critical points of $f|_{N_j^-}$ on the negative boundary N_j^- with Morse index k, $l_{k,N_j^-} = \left|\mathrm{Crit}^{-,k}_{N_j^-}(f)\right|$.

Let $\mathrm{Crit}^*(f) = \mathrm{Crit}^{\circ,*}(f) \cup \mathrm{Crit}^{+,*}_{N_1^+}(f) \cup \mathrm{Crit}^{-,*}_{N_1^-}(f)$.

Theorem 1.5. *There is a differential $\tilde{\partial}' : \mathrm{Crit}^k(f) \to \mathrm{Crit}^{k+1}(f)$ making $\left(\mathrm{Crit}^*(f) \otimes \mathbb{R}, \tilde{\partial}'\right)$ a chain complex. Moreover, $H^*\left(\mathrm{Crit}^\bullet(f) \otimes \mathbb{R}, \tilde{\partial}'\right)$ is isomorphic to the relative de Rham cohomology $H^*\left(M, N_2^- \cup N_1^+\right)$.*

In particular, for $N_1^+ = N^+$ and $N_1^- = \emptyset$, $H^*\left(\mathrm{Crit}^\bullet(f) \otimes \mathbb{R}, \tilde{\partial}'\right)$ is isomorphic to the relative de Rham cohomology $H^*(M, \partial M)$.

Corollary 1.6. *Set $b_i\left(M, N_2^- \cup N_1^+\right) = \dim\left(H^i\left(M, N_2^- \cup N_1^+\right)\right)$. Then we have the following Morse inequality:*

$$(-1)^k \sum_{i=0}^{k} (-1)^i b_i\left(M, N_2^- \cup N_1^+\right) \leq (-1)^k \sum_{i=0}^{k} (-1)^i \left(m_i + n_{i-1,N_1^+} + l_{i,N_2^-}\right).$$

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Remark 1.8.

1. *Theorem 1.5 is a generalisation of a result in [16].*
2. *Corollary 1.6 is a generalisation of a result in [18].*

1.4. Notation and organisation

In this article, we will generally use ϕ,ψ to denote differential forms; f a Morse function; u,v functions; ν,ω eigenforms; x,y,z critical points of f; p,q general points; and p_0 a fixed point.

This paper is organised as follows. In Section 2 we discuss the spectral theory for the Witten Laplacian in our setting. We then proceed to establish the exponential decay estimate for eigenforms of the Witten Laplacian in Section 3. Assuming two technical results whose proofs are deferred to Sections 7.1 and 7.2 and using a lemma proved in Section 4 about the Agmon distance, we prove Theorem 1.1, the Agmon estimate.

Section 4 concerns the Thom–Smale theory in our setting. More specifically, we define the Thom–Smale complex $C^*\left((W^u)',\tilde{\partial}'\right)$. Then we define a morphism between the Witten instanton complex and the Thom–Smale complex, $\mathcal{J}:\left(F_{Tf}^{[0,1],*},d_{Tf}\right)\mapsto C^*\left((W^u)',\tilde{\partial}'\right)$. We prove that $\mathcal{J}$ is well defined by using the Agmon estimate, deferring the proof that $\mathcal{J}$ is a chain map to Section 7.3.

In Section 5 we present a direct proof of the strong Morse inequalities (Theorem 1.2).

In Section 6 we give an application of our results, namely Theorem 1.5. Section 7 collects the proofs of several technical results. In the first two subsections we prove the lemmas used in the proof of Agmon estimate. In Section 7.3 we prove that our Thom–Smale complex is indeed a complex – that is, $\tilde{\partial}'^2=0$. The rest of the proof of Theorem 1.3 is in Sections 7.4 and 7.5. Finally, in Appendix Appendix A we discuss the Kodaira decomposition in a more general setting.

2. The Spectrum of the Witten Laplacian

In this section we study the spectral theory of the Witten Laplacian on noncompact manifolds. In particular, we establish the Kodaira decomposition and the Hodge theorem for the Witten Laplacian under our tameness condition.

2.1. Essential self-adjointness of $d_f+\delta_f$

Theorem 2.1. *On a complete Riemannian manifold, if*

$$\limsup_{p\to\infty}\frac{\left|\nabla^2 f\right|(p)}{\left|\nabla f\right|^2(p)}<\infty,$$

then $d_f+\delta_f$ is essentially self-adjoint.

Proof. Since $\limsup_{p\to\infty}\frac{\left|\nabla^2 f\right|(p)}{\left|\nabla f\right|^2(p)}<\infty$, $\square_f$ is bounded from below by Proposition 1.4. The rest of the proof is essentially the same as in [4, Section 4]; see also the proof of [11, Theorem 1.17]. $\square$

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2.2. On the spectrum of $\Box_{Tf}$

From now on we will assume that (M,g,f) is well tame. Let K be a compact subset, which can be taken to be a compact submanifold with boundaries that contains the closure of a ball of sufficiently large radius of M (we will make a more specific choice of K later in Section 4), such that $\epsilon_f(K) := \inf_{M-K}|\nabla f| > 0, c_f(K) := \sup_{M-K}\frac{|\nabla^2 f|}{|\nabla f|^2} < \infty$. Then on $M-K$,

$$|\nabla f| > \frac{1}{2}\epsilon_f, \qquad \left|\nabla^2 f\right| < 2c_f\,|\nabla f|^2. \tag{2.1}$$

Let $C_K = \max_K\left|\nabla^2 f\right|$. First we establish the following basic lemma:

Lemma 2.1. *Fix any $b \in (0,1)$. There exists $T_1 = T_1(c_f, C_R, \epsilon_f, b) \geq 0$ so that whenever $T > T_1$, $\phi \in \operatorname{Dom}(\Box_{Tf})$. Then*

$$\begin{aligned}\int_M (\Box_{Tf}\phi,\phi)\,\mathrm{dvol} \geq \int_M (\nabla\phi,\nabla\phi)\mathrm{dvol} + \int_{M-K} b^2T^2\,|\nabla f|^2\,(\phi,\phi)\mathrm{dvol}\\ -(C_R+TC_K)\int_K(\phi,\phi)\mathrm{dvol}.\end{aligned} \tag{2.2}$$

Here C_R is a constant depending only on the sectional curvature bound of g.

Proof. It suffices to show the inequality for a compactly supported smooth form. By Proposition 1.4, together with the Bochner–Weitzenböck formula, we have

$$\begin{aligned}\int_M (\Box_{Tf}\phi,\phi)\,\mathrm{dvol} \geq \int_M (\nabla\phi,\nabla\phi)\mathrm{dvol} - (C_R+TC_K)\int_K(\phi,\phi)\mathrm{dvol}\\ +\int_{M-K} e_T(p)(\phi,\phi)\mathrm{dvol},\end{aligned}$$

where $e_T = T^2\,|\nabla f|^2\left(1-\frac{4c_f}{T}-\frac{4C_R}{T^2\epsilon_f^2}\right)$. Thus, for any $b\in(0,1)$, define

$$T_1(K) := \max\left\{\frac{8c_f}{1-b^2}, \frac{\sqrt{8C_R}}{\epsilon_f\sqrt{1-b^2}}\right\}. \tag{2.3}$$

Then whenever $T > T_1$, one can see $1-\frac{4c_f}{T}-\frac{4C_R}{T^2\epsilon_f^2} > b^2$. Consequently,

$$\begin{aligned}\int_M (\Box_{Tf}\phi,\phi)\,\mathrm{dvol} \geq \int_M (\nabla\phi,\nabla\phi)\mathrm{dvol} + \int_{M-K}|bT\nabla f|^2\,(\phi,\phi)\mathrm{dvol}\\ -(C_R+TC_K)\int_K(\phi,\phi)\mathrm{dvol}.\end{aligned}$$

$\square$

Remark 2.2. *When (M,g,f) is strongly tame, we can take K to be sufficiently large so that c_f and $\frac{1}{\epsilon_f}$ are as small as we want. As a result, T_1 can be made as small as we want by an appropriate choice of K.*

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Theorem 2.2. *Let σ be the set of spectrum of $\square_{Tf}$. Then when $T>T_1$, $\sigma\cap\left[0,\left(\frac{b\epsilon_f}{2}\right)^2T^2\right]$ consists of a finite number of eigenvalues of finite multiplicity.*

Proof. Let $P: L^2\Lambda^*(M)\to L^2\Lambda^*(M)$ be the integral of the spectral measure of $\square_{Tf}$ on $\left[0,\left(\frac{b\epsilon_f}{2}\right)^2T^2\right]$. It suffices to prove that $L:=Im(P)$ is finite-dimensional. For any $\phi\in L$, we have

$$\int_M(\square_{Tf}\phi,\phi)\,dvol\le\left(\frac{b\epsilon_f}{2}\right)^2T^2\int_M|\phi|^2dvol. \tag{2.4}$$

Combining with formula (2.2), we have

$$\begin{aligned}\left(\frac{b\epsilon_f}{2}\right)^2T^2\int_M|\phi|^2dvol\ge\int_M(\nabla\phi,\nabla\phi)\mathrm{dvol}+\int_{M-K}|bT\nabla f|^2(\phi,\phi)\mathrm{dvol}\\ -(C_R+TC_K)\int_K(\phi,\phi)\mathrm{dvol},\end{aligned}$$

provided $T>T_1$. That is,

$$\begin{aligned}&\int_M(\nabla\phi,\nabla\phi)\mathrm{dvol}+\int_{M-K}|bT\nabla f|^2(\phi,\phi)\mathrm{dvol}\\ &\le\left(\frac{b\epsilon_f}{2}\right)^2T^2\left(1+\frac{4C_R}{(b\epsilon_f)^2T^2}+\frac{4C_K}{(b\epsilon_f)^2T}\right)\int_K(\phi,\phi)\mathrm{dvol}+\left(\frac{b\epsilon_f}{2}\right)^2T^2\int_{M-K}(\phi,\phi)\mathrm{dvol}.\end{aligned} \tag{2.5}$$

Since $|bT\nabla f|^2>\left(\frac{b\epsilon_f}{2}\right)^2T^2$ on $M-K$, when $T>T_1$,

$$\int_M(\nabla\phi,\nabla\phi)\mathrm{dvol}\le\left(\frac{b\epsilon_f}{2}\right)^2T^2\left(1+\frac{4C_R}{(b\epsilon_f)^2T^2}+\frac{4C_K}{(b\epsilon_f)^2T}\right)\int_K(\phi,\phi)\mathrm{dvol}. \tag{2.6}$$

Now define $Q: L\to L^2\Lambda^*(K)$, by $Qu=u|_K$. By formula (2.6), it's easy to see that Q is injective, and $Im(Q)\subset W^{1,2}(\Lambda^*K)$. Since $W^{1,2}(\Lambda^*K)\hookrightarrow L^2\Lambda^*(K)$ is compact, $\dim(L)=\dim(\mathrm{Im}(Q))$ must be finite. $\square$

We now state the important consequence of this section. By combining Theorems 2.1 and 2.2 with Proposition A.3 and decomposition (A.4), we have the following:

Theorem 2.3. *Assume that (M,g,f) is well tame. Then when $T>T_1$, we have the Kodaira decomposition*

$$L^2\Lambda^*(M)=\ker\square_{Tf}\oplus\mathrm{Im}\left(\bar{d}_{Tf}\right)\oplus\mathrm{Im}\left(\bar{\delta}_{Tf}\right).$$

Furthermore, the Hodge theorem holds:

$$H^*_{(2)}(M,d_{Tf})\cong\ker\square_{Tf}.$$

Remark 2.3. *If (M,g,f) is strongly tame, T_1 could be arbitrarily small, and hence Theorem 2.3 holds true for any $T>0$.*

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3. Exponential decay of eigenfunctions

In this section, we assume that (M,g,f) is well tame and $T > T_1$, where T_1 is described in Lemma 2.1.

Let $\tilde{g}_T := b^2T^2|\nabla f|^2 g$ be the Agmon metric on M. Let K be the compact set as in the previous section. In this and later sections, we define the Agmon distance $\rho_T(p)$ as the distance between p and K induced by $\tilde{g}_T$. Then we have $|\nabla\rho_T|^2 = b^2T^2|\nabla f|^2$ almost everywhere that $p \notin K$, where the gradient ∇ is induced by g.

For simplicity, denote $b^2T^2|\nabla f|^2$ by λ_T. We need the following two technical lemmas, whose proofs are postponed to Section 7.

Lemma 3.1. *Assume $w \in L^2(M), 0 \le u \in L^2(M)$ and $(\Delta + \lambda_T)u \le w$ outside the compact subset $K \subset M$ in the weak sense. That is,*

$$\int_{M-K} \nabla u \nabla v + \lambda_T uv\mathrm{dvol} \le \int_{M-K} w \cdot v\mathrm{dvol}, \quad \forall 0 \le v \in C_c^\infty(M-K).$$

Then for any $j \in \mathbb{N}$, there exists another compact subset $L \supset K$ of M such that

$$\begin{aligned}\int_{M-L} |u|^2\lambda_T \exp(2b\rho_{T,j})\mathrm{dvol} \le{}& C_2\int_{M-K}|w|^2\lambda_T^{-1}\exp(2b\rho_{T,j})\mathrm{dvol}\\ &+ C_1\int_{L-K}|u|^2\lambda_T\exp(2b\rho_{T,j})\mathrm{dvol}\end{aligned} \tag{3.1}$$

$a \in (0, \frac{\sqrt{2}}{2})$, for $C_1 = \frac{8(1+b^2)}{(1-b^2)^2}$, $C_2 = \frac{4}{(1-b^2)^2}$.

Here $\rho_{T,j} := \min\{\rho_T, j\}$.

Corollary 3.1. *If $w = cu$ for some $c > 0$ and $T > \frac{2\sqrt{1+c}}{b\epsilon_f}$, then*

$$I(u) := \int_M |u|^2\exp(2b\rho_T)\mathrm{dvol} < \infty.$$

Proof. With this choice of T, $\lambda_T > 1 + c$ outside K. Now replacing λ_T with $\lambda_T - c$ and w with 0 in Lemma 3.1, we get

$$\begin{aligned}&\int_{M-L}|u|^2\exp(2b\rho_{T,j})\mathrm{dvol} \le \int_{M-L}|u|^2(\lambda_T - c)\exp(2b\rho_{T,j})\mathrm{dvol}\\ &\le C_1\int_{L-K}|u|^2\lambda_T\exp(2b\rho_{T,j})\mathrm{dvol} \le C_1\int_{L-K}|u|^2\lambda_T\exp(4b)\mathrm{dvol} < \infty.\end{aligned}$$

Now let $j \to \infty$. By the monotone convergence theorem, we finish the proof. □

By refining this argument, we have the following corollary which will be used in the proof of our Agmon estimate for eigenforms:

Corollary 3.2. *If $0 \le u \in \mathrm{Dom}(\square_{Tf})$ and $\square_{Tf}u \le (c + T|\nabla^2 f|)u$ for some $c > 0$ and $T > \max\left\{\sqrt{\frac{3}{b^2\epsilon_f}}, \sqrt{\frac{2C_2c}{b^2\epsilon_f}}, 2c_fC_2\right\}$, then*

$$I(u) := \int_M |u|^2\exp(2b\rho_T)\mathrm{dvol} \le CT^2\|u\|^2,$$

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where the constant $C = C\left(C^L, c_f, \epsilon_f, b, c\right)$, $L = \{p \in M : \rho_T(p) \leq 2\}$ *and* $C^L > \max_L |\nabla f|^2$.

Proof. Following the proof of Lemma 3.1 given in Section 7.1, put $L = \{p \in M : \rho_T(p) \leq 2\}$. Since $u \in \mathrm{Dom}(\square_{Tf})$, then $|\nabla f| u \in L^2(M)$.

Then by Lemma 3.1 we deduce

$$\int_{M-L} |u|^2 \lambda_T \exp(2b\rho_{T,j}) \,\mathrm{dvol} \leq C_1 \int_{L-K} |u|^2 \lambda_T \exp(2b\rho_{T,j}) \,\mathrm{dvol} + C_2 \int_{M-K} \left(c + T\left|\nabla^2 f\right|\right) \lambda_T^{-1} |u|^2 \exp(2b\rho_{T,j}) \,\mathrm{dvol}$$

for C_1, C_2 as before.

Since $u \in L^2(M)$, $\int_{M-K} \left(c + T\left|\nabla^2 f\right|\right) \lambda_T^{-1} |u|^2 \exp(2b\rho_{T,j}) \,\mathrm{dvol} < \infty$.

We split the second integral on the right-hand side into two; the one over $L-K$ will be absorbed into the first term. The second term (we omit the volume form here) is

$$C_2 \int_{M-L} \left(c + T\left|\nabla^2 f\right|\right) \lambda_T^{-1} |u|^2 \exp(2b\rho_{T,j}) \leq \left(\frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}\right) \int_{M-L} |u|^2 \exp(2b\rho_{T,j}).$$

Combining the foregoing, we arrive at

$$\int_{M-L} |u|^2 \lambda_T \exp(2b\rho_{T,j}) \leq C_1 \int_{L-K} |u|^2 \left(\lambda_T + \frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}\right) \exp(2b\rho_{T,j}) + C_2 \left(\frac{4c}{b^2 T^2 \epsilon_f^2} + \frac{2c_f}{T}\right) \int_{M-L} |u|^2 \exp(2b\rho_{T,j}).$$

Thus,

$$\int_{M-L} |u|^2 \left(\lambda_T - \frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}\right) \exp(2b\rho_{T,j}) \leq 2C_1 \left(C^L b^2 T^2 + \frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}\right) e^{4b} \|u\|^2,$$

where $C^L > \max_L |\nabla f|^2$. If $T > \max\left\{2\frac{\sqrt{3}}{b\epsilon_f}, \frac{2\sqrt{C_2 c}}{b\epsilon_f}, 2c_f C_2\right\}$, then

$$\lambda_T - \left(\frac{4C_2 c}{b^2 T^2 \epsilon_f^2} + \frac{2C_2 c_f}{T}\right) > 1$$

outside L. Hence

$$\int_{M-L} |u|^2 \exp(2b\rho_{T,j}) \,\mathrm{dvol} \leq 2C_1 \left(C^L b^2 T^2 + C_2 \left(\frac{2c}{b^2 T^2 \epsilon_f} + \frac{2c_f}{T}\right)\right) e^{4b} \|u\|^2,$$

and consequently

$$\int_M |u|^2 \exp(2b\rho_{T,j})\,\mathrm{dvol} \le \left[2C_1\left(C^L b^2T^2 + \frac{2C_2c}{b^2T^2\epsilon_f} + \frac{2C_2c_f}{T}\right)+1\right] e^{4b}\|u\|^2,$$

for $T > \max\left\{\frac{2\sqrt{3}}{b\epsilon_f}, \frac{2\sqrt{C_2c}}{b\epsilon_f}, 2c_fC_2\right\}$.

Now let $j \to \infty$. By the monotone convergence theorem again, we finish the proof. $\square$

Remark 3.2. *It may seem that C^L and C_L depend on T, as $L = \{p \in M : \rho_T(p) \le 2\}$. However, notice that as T becomes bigger, L gets smaller. Hence we can choose $C^L > \max_{p\in L}|\nabla f|(p)$ and $C_L > \max_{p\in L}\left|\nabla^2 f(p)\right|$, which are then independent of T.*

Lemma 3.3 (De Giorgi–Nash–Moser estimates)**.** *For $r > 0$, let $B_r(p)$ be the geodesic ball around p with radius r (in the metric g). Let $0 \le u \in L^2(M)$, and $\Delta u \le cu$ on $B_{2r}(p)$ in the weak sense for some constant $c \ge 0$. Then there exists constant $C_3(n,c,r_0,R) > 0$ depending only on the dimension n, the Sobolev constant (which depends on the injectivity radius lower bound r_0 and curvature bound on R) and c, such that for $r \le r_0$*

$$\sup_{y\in B_r(p)} u(y) \le \frac{C_3}{r^{n/2}}\|u\|_{L^2(B_{2r}(p))}.$$

With this preparation we are now ready to prove our first main estimate for the eigenforms of $\square_{Tf}$.

Proof of Theorem 1.1. Consider an eigenform ω of $\square_{Tf}$. That is, $\square_{Tf}\omega = \mu(T)\omega$, where the eigenvalue $\mu(T)$ satisfies $|\mu(T)| \le c$ for some constant c. Then letting $u = g(\omega,\omega)^{1/2}$, by a straightforward computation using Bochner's formula (for forms) and Kato's inequality, we have

$$\square_{Tf} u \le \left(c + |R| + T\left|\nabla^2 f\right|\right) u,$$

where $|R|$ is the upper bound of curvature tensor. Hence by Corollary 3.2 we have, for $T \ge \max\left\{\frac{2\sqrt{3}}{b\epsilon_f}, \frac{2\sqrt{C_2(c+|R|)}}{b\epsilon_f}, 2c_fC_2\right\}$,

$$I(u) = \int_M |u|^2 \exp(2b\rho_T)\mathrm{dvol} \le CT^2\|u\|^2,$$

where the constant $C = C\left(C^L, c_f, \epsilon_f, b, c, |R|, n\right)$.

Recall that for the compact set K, formula (2.1) is satisfied. Hence by Proposition 1.4, the conditions of Lemma 3.3 are satisfied for u on $M - K$. Namely, for $T > T_1$,

$$\Delta u \le (c + |R|)\, u$$

on $M - K$. Also, the Agmon distance $\rho_T(p)$ is the distance between p and K induced by $\tilde{g}_T$ and $L = \{p \in M : \rho_T(p) \le 2\}$. Suppose $p \in M - L$. Denote by $\tilde{B}_r(p)$ the $\tilde{g}_T$-geodesic ball around p with radius r. Set $l = \sup_{q\in\tilde{B}_2(p)}|T\nabla f|(q)$ and $r = 1/(2l)$. Then one can easily verify that $B_{2r}(q) \subset \tilde{B}_2(p)$, for all $q \in \tilde{B}_1(p)$.

Choose $q_0 \in \overline{\tilde{B}_2(p)}$ so that $|T\nabla f|(q_0) \in (l/2,l]$. By Lemma 3.1 and the de Giorgi–Nash–Moser estimate (Lemma 3.3), we have

$$\begin{aligned}|u(p)|^2 \exp(2b\rho_T(p)) &\leq \frac{C_3(n,c,r_0,R)}{r^n}\|u\|^2_{L^2(B_{2r}(p))}\exp(2b\rho_T(p))\\ &\leq \frac{C_4(n,c,r_0,R)}{r^n}\int_{\tilde{B}_2(p)}|u|^2(q)\exp(2b\rho_T(q))\mathrm{dvol}\\ &\leq C_5\left(C^L,c_f,\epsilon_f,b,c,R,n,r_0\right)|T\nabla f(q_0)|^n I(u).\end{aligned}$$

We will prove that

$$|\nabla f(q_0)|^2 \leq \sup_{p'\in K}|\nabla f|^2(p')\exp\left(\frac{2c_f}{bT}\rho_T(q_0)\right) \tag{3.2}$$

in Lemma 4.6. Hence,

$$|\nabla f(q_0)|^2 \leq \sup_{p'\in K}|\nabla f|^2(p')\exp\left(\frac{2c_f}{bT}\rho_T(q_0)\right) \leq \sup_{p'\in K}|\nabla f|^2(p')\exp(\epsilon\rho_T(q_0))$$

for any small ϵ, provided $T \geq \frac{2c_f}{b\epsilon}$. It follows then that

$$|u(p)|^2 \leq C_6\left(C^L,c_f,\epsilon_f,a,b,c,R,n,r_0\right)I(u)T^n\exp(-2a\rho_T(p))$$

for any $a<b$, provided $T \geq \frac{nc_f}{b(b-a)}$. Hence if

$$T \geq T_2(K) := \max\left\{\frac{2\sqrt{3}}{b\epsilon_f},\frac{2\sqrt{C_2(c+|R|)}}{b\epsilon_f},2c_fC_2,\frac{nc_f}{b(b-a)}\right\}, \tag{3.3}$$

then we have

$$|u(p)|^2 \leq C_7\left(C^L,c_f,\epsilon_f,a,b,c,r_0,|R|,n\right)T^{n+2}\exp(-2a\rho_T(p))\|u\|^2. \qquad \square$$

Remark 3.4. *The foregoing proof gives the inequality for $p \in M-L=\{\rho_T(p)>r_0\}$ for some constant r_0 independent of T, which is what we need for later applications. For $p\in L$, using the same reasoning as in Remark 3.2, there exists a constant $C>0$, which is independent of T, such that*

$$\Delta u \leq CTu \tag{3.4}$$

for all $p \in L$. Therefore via Moser iteration as in Lemma 3.3 and similar arguments as before, one can show that

$$|u|^2(p) \leq C'T^n\|u\|^2_{L^2} \leq C'\exp(2a)T^n\exp(-a\rho_T)\|u\|^2_{L^2}.$$

Remark 3.5. *When (M,g,f) is strongly tame, T_2 can be arbitrarily small.*

4. Thom–Smale theory

In this and the next section, we assume that f is a Morse function on M. Moreover, let K be a suitable compact subset of M such that $\epsilon_f(K)>0$, $\epsilon>0$ is small enough (to be

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determined later) and $T_5(\epsilon,K) := \frac{c_f}{\epsilon}$. Then outside K we have

$$T\left|\nabla^2 f\right| \leq \epsilon T^2 \left|\nabla f\right|^2, \tag{4.1}$$

provided $T \geq T_5$.

Remark 4.1. *We can take T_5 to be arbitrarily small if (M,g,f) is strongly tame.*

In this section, we always assume that $T \geq T_5$. Under these conditions we will define the Thom–Smale complex $\left(C_*(W^u),\tilde{\partial}\right)$, but we leave the proof that $\tilde{\partial}^2 = 0$ to Section 7.3. The remainder of this Section is devoted to the pairing between the Thom–Smale complex and the Witten instanton complex.

Before defining the Thom–Smale complex, there is still a subtle issue for noncompact cases. The gradient vector field $-\nabla f$ may not be complete – that is, its flow curves may not exist for all time. But notice that if we rescale the vector field by some positive function, the corresponding integral curves will simply be reparameterisations of the original integral curves.

For this purpose, we fix a positive smooth function F such that

$$F|_{M-K} = \frac{1}{bT\left|\nabla f\right|^2}.$$

Then we have the following:

Lemma 4.2.

1.

$$bT|f(p)-f(q)| \leq \tilde{d}_T(p,q), \quad \forall p,q \in M. \tag{4.2}$$

2. *Let $\tilde{\Phi}^t$ be the flow generated by $Y_f := -F\nabla f$, and $p \in M$. If the flow line $\tilde{\Phi}^t(p)$, $t \in [s_1,s_2]$ is outside K, then it is the minimal geodesic connecting $\tilde{\Phi}^{s_1}(p)$ and $\tilde{\Phi}^{s_2}(p)$ with respect to metric $\tilde{g}_T$. Moreover,*

$$\tilde{d}_T\left(\tilde{\Phi}^{s_1}(p),\tilde{\Phi}^{s_2}(p)\right) = bT\left|f\left(\tilde{\Phi}^{s_1}(p)\right) - f\left(\tilde{\Phi}^{s_2}(p)\right)\right| = |s_2 - s_1|. \tag{4.3}$$

3. *Y_f is a complete vector field.*

Proof. (1) Let $\gamma : \left[0,\tilde{d}_T(p,q)\right] \to M$ be the minimal geodesic connecting p and q with respect to $\tilde{g}_T$. Let $\tilde{\nabla}^T$ be the Levi–Civita connection induced by $\tilde{g}_T$. One computes

$$\begin{aligned}\frac{d}{ds}f(\gamma(s)) &= \tilde{g}_T\left(\tilde{\nabla}^T f,\gamma'(s)\right) = \tilde{g}_T\left(\frac{\nabla f}{b^2T^2\left|\nabla f\right|^2},\gamma'(s)\right)\\ &\leq \sqrt{\tilde{g}_T\left(\frac{\nabla f}{b^2T^2|\nabla f|^2},\frac{\nabla f}{b^2T^2|\nabla f|^2}\right)} = \frac{1}{bT}.\end{aligned}$$

Consequently, $bT|f(p)-f(q)| \leq \tilde{d}_T(p,q)$.

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(2) We give a direct proof (see [13, Lemma A2.2] for another).

We first show that $\gamma(s) := \tilde{\Phi}^s(p)$, $s \in [s_1,s_2]$, is a geodesic. Since $\tilde{g}_T(Y_f,Y_f) = 1$ outside K, we let $\tilde{e}_1^T(s),...,\tilde{e}_n^T(s)$ be a local orthomormal frame on γ with $\tilde{e}_1^T = Y_f$. One can easily show that outside K, $-Y_f/(bT)$ is the gradient of f with respect to $\tilde{g}_T$. In order to prove $\gamma'' = 0$, it suffices to prove $\tilde{g}_T\left(\gamma'',\tilde{e}_i^T\right) = 0, i \geq 2$.

Indeed,

$$\begin{aligned}
\tilde{g}_T\left(\gamma'',\tilde{e}_i^T\right) &= \tilde{g}_T\left(\tilde{\nabla}^T_{Y_f} Y_f,\tilde{e}_i^T\right) \\
&= -\tilde{g}_T\left(Y_f,\left[Y_f,\tilde{e}_i^T\right]\right) \\
&= \left[Y_f,\tilde{e}_i^T\right] f \\
&= -Y_f \tilde{g}_T\left(\tilde{e}_i^T,\frac{Y_f}{bT}\right) + \tilde{e}_i^T \tilde{g}_T\left(Y_f,\frac{Y_f}{bT}\right) \\
&= 0.
\end{aligned}$$

We now prove that γ is the shortest geodesic connecting $\gamma(s_1)$ and $\gamma(s_2)$ in $(M,\tilde{g}_T)$. Assume that $\sigma : [s_1,s_2'] \mapsto M$ is another normal geodesic connecting $\gamma(s_1)$ and $\gamma(s_2)$ induced by $\tilde{g}_T$. Then $\tilde{g}_T(\sigma'(s_1),Y_f) < 1$. Set $\alpha(s) = f\circ\gamma(s)$ and $\beta(s) = f\circ\sigma(s)$; then we have $\alpha(s_1) = \beta(s_1)$, $\alpha'(s) = -1$ and $\beta'(s) = -\tilde{g}_T(\sigma'(s),Y_f\circ\gamma(s)) \geq -1$. Hence by a comparison theorem in ordinary differential equations, we must have $\alpha(s) \leq \beta(s)$. Now $\sigma(s_2') = \gamma(s_2)$ and $s_2' - s_1 = Length(\sigma)$. Thus $a(s_2') \leq \beta(s_2') = \alpha(s_2)$. Since α is decreasing, we must have $s_2' \geq s_2$.

Therefore, $\tilde{\Phi}^s(y)$, $s \in [s_1,s_2]$, is one of the shortest geodesics connecting y and $\tilde{\Phi}^t(y)$.

Hence $\tilde{d}_T\left(\tilde{\Phi}^{s_1}(p),\tilde{\Phi}^{s_2}(p)\right) = |s_2 - s_1|$. Moreover, since $\frac{\partial}{\partial s} f\left(\tilde{\Phi}^s(p)\right) = Y_f f = g(\nabla f,Y_f) = \frac{1}{bT}$, we have

$$bT\left|f\left(\tilde{\Phi}^{s_1}(p)\right) - f\left(\tilde{\Phi}^{s_2}(p)\right)\right| = |s_2 - s_1|.$$

(3) To prove that Y_f is complete, we show that for any $p \in M$, there exists a uniform constant $\epsilon_0 > 0$ such that $\tilde{\Phi}^t(p)$ is well defined on $(-\epsilon_0,\epsilon_0)$.

Recall that $L := \left\{p \in M : \tilde{d}_T(p,K) \leq 2\right\}$. It suffices to show that for any $p \in M - L$, $\tilde{\Phi}^t(p)$ is well defined on $(-1,1)$, as L is compact.

But this is clear: on $M - K$, $bTF^{-1}g(Y_f,Y_f) = \tilde{g}_T(Y_f,Y_f) = 1$, and $\left(M,bTF^{-1}g\right)$ is complete, and $\tilde{\Phi}^t(p)$, $t \in (-1,1)$, is a geodesic inside $M - K$ with respect to $bTF^{-1}g$. □

Now we can talk about the unstable and stable manifolds of Y_f.

Let x be a critical point of the Morse function f and $W^s(x)$ and $W^u(x)$ be the stable and unstable manifold of x, respectively, with respect to flow $\tilde{\Phi}^t$ defined in Lemma 4.2. (See [23, Chapter 6] for a precise definition of stable and unstable manifolds.) We will further assume that f satisfies the Smale transversality condition, namely that $W^s(x)$ and $W^u(y)$ intersect transversally. Then the Thom–Smale complex $\left(C_*(W^u),\tilde{\partial}\right)$ is defined by

$$C_*(W^u) = \oplus_{x\in \mathrm{Crit}(f)} \mathbb{R} W^u(x),$$

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and

$$C_i(W^u) = \oplus_{x\in \mathrm{Crit}(f), n_f(x)=i}\mathbb{R}W^u(x).$$

To define the boundary operator, let x and y be critical points of f, with $n_f(y) = n_f(x) - 1$.

For $x \in \mathrm{Crit}(f)$, set

$$\tilde{\partial} W^u(x) = \sum_{y\in \mathrm{Crit}(f), n_f(y)=n_f(x)-1} m(x,y)W^u(y).$$

Here the integer $m(x,y)$ is the signed counts of the flow lines in $W^s(y)\cap W^u(x)$.

In order to see that the integer $m(x,y)$, and hence the coboundary operator, is well defined, we now make a more judicious choice of K. Fix any $p_0 \in M$. Let $\tilde{d}$ be the distance function induced by (the Agmon metric) $|\nabla f|^2 g$, and set

$$D = \sup_{y\in \mathrm{Crit}(f)} \tilde{d}(y,p_0) + 2 \sup_{y,z\in \mathrm{Crit}(f)} \tilde{d}(y,z). \tag{4.4}$$

We choose K so that

$$\tilde{B}_{D+1}(p_0) \subset K^\circ,$$

where K° denotes the interior of K and $\tilde{B}_r(p_0) := \left\{p \in M : \tilde{d}(p,p_0) \le r\right\}$. From the definition of D, it is clear that all critical points are contained in K°. Moreover, we make the following remark:

Remark 4.3. *The choice of K together with equation (4.3) guarantees that for any $x,y \in \mathrm{Crit}(f)$, $W^s(x)\cap W^u(y) \subset K^\circ$. See also Lemma 7.6 for more detail.*

Thus, just as in the compact case, by transversality $m(x,y)$ is well defined.

We will prove in Section 7.3 that under our tameness condition, $\tilde{\partial}^2 = 0$. Thus, $\left(C_*(W^u),\tilde{\partial}\right)$ is a complex.

Let $F_{Tf}^{[0,1],*}$ be the space spanned by the eigenforms of $\Box_{Tf}$ with eigenvalue lying in $[0,1]$. By Theorem 2.2, $F_{Tf}^{[0,1],*}$ is finite-dimensional when T is big enough. By previous discussions, the cohomology of the Witten instanton complex is $H^*_{(2)}(M,d_{Tf})$ when T is large enough.

To prove Theorem 1.3, we now consider the chain map $\mathcal{J} : \left(F_{Tf}^{[0,1],*}, d_{Tf}\right) \mapsto C^*\left((W^u)',\tilde{\partial}'\right)$. Here $C^*\left((W^u)',\tilde{\partial}'\right)$ denotes the dual chain complex. Let $W^u(x)'$ be the dual basis of $W^u(x)$. Then

$$\mathcal{J}\omega = \sum_{x\in \mathrm{Crit}(f)} W^u(x)' \int_{W^u(x)} \exp(Tf)\omega.$$

However, there is a technical issue here we need to address. When $\overline{W^u(x)}$ is compact, the integral $\int_{W^u(x)} \exp(Tf)\omega$ is clearly well defined, but $\overline{W^u(x)}$ here may be noncompact. We will be content here only with the well-definedness of the map, and leave the proof that $\mathcal{J}$ is indeed a chain map to Section 7.3 (see Corollary 7.2).

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Let $r>0$ be small enough and $B_r^{n_f(x)}(x)\subset K$ be the $n_f(x)$-dimensional ball in $W^u(x)$ with center x and radius r with respect to metric g. As before, let $\tilde{\Phi}^t$ be the flow generated by $-F\nabla f$. Then $W^u(x)=\cup_{t>0}\tilde{\Phi}^t\left(B_r^{n_f(x)}(x)\right)$. Moreover, by the definition of unstable manifold, if $t_1<t_2$, then $\tilde{\Phi}^{t_1}\left(B_r^{n_f(x)}(x)\right)\subset\tilde{\Phi}^{t_2}\left(B_r^{n_f(x)}(x)\right)$.

Therefore, for any $\omega\in F_{Tf}^{[0,1],*}$,

$$\begin{aligned}\left|\int_{W^u(x)}\exp(Tf)\omega\right| &= \left|\lim_{t\to\infty}\int_{(\tilde{\Phi}^t)\left(B_r^{n_f(x)}(x)\right)}\exp(Tf)\omega\right|\\ &\leq C\exp(Tf(x))\lim_{t\to\infty}\int_{B_r^{n_f(x)}(x)}|\omega|\circ\tilde{\Phi}^t\left|\det\left(\left(\tilde{\Phi}^t\right)_*\right)\right|\mathrm{dvol}_{W^u(x)}.\end{aligned}$$

The well-definedness of $\mathcal{J}$ is now reduced to the following two technical lemmas, as well as Theorem 1.1 and the well tameness of (M,g,f).

Lemma 4.4. *Suppose $t>0$ is big enough and $y\in B_r^{n_f(x)}(x)-\tilde{\Phi}^{-t}K$. Then*

$$\left|\rho_T\left(\tilde{\Phi}^t(y)\right)-t\right|<T\sup_{p\in K}|\nabla f|\,\mathrm{diam}(K),$$

where $\mathrm{diam}(K)$ *is the diameter of K with respect to metric g.*

Proof. For any $y\in B_r^{n_f(x)}(x)-\tilde{\Phi}^{-t}K$, equation (4.3) and the triangle inequality give

$$\begin{aligned}\left|\rho_T\left(\tilde{\Phi}^t(y)\right)-t\right| &= \left|\tilde{d}_T\left(\tilde{\Phi}^t(y),K\right)-\tilde{d}_T\left(\tilde{\Phi}^t(y),y\right)\right|\\ &\leq T\sup_{p\in K}|\nabla f|\,\mathrm{diam}(K),\end{aligned}$$

where $\tilde{d}_T$ is the distance induced by $\tilde{g}_T$. ∎

Lemma 4.5. *Fix any $y\in B_r^{n_f(x)}(x)-\tilde{\Phi}^{-t}K$ and set $p=\tilde{\Phi}^t(y)$. We have*

$$\left|\tilde{\Phi}_*^t(y)\right|\leq C_7(T)\exp\left(\frac{6\epsilon\rho_T(p)}{b}\right).$$

Hence,

$$\left|\det\left(\Phi^t\right)_*(y)\right|\leq C_7(T)\exp\left(\frac{6n_f(x)\epsilon\rho_T(p)}{b}\right).$$

Here C_7 is a constant independent of y. In particular, for the fixed $a\in(0,1)$ in Theorem 1.1 and $b\in(a,1)$, any choice of $0<\epsilon\leq\frac{ab}{12n}$ will guarantee that $\mathcal{J}$ is well defined for $T>T_5(\epsilon)$.

Proof. Let e be a unit tangent vector of $W^u(x)$ at y and extend e to a local unit vector field (still denoted by e) of $W^u(x)$ near y. Noting that from Formula (4.1)

$$\left|\nabla_{\tilde{\Phi}_*^t e}\left(\tilde{\Phi}^t\right)_*(Y_f)\right|\leq\frac{3\epsilon}{b}\left|\left(\tilde{\Phi}^t\right)_*e\right|,$$

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we have

$$\begin{aligned}
&\left|\frac{\partial}{\partial t} g\left(\left(\tilde{\Phi}^t\right)_* e(y), \left(\tilde{\Phi}^t\right)_* e(y)\right)\right| \\
&= 2\left|g\left(\nabla_{\left(\tilde{\Phi}^t\right)_* e(y)} \left(\tilde{\Phi}^t\right)_* Y_f, \left(\tilde{\Phi}^t\right)_* e(y)\right)\right| \\
&\leq \frac{6\epsilon}{b}\left|g\left(\left(\tilde{\Phi}^t\right)_* e(y), \left(\tilde{\Phi}^t\right)_* e(y)\right)\right|.
\end{aligned}$$

By a classical result in ordinary differential equations, we have

$$g\left(\left(\tilde{\Phi}^t\right)_* e(y), \left(\tilde{\Phi}^t\right)_* e(y)\right) \leq C_8 \exp\left(\frac{6\epsilon t}{b}\right).$$

Our lemma follows from Lemma 4.4. □

Now when (M,g,f) is well tame, we set T_0 to be the smallest nonnegative number such that for all $\delta > 0$,

1. whenever $T \geq T_0 + \delta$, Theorem 1.1 holds true for the Agmon distance with respect to some compact subset $K(\delta) \subset M$ depending on δ,
2. Theorem 2.3 holds true whenever $T > T_0$ and
3. the map $\mathcal{J}$ is well defined whenever $T > T_0$.

Fix a compact set K as before. Then

$$T_0 \leq \max\{T_1(K), T_2(K), T_5(K)\}. \tag{4.5}$$

(Compare definition (2.3) for the description of T_1 and definition (3.3) for T_2.) Moreover, if (M,g,f) is strongly tame, $T_0 = 0$.

We note in passing the following lemma, which plays an important role in estimating the eigenforms previously:

Lemma 4.6. *Suppose $T \geq T_0$. Then for any $q \in M$,*

$$|\nabla f|^2(q) \leq \sup_{p\in K} |\nabla f|^2(p) \exp\left(\frac{2c_f}{bT}\rho_T(q)\right).$$

Proof. Let $\gamma : [0, \rho_T(q)] \mapsto M$ be a normal minimal $\tilde{g}_T$-geodesic connecting K and q. Then we have $g(\gamma', \gamma') = \frac{1}{b^2T^2|\nabla f|^2}$ outside K.

Let $h(t) = |\nabla f|^2 \circ \gamma$; then

$$h'(t) = 2g(\nabla_{\gamma'}\nabla f, \nabla f) \leq \frac{2}{bT}\left|\nabla^2 f\right| \leq \frac{2c_f}{bT}|\nabla f|^2 = \frac{2c_f}{bT} h(t),$$

and hence $|\nabla f|^2(q) \leq |\nabla f|^2 \circ \gamma(0) \exp\left(\frac{2c_f}{bT}\rho_T(q)\right)$. □

Now we give a direct proof of the isomorphism of $H^*_{(2)}(M, d_{Tf})$ and $H^*_{dR}(M, U_c)$ under the assumption that f is proper:

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Theorem 4.1. *Assume that f is proper. Set $I = \inf_{p\in K} f(p), S = \sup_{p\in K} f(p)$ and fix $c > |I| + |S| + 2$. Then for $U_c = \{p \in M : f(p) < -c\}$, $\left(\Omega^*_{(2)}(M), d_{Tf}\right)$ and $(\Omega^*(M,U_c),d)$ are quasi-isomorphic.*

Proof. We may as well set $K = f^{-1}[I,S]$. Motivated by [10], consider $(Cone^*, d_C)$, where $Cone^j = \Omega^j(M) \oplus \Omega^{j-1}(U_c)$:

$$d_C(\phi,\phi') = (d\phi, -d|_{U_c}\phi' + \phi|_{U_c}).$$

Then $(Cone^*, d_C)$ and $(\Omega^*(M,U_c),d)$ are quasi-isomorphic.

Set $U'_c = \{p \in M : f(p) > c\}$ and $U = U_c \cup U'_c$. Let $\bar\Phi^t$ be the flow in U generated by $X_f = -\frac{\nabla f}{bT|\nabla f|^2}$ on U_c, and $X_f = \frac{\nabla f}{bT|\nabla f|^2}$ on U'_c.

Define a map $\mathcal{L} : F^{[0,1],j}_{Tf} \mapsto Cone^j$:

$$\omega \mapsto \left(\exp(Tf)\omega, -\int_0^\infty \left(\bar\Phi^s\right)^* \left(\exp(Tf)\iota_{X_f}\omega\right) ds|_{U_c}\right).$$

By Theorem 1.1 and a similar argument as in Lemma 4.4, we have

$$\left|\left(\bar\Phi^s\right)^* \iota_{X_f}\omega\right| \le C\exp\left(-\int_0^s aT dt\right) \le C\exp(-aTs).$$

Hence, $\mathcal{L}$ is well defined.

To see that $\mathcal{L}$ is a chain map, one computes

$$\begin{aligned}
\mathcal{L}(d_{Tf}\omega) &= \left(\exp(Tf)d_{Tf}\omega, -\int_0^\infty \left(\bar\Phi^s\right)^* \left(\exp(Tf)\iota_{X_f} d_{Tf}\omega\right) ds\right)\\
&= \left(\exp(Tf)d_{Tf}\omega, -\int_0^\infty \left(\bar\Phi^s\right)^* \left(\iota_{X_f}\exp(Tf) d_{Tf}\omega\right) ds\right)\\
&= \left(d(\exp(Tf)\omega), -\int_0^\infty \left(\bar\Phi^s\right)^* \left(\iota_{X_f} d(\exp(Tf)\omega\right) ds\right)\\
&= \left(d(\exp(Tf)\omega), -\int_0^\infty \left(\bar\Phi^s\right)^* \left(L_{X_f}(\exp(Tf)\omega) - d\iota_{X_f}(\exp(Tf)\omega\right) ds\right)\\
&= \left(d(\exp(Tf)\omega), -\int_0^\infty \left[\frac{d}{ds}\left(\bar\Phi^s\right)^*(\exp(Tf)\omega) - \left(\bar\Phi^s\right)^*\left(d(\exp(Tf)\iota_{X_f}\omega\right)\right] ds\right)\\
&= \left(d(\exp(Tf)\omega), \exp(Tf)\omega + d\int_0^\infty \left(\bar\Phi^s\right)^* \left(\exp(Tf)\iota_{X_f}\omega\right) ds\right)\\
&= d_C\mathcal{L}(\omega).
\end{aligned}$$

Hence $\mathcal{L}$ induces a homomorphism (still denote it by $\mathcal{L}$) between $H^*\left(\Omega^\bullet_{(2)}(M), d_{Tf}\right)$ and $H^*(Cone^\bullet, d_C)$. The proof of the fact that $\mathcal{L}$ is a bijection is tedious, and will be given in Section 7.6. □

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5. Morse inequalities

In this section we assume that $T \geq T_0$. In fact, we also assume that in a neighbourhood U_x of critical points x of f, we have coordinate system $z = (z_1, ..., z_n)$ such that for $k = n_f(x)$,

$$f = f(x) - z_1^2 - \cdots - z_k^2 + z_{k+1}^2 + ... + z_n^2, \qquad g = dz_1^2 + \cdots + dz_n^2. \tag{5.1}$$

This is a generic condition. Without loss of generality, we assume that U_x is a Euclidean open ball around x with radius 1. Also, these open sets are disjoint.

Recall that m_i denotes the number of critical points of f with Morse index i. We have the following proposition:

Proposition 5.1. *There exists $T_3 \geq T_0$ big enough (see formula (4.5) for the definition of T_0), so that whenever $T \geq T_3$, the number of eigenvalues (counted with multiplicity) in $[0,1]$ of $\square_{Tf}|_{\Omega^i_{(2)}(M)}$ equals m_i – that is, $\dim F^{[0,1],*}_{Tf} = m_i$.*

The proof of Proposition 5.1 follows from that of [23, Proposition 5.5], except for the proof of the following proposition, which requires a slight modification using the well-tame condition:

Proposition 5.2. *There exist constants $C > 0$, $T_4 > 0$ such that for any smooth form $\phi \in \Omega^*_{(2)}(M)$ with $\operatorname{supp}(\phi) \subset M - \cup_{x \in \operatorname{Crit} f} U_x$ and $T \geq T_4$, one has*

$$\|\square_{Tf}\phi\|_{L^2} \geq CT\|\phi\|_{L^2}.$$

Here $\operatorname{supp}(\phi)$ denotes the support of ϕ.

Proof. Since f is well tame, there exist $\delta_1, \delta_2 > 0$ such that $|\nabla f| \geq \delta_1$ and $\left|\nabla^2 f\right| \leq \delta_2 |\nabla f|^2$ on $M - \cup_{x \in \operatorname{Crit} f} U_x$. Then our proposition follows from the same argument as in [23, Proposition 4.7]. □

On the other hand, $\left(F^{[0,1],*}_{Tf}, d_{Tf}\right)$ forms a complex, the so-called Witten instanton complex, whose cohomology is $H^*_{(2)}(M, d_{Tf})$, when T is big enough by Theorem 2.3. As a result, our Theorem 1.2 (the strong Morse inequalities) follows from Proposition 5.1 and our Hodge theorem when $T > T_3$. For the case of $T \in (T_0, T_3]$, see Section 7.

6. An application of Theorem 1.3

Let (M, g) be an oriented, compact Riemannian manifold with boundary ∂M. Set $\tilde{M} := M \cup T$, where $T \cong \partial M \times [1, \infty)$. Then we can extend the metric g to $\tilde{M}$, so that near infinity, the metric g on $\tilde{M}$ is of product type – that is, $g_{\partial M} + dr^2$. Clearly $\left(\tilde{M}, g\right)$ has bounded geometry.

We have the following technical lemma:

Lemma 6.1. *Given a transversal Morse function f and a partition of boundaries $N^+ = N_1^+ \sqcup N_2^+$, $N^- = N_1^- \sqcup N_2^-$, one can extend f to a function $\tilde{f}$ on $\tilde{M}$ such that*

1. $\left|\tilde{f}\right|(x) \to \infty$ *as* $x \to \infty$,

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2. $\tilde{f} < 0$ *on* $\left(N_1^+ \sqcup N_2^-\right) \times (2,\infty)$,
3. $\tilde{f}$ *has critical points* $\mathrm{Crit}^*\left(\tilde{f}\right) = \mathrm{Crit}^{\circ,*}(f) \cup \mathrm{Crit}^{+,*}_{N_1^+}(f) \cup \mathrm{Crit}^{-,*}_{N_1^-}(f)$ *and*
4. $\left(\tilde{M}, g, \tilde{f}\right)$ *is well tame.*

Proof. We use $x = (x', r), x' \in \partial M, r \in (0,1]$ to denote $x \in \partial M \times (0,1]$. Since f is a transversal Morse function, there exists $s_0 < 1$ such that $\left|\frac{\partial f}{\partial r}(x', r)\right| \neq 0$ on $\partial M \times (s_0, 1]$.

Hence, by considering the Taylor expansion of $f(x', r)$ in r, there is a smooth function θ on $\partial M \times (s_0, 1]$ such that

1. $f(x', r) = f(x', 1) + \frac{\partial f}{\partial r}(x', 1)\theta(x', r)$,
2. $\theta(x', r) = (r-1) + o((r-1))$ near $\partial M \times \{1\}$ and
3. $\frac{\partial \theta}{\partial r}(x', r) = 1 + o(1)$.

Assume that s_0 is close enough to 1 so that on $\partial M \times (s_0, 1]$, $\theta(x', r) < \min\left\{-(r-1)^2, 1/2(r-1)\right\}$, $\frac{\partial \theta}{\partial r}(x', r') > 0$. Let η_1 be a smooth function on $(-\infty, \infty)$ such that

1. $0 < \eta < 1$ on $(s_0, 1)$ and $\eta \equiv 0$ on $(-\infty, s_0)$, $\eta \equiv 1$ on $(1, \infty)$; and
2. $\eta'(r) > 0$, for all $r \in (s_0, 1)$.

Then for all $x' \in N_1^+ \cup N_1^-$, let

$$\tilde{f}(x', r) = f(x', 1) + \frac{\partial f}{\partial r}(x', 1)\left((1-\eta(r))\theta(x', r) - \eta(r)(r-1)^2\right);$$

and for all $x' \in N_2^+ \cup N_2^-$, let

$$\tilde{f}(x', r) = f(x', 1) + \frac{\partial f}{\partial r}(x', 1)((1-\eta(r))\theta(x', r) + \eta(r)1/2(r-1)).$$

One easily verifies that $\tilde{f}$ satisfies our conditions. □

Since the Thom–Smale complex $\left(C^*(W^u), \tilde{\partial}'\right)$ of $\tilde{f}$ induces a differential operator $\tilde{\partial}'$ on $\mathrm{Crit}^*(f)$, Theorem 1.5 follows from Theorem 1.3.

7. The Agmon estimate

In this section we first carry out the main technical estimates of the paper. Then in Section 7.3 we establish the Stokes formula for the Thom–Smale complex in our setting and deduce among its consequences that the square of the coboundary operator for the Thom–Smale complex is zero. The remaining subsections are devoted to the rest of the proofs for Theorems 1.3 and 4.1.

7.1. Proof of Lemma 3.1

Proof. Our proof is adapted from that of [1, Theorem 1.5].

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Let $L=\{p\in M:\rho_T(p)\le 2\}$. Let $\eta_k\in C_c^\infty(\mathbb{R})$ (k large enough) be a smooth bump function such that

$$\eta_k(t)=\begin{cases}0 & \text{if } |t|<1 \text{ or } |t|>k+1,\\ 1 & \text{if } |t|\in(2,k),\end{cases}$$

and $|\eta_k'(t)|\le 2$, $\eta_k(t)\in[0,1]$, for all $t\in\mathbb{R}$.

Set $\rho_{T,j}=\min\{\rho_T,j\}$, and

$$\lambda_{T,j}=\begin{cases}\lambda_T & \text{if } \rho_T<j,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $|\nabla\rho_{T,j}|^2=\lambda_{T,j}$ almost everywhere and $\lambda_T\ge\lambda_{T,j}$.

Now set $\varphi_{k,j}=(\eta_k\circ\rho_T)\exp(b\rho_{T,j})$. Then by assumption we have

$$\int_M \nabla u\nabla\left(\varphi_{k,j}^2u\right)+\lambda_T\left(u\varphi_{k,j}\right)^2\mathrm{dvol}\le\int_M w\varphi_{k,j}^2u\mathrm{dvol}.$$

Noting that $\nabla u\nabla\left(\varphi_{k,j}^2u\right)=|\nabla(\varphi_{k,j}u)|^2-|\nabla\varphi_{k,j}|^2u^2\ge-|\nabla\varphi_{k,j}|^2u^2$, we have

$$\int_{M-K}\left(\lambda_T|u\varphi_{k,j}|^2-|u|^2|\nabla\varphi_{k,j}|^2\right)\mathrm{dvol}\le\int_{M-K}wu\varphi_{k,j}^2\mathrm{dvol}. \tag{7.1}$$

Omitting the volume form dvol, since

$$\int_{M-K}wu\varphi_{k,j}^2\le\frac{1}{1-b^2}\int_{M-K}(\lambda_T)^{-1}w^2\varphi_{k,j}^2+\frac{1-b^2}{4}\int_{M-K}\lambda_Tu^2\varphi_{k,j}^2$$

and

$$\begin{aligned}|\nabla\varphi_{k,j}|^2&\le\frac{1+b^2}{2}(\eta_k\circ\rho_T)^2|\nabla\rho_{T,j}|^2\exp(2b\rho_{T,j})+\frac{1+b^2}{1-b^2}\left(\eta_k'\circ\rho_T\right)^2|\nabla\rho_T|^2\exp(2b\rho_{T,j})\\&=\frac{1+b^2}{2}(\eta_k\circ\rho_T)^2\lambda_{T,j}\exp(2b\rho_{T,j})+\frac{1+b^2}{1-b^2}\left(\eta_k'\circ\rho_T\right)^2\lambda_T\exp(2b\rho_{T,j}),\end{aligned}$$

by formula (7.1) we have

$$\begin{aligned}&\frac{3+b^2}{4}\int_{M-K}\lambda_T(\eta_k\circ\rho_T)^2u^2\exp(2b\rho_{T,j})-\frac{1+b^2}{2}\int_{M-K}\lambda_{T,j}(\eta_k\circ\rho_T)^2u^2\exp(2b\rho_{T,j})\\&\le\frac{1}{1-b^2}\int_{M-K}w^2(\eta_k\circ\rho_T)^2\lambda_T^{-1}\exp(2b\rho_{T,j})\\&+\frac{1+b^2}{1-b^2}\int_{M-K}u^2\left(\eta_k'\circ\rho_T\right)^2\lambda_T\exp(2b\rho_{T,j})\\&\le\frac{1}{1-b^2}\int_{M-K}w^2\lambda_T^{-1}\exp(2b\rho_{T,j})\\&+2\frac{1+b^2}{1-b^2}\int_{L-K}u^2\lambda_T\exp(2b\rho_{T,j})+2\frac{1+b^2}{1-b^2}\int_{\tilde{B}_{k+1}-\tilde{B}_k}u^2\lambda_T\exp(2bj).\end{aligned} \tag{7.2}$$

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Letting $k\to\infty$, by the monotone convergence theorem and the fact that $\int_M \lambda_T u^2<\infty$, we have

$$\begin{aligned}
&\frac{3+b^2}{4}\int_{M-L}\lambda_T u^2\exp(2b\rho_{T,j})-\frac{1+b^2}{2}\int_{M-L}\lambda_{T,j}u^2\exp(2b\rho_{T,j})\\
&\leq\frac{1}{1-b^2}\int_{M-K}w^2\lambda_T^{-1}\exp(2b\rho_{T,j})+2\frac{1+b^2}{1-b^2}\int_{L-K}u^2\lambda_T\exp(2b\rho_{T,j}).
\end{aligned}$$ □

7.2. Proof of Lemma 3.3

Proof. The proof is a standard argument of Moser iteration, and we present it here for convenience.

The starting point is the differential inequality

$$cu\geq\Delta u \tag{7.3}$$

weakly on $B_{2r}(p)$.

Set $r_1=2r$, $r_{k+1}=r_k-(1/2)^k r$ and $n_k=(n/(n-2))^{k-1}$.

Let $\eta_k\in C_c^\infty(B_{2r})$ be bump functions such that

$$\eta_k=\begin{cases}1 & \text{on } B_{r_{k+1}},\\ 0 & \text{on } B_{2r}-B_{r_k},\end{cases}$$

and $|\nabla\eta_k(q)|<\frac{2}{r_{k+1}-r_k}$, $\eta_k(q)\in[0,1]$, for all $q\in B_{2r}$.

Set $u_m=\min\{u,m\}$ and $\phi_1=\eta_1^2u_m\in H_0^1(B_{2r})$. Notice that $\phi_1=0$ and $\nabla\phi_1=0$ in $\{u\geq m\}$. Hence by formula (7.3) we have

$$\begin{aligned}
&\int_{B_{r_1}}c(u_m)^2dvol\geq\int_{B_{2r}}cu\phi_1dvol\geq\int_{B_{2r}}\nabla u\nabla\phi_1dvol\\
&=\int_{B_{2r}}\eta_1^2|\nabla u_m|^2+2\eta_1\nabla\eta\nabla u_mu_mdvol\\
&\geq\int_{B_{2r}}\eta_1^2|\nabla u_m|^2-1/2\eta_1^2|\nabla u_m|^2-2|\nabla\eta u_m|^2dvol\\
&\geq\int_{B_{2r}}\eta_1^2|\nabla u_m|^2-1/2\eta_1^2|\nabla u_m|^2-2|\nabla\eta u_m|^2dvol\\
&\geq 1/2\int_{B_{r_2}}|\nabla u_m|^2dvol-4/(r_2-r_1)^2\int_{B_{r_1}}|u_m|^2dvol.
\end{aligned}$$

Hence we have

$$\int_{B_{r_2}}|\nabla u_m|^2dvol\leq\left(2c+8/(r_1-r_2)^2\right)\int_{B_{r_2}}c(u_m)^2dvol\leq C(n)/(r_1-r_2)^2\int_{B_{r_2}}c(u_m)^2dvol.$$

By Sobolev inequality,

$$\left(\int_{B_{r_1}}|u_m|^{2n_2}dvol\right)^{1/n_2}\leq C(n)/(r_1-r_2)^2\int_{B_{r_2}}c(u_m)^2dvol.$$

That is,

$$\|u_m\|_{L^{2n_2}(B_{r_2})} \leq (C(n)/(r_1 - r_2))\|u_m\|_{L^{2n_1}(B_{r_1})}.$$

Let $m \to \infty$; then we have

$$\|u\|_{L^{2n_2}(B_{r_2})} \leq (C(n)/(r_1 - r_2))\|u\|_{L^{2n_1}(B_{r_1})}.$$

Consider $\phi_k = \eta_k^2 \left(u_m^{2n_k - 1}\right) \in H_0^1(B_{2r})$. By the same arguments as before, we have

$$\|u\|_{L^{2n_{k+1}}(B_{r_{k+1}})} \leq (C(n)/(r_k - r_{k+1}))^{1/(n_k)}\|u\|_{L^{2n_k}(B_{r_k})}.$$

As a consequence,

$$\begin{aligned}
\|u\|_{L^\infty(B_r)} &= \lim_{k\to\infty} \|u\|_{L^{2n_k}(B_{r_k})} \\
&\leq C\Pi_{k=1}^\infty (C(n)/(r_k - r_{k+1}))^{1/(n_k)}\|u\|_{L^2(B_{2r})} \\
&= C(C(n)/r)^{\left(\sum_{k=1}^\infty 1/(n_k)\right)} 2^{\sum_{k=1}^\infty k/n_k}\|u\|_{L^2(B_{2r})} \\
&\leq C/r^{n/2}\|u\|_{L^2(B_{2r})}.
\end{aligned}$$

□

We state two lemmas that will be needed shortly.

Lemma 7.1. *Suppose that $u,w \in L^2(M)$ and $\square_{Tf} u \leq w$ in the weak sense (and $u \geq 0$). For $r > 0$ small enough, $p \notin L$, let $B_r(p)$ be the geodesic ball around p with radius r induced by g. Then*

$$\sup_{y\in B_r(p)} u(y) \leq \frac{C_2}{r^{n/2}} \left(\|u\|_{L^2(B_{2r}(p))} + \|w\|_{L^2(B_{2r}(p))}\right),$$

where $C_2 > 0$ is a constant that depends only on the dimension n, the injectivity radius lower bound r_0 and the curvature bound.

Proof. The proof is actually similar to the proof of Lemma 3.3, requiring only some slight modification. See [12, Theorem 4.1] for a reference. □

By the same argument as the proof of Theorem 1.1, we have the following:

Lemma 7.2. *Let (M,g,f) be well tame, with $w \in L^2(M)$ satisfying*

$$\int_M \lambda_T^{-1}|w|^2 \exp(a''\rho_T) dvol < \infty$$

for some $a'' \in (0,b)$. If $\phi \in L^2(M)$ is a weak solution of $\square_{Tf}\phi \leq w$, then

$$|\phi(p)| \leq C\exp(-a''\rho_T(p)).$$

7.3. On the Thom–Smale complex

In this subsection, we will show that the Thom–Smale complex defined in Section 4 is indeed a complex. The key here is to establish the analogue of the so called Stokes formula in our setting. We use a doubling construction to reduce it to the compact case and make essential use of the uniform lower bound of $|\nabla f|$ outside suitably chosen compact sets, which guarantees that the flow lines coming out of the compact region will never return (see also Remark 7.7).

Intuitively the idea may be explained as follows. When the Morse function f is proper, such compact regions can be chosen to be the sublevel set $a \le f \le b$. Since f decreases along its negative gradient flow, a flow line out of the region will obviously not return. In general, however, f may not be proper, but it turns out that the Agmon distance is a good replacement. Indeed, when f is proper, f measures the Agmon distance between its level sets.

First we recall the Stokes formula in the compact case. The following is a restatement of [15, Proposition 6]:

Proposition 7.3. *Let (N,g) be a compact Riemannian manifold (without boundary) and f be a Morse function. Assume that (N,g,f) satisfies the Thom–Smale transversality condition. Then for any critical point $x \in \mathrm{Crit}(f)$ with Morse index $n_f(x)$ and any $\phi \in \Omega^{n_f(x)-1}(M)$, we have the following so-called Stokes formula:*

$$\int_{W^u(x)} d\phi = \sum_{y \in \mathrm{Crit}(f),\, n_f(y)=n_f(x)-1} m(x,y) \int_{W^u(y)} \phi.$$

For our noncompact case with tame conditions and Thom–Smale transversality, we similarly have the following:

Proposition 7.4. *For any critical point $x \in \mathrm{Crit}(f)$ with Morse index $n_f(x)$ and any $\phi \in \Omega_c^{n_f(x)-1}(M)$, we have the following Stokes formula:*

$$\int_{W^u(x)} d\phi = \sum_{y \in \mathrm{Crit}(f),\, n_f(y)=n_f(x)-1} m(x,y) \int_{W^u(y)} \phi.$$

Before giving the proof of this proposition, we first draw a couple of consequences.

Corollary 7.1. *Let $\tilde{\partial} : C_*(W^u) \mapsto C_{*-1}(W^u)$ be the map constructed in Section 4; then $\tilde{\partial}^2 = 0$.*

Proof. Otherwise, $\tilde{\partial}^2 W^u(x) \neq 0$. Then there exists $\phi \in \Omega_c^{n_f(x)-2}(M)$ such that

$$\int_{\tilde{\partial}^2 W^u(x)} \phi \neq 0.$$

But by Proposition 7.4,

$$\int_{\tilde{\partial}^2 W^u(x)} \phi = \int_{W^u(x)} d^2\phi = 0,$$

which is a contradiction. ∎

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Corollary 7.2. *Set $\omega \in F_{Tf}^{[0,1],n_f(x)-1}$. Then we have*

$$\int_{W^u(x)} \exp(Tf) d_{Tf}\omega = \sum_{y\in \mathrm{Crit}(f), n_f(y)=n_f(x)-1} m(x,y) \int_{W^u(y)} \exp(Tf)\omega.$$

In particular, the map $\mathcal{J}$ introduced in Section 4 is a chain map.

Proof. By Theorem 1.1 and Lemma 4.5, for any $\epsilon > 0$ there exists $\phi \in \Omega_c^{n_f-1}(M)$ such that for any $y \in \mathrm{Crit}(f)$ with $n_f(y) = n_f(x) - 1$,

$$\int_{W^u(x)} |\exp(Tf) d_{Tf}\omega - d\phi| < \epsilon, \qquad \int_{W^u(y)} |\exp(Tf)\omega - \phi| < \epsilon.$$

Now the corollary follows from Proposition 7.4. □

We now turn to the proof of Proposition 7.4. We start with the following observation:

Lemma 7.5. *Let $(N,\partial N)$ be a compact manifold with boundary. Moreover, assume that near the boundary ∂N, the manifold is of product type $(0,1] \times \partial N$. Suppose that f is a Morse function on $N - [1/2,1] \times \partial N$. Then there exists a Morse function $\bar{f}$ on N such that $\bar{f}|_{N-[1/4,1]\times\partial N} = f$ and $\tilde{f}\Big|_{[3/4,1]\times\partial N} = r$. Here r is the standard coordinate on the $(0,1]$ factor.*

The proof is essentially the same as that of [19, Theorem 2.5].

Recall from Section 4 that $\tilde{d}$ denotes the distance function induced by the Agmon metric $|\nabla f|^2 g$. Let Φ^t denote the flow generated by $-\nabla f$. By reparameterisation, the results in Lemma 4.2 can be restated for Φ^t and $\tilde{d}$. Namely, we have

$$|f(p) - f(q)| \le \tilde{d}(p,q), \quad \forall p,q \in M, \tag{7.4}$$

and

$$\tilde{d}\left(\Phi^{t_1}(p), \Phi^{t_2}(p)\right) = \left| f\left(\Phi^{t_1}(p)\right) - f\left(\Phi^{t_2}(p)\right)\right|. \tag{7.5}$$

Set

$$D = \sup_{y\in \mathrm{Crit}(f)} \tilde{d}(y,p_0) + 2 \sup_{y,z\in \mathrm{Crit}(f)} \tilde{d}(y,z)$$

(compare equation (4.4)).

Lemma 7.6. *For any fixed $x \in \mathrm{Crit}(f)$ and any $\bar{D} > D$, let $\tilde{B}_{\bar{D}}(x)$ be the ball centered at x with radius $\bar{D}$ in the distance $\tilde{d}$ and $\tilde{B}^\circ_{\bar{D}}(x)$ be the interior of $\tilde{B}_{\bar{D}}(x)$. Then for any $y,z \in \mathrm{Crit}(f)$, $W^u(y) \cap W^s(z) \subset \tilde{B}^\circ_{\bar{D}}(x)$. Moreover, if $p \notin \tilde{B}_{\bar{D}}(x)$ lies in the unstable manifold $W^u(x)$, then $\{\Phi^t(p) : t \ge 0\} \cap \tilde{B}_{\bar{D}}(x) = \emptyset$.*

Proof. Since f is decreasing along the flow Φ^t, by equation (7.5), for any $p \in W^u(y) \cap W^s(z)$,

$$\tilde{d}(y,p) = f(y) - f(p) \le f(y) - f(z) = \tilde{d}(y,z).$$

Hence

$$\tilde{d}(x,p) \leq \tilde{d}(x,y) + \tilde{d}(y,p) \leq \tilde{d}(x,y) + \tilde{d}(y,z) \leq D.$$

Similarly, if $q \notin \tilde{B}_{\bar{D}}(x)$ lies in the unstable manifold $W^u(x)$, then for any $t \geq 0$,

$$\tilde{d}\left(x,\Phi^t(q)\right) = f(x) - f\left(\Phi^t(q)\right) \geq f(x) - f(q) = \tilde{d}(x,q) \geq \bar{D},$$

as desired. □

Now we are ready to prove Proposition 7.4.

Proof of Proposition 7.4. We reduce it to the compact case by a doubling construction and make use of Proposition 7.3.

For any $\phi \in \Omega_c^{n_f(x)}(M)$, define

$$\bar{D} := \sup_{p \in \mathrm{Crit} f \cup \mathrm{supp}(\phi)} \tilde{d}(p,p_0) + 2 \sup_{p,q \in \mathrm{Crit} f \cup \mathrm{supp}(\phi)} \tilde{d}(p,q).$$

We can find a compact submanifold $(N,\partial N)$ with boundary, such that $\tilde{B}_{\bar{D}}(x) \subset N^\circ$. Here $\mathrm{supp}(\phi)$ denotes the support of ϕ and N° denotes the interior of N. Thus, $\mathrm{supp}(\phi) \subset \tilde{B}^\circ_{\bar{D}}(x)$.

Now consider the double $(DN = N^+ \cup N^-, g_{DN})$ of N, $g_{DN}|_{\tilde{B}_{\bar{D}}(x)} = g$. By Lemma 7.5, we can find a Morse function $\bar{f}$ on DN such that $\bar{f}|_{\tilde{B}_{\bar{D}}(x)} = f$. We may as well assume that $\left(DN, g_{DN}, \bar{f}\right)$ satisfies the Thom–Smale transversality condition. Then for any $y,z \in \mathrm{Crit}\left(\bar{f}\right)$ with Morse index $n_{\bar{f}}(y) = n_{\bar{f}}(z) + 1$, let $m_{DN}(y,z)$ be the signed count of the number of flow lines in $W^u_{DN}(y) \cap W^s_{DN}(z)$, where W^s_{DN} and W^u_{DN} denote, respectively, the stable and unstable manifolds with respect to $\bar{f}$ on DN.

We make the following observations:

1. By Lemma 7.6 and its proof, if $z \in \tilde{B}_{\bar{D}}(x)$ is a critical point of $\bar{f}$ with $n_{\bar{f}}(z) = n_{\bar{f}}(x) - 1$, we have $m_{DN}(x,z) = m(x,z)$. Indeed, suppose γ is a flow line on DN connecting x and z and is not contained in $\tilde{B}_{\bar{D}}(x)$. Let $w \in \gamma \cap \partial \tilde{B}_{\bar{D}}(x)$ be the place where γ first meets $\partial \tilde{B}_{\bar{D}}(x)$. Then

$$D \geq \tilde{d}(x,z) \geq f(x) - f(z) = \bar{f}(x) - \bar{f}(z) > \bar{f}(x) - \bar{f}(w) = f(x) - f(w) = \tilde{d}(x,w) = D,$$

   which is a contradiction. Here the strict inequality follows from the fact that $\bar{f}$ decreases along its flow lines, and the second-to-last equation follows from the fact that the part of flow lines of $\bar{f}$ inside $\tilde{B}_{\bar{D}}(x)$ coincides with flow lines of f as $g_{DN}|_{\tilde{B}_{\bar{D}}(x)} = g$, $\bar{f}\big|_{\tilde{B}_{\bar{D}}(x)} = f$.

   As a result, flow lines (if they exist) connecting x and z in DN must be contained in $\tilde{B}_{\bar{D}}(x)$. By Lemma 7.6, they are exactly flow lines connecting x and z in M. Therefore, $m_{DN}(x,z) = m(x,z)$.

2. If $z \notin \tilde{B}_{\bar{D}}(x)$ is a critical point of $\bar{f}$, and $W^s_{DN}(z) \cap W^u_{DN}(x) \neq \emptyset$, then $W^u_{DN}(z) \cap \mathrm{supp}(\phi) = \emptyset$.

   To see why, let γ be a flow line connecting x and z in DN, and let $w \in \gamma \cap \partial \tilde{B}_{\bar{D}}(x)$ be the first place where γ meets $\partial \tilde{B}_{\bar{D}}(x)$. By equation (7.5), $\bar{f}(x) - \bar{f}(z) > \bar{f}(x) - \bar{f}(w) =$

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$f(x) - f(w) = \tilde{d}(x,w) = \bar{D}$. Hence

$$\bar{f}(z) < \inf_{p \in \mathrm{supp}(\phi)} \bar{f}(p). \tag{7.6}$$

Otherwise, there is $p \in \mathrm{supp}(\phi)$ such that $\bar{f}(z) \geq \bar{f}(p)$. Then by formula (7.4), $\bar{D} \geq \tilde{d}(x,p) \geq f(x) - f(p) = \bar{f}(x) - \bar{f}(p) \geq \bar{f}(x) - \bar{f}(z) > \bar{D}$. By formula (7.6), $W^u_{\bar{D}N}(z) \cap \mathrm{supp}(\phi) = \emptyset$.

As a result, by Proposition 7.3,

$$\begin{aligned}
\int_{W^u(x)} d\phi = \int_{W^u_{\bar{D}N}(x)} d\phi &= \sum_{z \in \mathrm{Crit}\left(\bar{f}\right), n_{\bar{f}}(z) = n_{\bar{f}}(x) - 1} m_{\bar{D}N}(x,z) \int_{W^u_{\bar{D}N}(z)} \phi \\
&= \sum_{y \in \mathrm{Crit}(f), n_f(y) = n_f(x) - 1} m_{\bar{D}N}(x,y) \int_{W^u(y)} \phi \quad \text{(by Observation 2)} \\
&= \sum_{y \in \mathrm{Crit}(f), n_f(y) = n_f(x) - 1} m(x,y) \int_{W^u(y)} \phi \quad \text{(by Observation 1)},
\end{aligned}$$

as claimed. □

Remark 7.7. *Here we have made essential use of the fact that $|\nabla f|$ has a positive lower bound outside some compact set K_0. Indeed, in this case, $\left(M, |\nabla f|^2 g\right)$ is complete, and hence $\tilde{B}_r(p)$ is compact for all $r > 0$, $p \in M$. Therefore one can always find a compact manifold with boundary N containing $\tilde{B}_{\bar{D}}(x)$. Moreover, by our choice of $\bar{D}$, for all $q \in \left(M - \tilde{B}_{\bar{D}}(x)\right) \cap W^u(x)$, $f(q) < \inf_{q' \in \mathrm{supp}(\phi) \cup \mathrm{Crit}(f)} f(q')$. Therefore, since f is decreasing along the flow, once a flow line escapes $\tilde{B}_{\bar{D}}(x)$ it never flows back to* $\mathrm{supp}(\phi) \cup \mathrm{Crit}(f)$. *Consequently, we have Lemma 7.6, Observations 1 and 2.*

7.3.1. A counterexample. To close out this subsection, we present a counterexample provided by Shu Shen (whom we thank) which shows that if we drop the condition that $|\nabla f|$ have a positive lower bound near infinity, the conclusion $\tilde{\partial}^2 = 0$ can fail.

Consider the following heart-shaped topological sphere S, with f being the height function. Then we have four critical points x,y,z,w, as indicated. Let γ be a flow line connecting y and w, and remove a point p on γ. Make a conformal change of metric near the point p so that $S - \{p\}$ is complete under this new metric. Now one can check that $|\nabla f(q)| \to 0$, as $q \to p$. On the other hand, since the flow line is invariant under the conformal change of metric, $\gamma - \{p\}$ is still a (broken) flow line. And in this case, $\tilde{\partial}^2 x = w$, which is nonzero.

In our previous arguments, the fact that $|\nabla f|$ has a positive lower bound near infinity plays a crucial role (see Remark 7.7).

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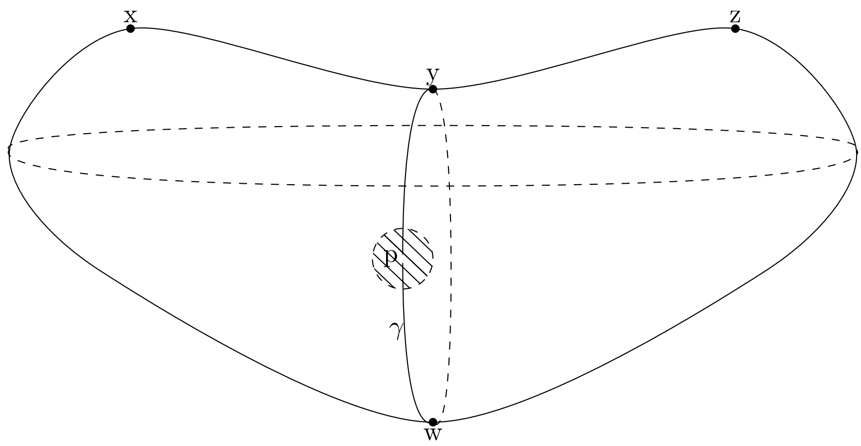

7.4. Isomorphism of $H^*(C^\bullet(W^u),\tilde{\partial}')$ and $H^*_{dR}(M,U_c)$

For simplicity, we assume that f is a self-indexed Morse function – that is, if x is a critical point of f with Morse index i, we require $f(x) = i$.

Let $V_i = f^{-1}\left(-\infty, i+\frac{1}{2}\right]$, $0 \leq i \leq n$.

Recall our assumption that in a neighbourhood U_x of critical points x of f, we have a coordinate system $z = (z_1,\dots,z_n)$ such that

$$f = f(x) - z_1^2 - \cdots - z_{n_f(x)}^2 + z_{n_f(x)+1}^2 + \cdots + z_n^2,$$

$$g = dz_1^2 + \cdots + dz_n^2.$$

Moreover, U_x is a Euclidean open ball around x with radius 1. Also, these open balls are disjoint.

We have the following observation:

Lemma 7.8. *V_0 can be written as a disjoint union of $\cup_{x\in \mathrm{Crit}(f), n_f(x)=0}\tilde{U}_x$ and V, where V is some open subset diffeomorphic to U_c and $\tilde{U}_x$ is a Euclidean ball around x with radius $\frac{1}{2}$. Also, V_n is diffeomorphic to M.*

Proof. Define $X_f := \frac{\nabla f}{|\nabla f|^2}$ and let Φ^t be the flow generated by X_f. Then we have

$$\left(\Phi^{c+\frac{1}{2}}(U_c)\right) \cap \left(\cup_{x\in \mathrm{Crit}(f), n_f(x)=0}\tilde{U}_x\right) = \emptyset.$$

This is for the following reasons:

- If $f(p) \leq c - \frac{1}{2}$, then $f\left(\Phi^{c+\frac{1}{2}}(p)\right) < 0$. Hence $\Phi^{c+\frac{1}{2}}(p) \notin \cup_{x\in \mathrm{Crit}(f), n_f(x)=0}\tilde{U}_x$.
- If $c - \frac{1}{2} \leq f(p) < c$, and if $\Phi^{c+\frac{1}{2}}(p) \in \tilde{U}_x$ for some $x \in \mathrm{Crit}(f)$ with Morse index $n_f(x) = 0$, then $\Phi^{c+\frac{1}{2}}(p) \in W^s(x)$, which implies $p \in W^s(x)$. But this is impossible, since $f(p) < -c < 0 = f(x)$.

We can similarly prove that V_n is diffeomorphic to M. □

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Let $C_*(V_i,U_c)$ be the complex of relative singular chains. Then we have

$$C_*(V_n,U_c) \supset C_*(V_{n-1},U_c) \supset \cdots C_*(V_0,U_c).$$

By Lemma 7.8 and a spectral sequence argument similar to the proof of [2, Theorem 1.6], one can show that

$$H_*\left(C^\bullet(W^u),\tilde{\partial}\right) \simeq H_*(M,U_c).$$

Thus, it follows from the universal coefficient theorem that

$$H^*\left(C^\bullet(W^u),\tilde{\partial}'\right) \simeq H^*_{dR}(M,U_c).$$

7.5. Isomorphism of $H^*_{(2)}(M,d_{Tf})$ and $H^*(C^\bullet(W^u),\tilde{\partial}')$

We will first show that the chain map $\mathcal{J}: \left(F^{[0,1],*}_{Tf},d_{Tf}\right) \mapsto C^*\left((W^u)',\tilde{\partial}'\right)$ defined in Section 4 is in fact an isomorphism when T is sufficiently large. Hence $\mathcal{J}$ induces an isomorphism between $H^*_{(2)}(M,d_{Tf})$ and $H^*(C^\bullet(W^u),\partial)$ in that case.

More precisely, the arguments follow those in [23, Chapter 6], with a necessary modification which we will only indicate here. The basic idea is to construct an explicit map which approximates the inverse of $\mathcal{J}$ (up to a constant multiple) as $T \to \infty$. Therefore, there exists $T_6 > T_0$ such that $\mathcal{J}$ is an isomorphism whenever $T > T_6$. (We point out that the explicit description of T_6 is more involved than that of T_0.)

In fact, the modification we need is a more refined estimate in [23, Theorem 6.7]. Namely, we have

$$|\mathcal{P}\tau_{x,T} - \tau_{x,T}| \le C\exp\left(-a'T\sqrt{\rho^2+1}\right)\|\tau_{x,T}\|_{L^2}, \tag{7.7}$$

where $\mathcal{P}$ is the orthogonal projection from $L^2\Lambda(M)$ to $F^{[0,1],*}$, and C, $a' < a$ are positive constants.

Here $\tau_{x,T}$ is defined as follows (and the explicit map from $C^*\left((W^u)',\tilde{\partial}'\right)$ to $\left(F^{[0,1],*}_{Tf},d_{Tf}\right)$ assigns a normalising multiple of $\mathcal{P}\tau_{x,T}$ to $W^u(x)^*$). Notice that in Section 5, we require that in a neighbourhood U of x, the metric and Morse function be of the form of equation (5.1). Let α_x be a bump function whose support is contained in U and $\alpha_x \equiv 1$ in a neighbourhood V of x, and set

$$\tau_{x,T} = \alpha_x \exp\left(-T^2|z|^2\right) dz_1 \wedge \cdots \wedge dz_{n_f(x)}.$$

Then $\Box_{Tf}\tau_{x,T} = 0$ in V and $M - U$.

To obtain estimate (7.7), pick a bump function η with compact support such that $\eta \equiv 1$ on K. Then our Agmon estimate yields

$$|(1-\eta)(\mathcal{P}\tau_{x,T} - \tau_{x,T})| \le C\exp(-aT\rho)\|\tau_{x,T}\|_{L^2}.$$

On the other hand, the estimate

$$|\eta(\mathcal{P}\tau_{x,T} - \tau_{x,T})| \le C\exp(-cT)\|\tau_{x,T}\|_{L^2}$$

follows from exactly the same argument in the proof of [23, Theorem 6.7].

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Now it remains to prove that when $T \in (T_0, T_6]$, $H^*_{(2)}(M, d_{Tf})$ and $H^*(C^\bullet(W^u), \partial)$ are still isomorphic.

We only present the proof for the case when (M, g, f) is strongly tame (the case for well tame being exactly the same except in notation). In this case, we have $T_0 = 0$. The idea is to show that if $S > 0$, then for any $T \in [7/8S, S]$, $H^*_{(2)}(M, d_{Tf})$ and $H^*_{(2)}(M, d_{Sf})$ are isomorphic. Hence $H^*_{(2)}(M, d_{Tf})$ is independent of $T \in (0, \infty)$, which finishes the proof of isomorphism of $H^*_{(2)}(M, d_{Tf})$ and $H^*(C^\bullet(W^u), \partial)$.

For simplicity, we prove that $H^*_{(2)}(M, d_{7f})$ and $H^*_{(2)}(M, d_{8f})$ are isomorphic, the general case being similar.

Thus fix coefficients $a = \frac{63}{64}$ and $b = \frac{127}{128}$ in Lemma 2.1 and Theorem 1.1.

Define $M_f : \left(F^{*,[0,1]}_{8f}, d_{8f}\right) \mapsto \left(\Omega^*_{(2)}(M), d_{7f}\right)$; for all $w \in F^{*,[0,1]}_{8f}$, $M_f(w) = \exp(f) w$. Similarly, $M_{-f} : \left(F^{*,[0,1]}_{7f}, d_{7f}\right) \mapsto \left(\Omega^*_{(2)}(M), d_{8f}\right)$; for all $w \in F^{*,[0,1]}_{7f}$, $M_{-f}(w) = \exp(-f) w$.

Clearly these are chain maps once we check that M_f and M_{-f} are well defined. To this end, we verify that $|f(p)| \le \sup_{q \in K} |f(q)| + \frac{1}{bT} \rho_T(p)$. Indeed, let $\gamma : [0, \rho_T(p)]$ be a normal minimal geodesic connecting K and p, in the metric $\tilde{g}_T$. Then

$$\left| \frac{d}{dt} f \circ \gamma(t) \right| = \left| \left\langle \tilde{\nabla} f, \gamma' \right\rangle_{\tilde{g}_T} \right| \le \frac{1}{bT}.$$

Now the L^2 bound of $M_f(w)$ (resp., $M_{-f}(w)$) follows by Theorem 1.1 and the standard volume comparison. Hence M_f induces a homomorphism (still denote it by M_f) from $H^*_{(2)}(M, d_{8f})$ to $H^*_{(2)}(M, d_{7f})$.

Our next step is to show that M_f is injective. Suppose we have $w \in \ker(\Box_{8f})$ such that $M_f w$ is exact, which means that we can find $\alpha \in \operatorname{Im}(\delta_{7f})$ such that $\exp(f) w = d_{7f} \alpha (= (d_{7f} + \delta_{7f}) \alpha)$.

Thus

$$\begin{aligned} \Box_{7f} \alpha &= (d_{7f} + \delta_{7f}) \exp(f) w = \exp(f) d_{8f} w + \exp(2f) \delta_{6f} w \\ &= 0 + \exp(2f) (\delta_{8f} w - \iota_{2f} w) = -\exp(2f) \iota_{2f} w. \end{aligned}$$

By Lemma 7.2, $|\alpha| \le C \exp(-1/3 \rho_7)$. Consequently, $\exp(-f) \alpha \in L^2 \Lambda^*(M)$, and $w = d_{8f} \exp(-f) \alpha$ is exact.

As a result, M_f is injective. Similarly, M_{-f} is injective. Therefore, $H^*_{(2)}(M, d_{8f})$ and $H^*_{(2)}(M, d_{7f})$ are isomorphic.

7.6. $\mathcal{L}$ is bijective

As promised, here we finish the proof of Theorem 4.1 by presenting the details of the argument that the map $\mathcal{L}$ defined in the proof is a bijection.

First we show that $\mathcal{L}$ is injective. Set $\omega \in F^{[0,1],j}_{Tf}$ such that $\mathcal{L}(\omega)$ is exact. Then there exist $\phi \in \Omega^{j-1}(M)$ and $\phi' \in \Omega^{j-2}(U_c)$ such that

$$\exp(Tf) \omega = d\phi, \qquad \omega' = \phi - d\phi',$$

where $\omega' = -\int_0^\infty \left(\bar{\Phi}^s\right)^* \left(\exp(Tf) \iota_{X_f} \omega\right) ds$.

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Let $c' = c+2$, and then choose a smooth function $\chi : M \mapsto \mathbb{R}$ such that $\chi|_{U_{c'}} = 1$ and $\chi|_{M-U_c} = 0$. Then denoting $\psi = \phi - d(\chi\phi')$, we have

$$d\psi = \exp(Tf)\omega, \qquad \psi|_{U_{c'}} = \omega'.$$

Also, on $U_{c'}$,

$$\begin{aligned}\iota_{X_f}\psi &= -\iota_{X_f}\int_0^\infty \left(\bar{\Phi}^s\right)^* \left(\exp(Tf)\iota_{X_f}\omega\right) ds \\ &= -\int_0^\infty \left(\bar{\Phi}^s\right)^* \left(\exp(Tf)\iota_{X_f}\iota_{X_f}\omega\right) ds = 0.\end{aligned} \tag{7.8}$$

Next, choose a smooth function η such that $\eta = 0$ on K and $\eta = 1$ in $f^{-1}((S+1,\infty)\cup(-\infty,I-1))$. Now we would like to construct ψ' satisfying $\exp(-Tf)\psi' \in L^2\Lambda^*(M)$ and $d\psi' = \exp(Tf)\omega$. For this purpose, we set

$$\psi'(p) = \psi(p) - d\left(\eta\int_{bT(-f(p)+I/2+S/2)}^0 \left(\bar{\Phi}^s\right)^* \iota_{X_f}\psi ds\right), \quad p \in f^{-1}(S,\infty),$$

$$\psi'(p) = \psi(p) - d\left(\eta\int_0^{bT(-f(p)+I/2+S/2)} \left(\bar{\Phi}^s\right)^* \iota_{X_f}\psi ds\right), \quad p \in f^{-1}(-\infty,I),$$

$$\psi'(p) = \psi(p), \quad p \in f^{-1}[I,S].$$

By equation (7.8), we have $\psi' = \psi$ on $U_{c'}$.

Thus for $p \in U'_{c'}$,

$$\begin{aligned}\iota_{X_f}\psi' &= \iota_{X_f}\psi - \iota_{X_f} d\left(\eta\int_{bT(-f(p)+I/2+S/2)}^0 \left(\bar{\Phi}^s\right)^* \iota_{X_f}\psi ds\right) \\ &= \iota_{X_f}\psi - \iota_{X_f} d\left(\int_{bT(-f(p)+I/2+S/2)}^0 \left(\bar{\Phi}^s\right)^* \iota_{X_f}\psi ds\right) \quad (\text{since } \eta \equiv 1 \text{ on } U'_{c'}) \\ &= \iota_{X_f}\psi - \left(\int_{bT(-f(p)+I/2+S/2)}^0 \iota_{X_f} d\left(\bar{\Phi}^s\right)^* \iota_{X_f}\psi ds\right) \\ &\quad - bT\iota_{X_f} df\left(\bar{\Phi}^{bT(-f(p)+I/2+S/2)}\right)^* \iota_{X_f}\psi \\ &= \iota_{X_f}\psi - \left(\int_{bT(-f(p)+I/2+S/2)}^0 \frac{d}{ds}\left(\bar{\Phi}^s\right)^* \iota_{X_f}\psi ds - \left(\bar{\Phi}^{bT(-f(p)+I/2+S/2)}\right)\right)^* \iota_{X_f}\psi \\ &= \iota_{X_f}\psi - \iota_{X_f}\psi + \left(\bar{\Phi}^{bT(-f(p)+I/2+S/2)}\right)^* \iota_{X_f}\psi - \left(\bar{\Phi}^{bT(-f(p)+I/2+S/2)}\right)^* \iota_{X_f}\psi \\ &= 0.\end{aligned}$$

As a consequence,

$$\frac{d}{ds}\left(\bar{\Phi}^s\right)^* \psi'|_{s=t} = \iota_{X_f}\left(\bar{\Phi}^t\right)^* d\psi' = \iota_{X_f}\left(\bar{\Phi}^t\right)^* \exp(Tf)\omega$$

on $U_{c'} \cup U'_{c'}$.

Therefore, on U_c', we have

$$\begin{aligned}\left|\left(\bar{\Phi}^t\right)^*\psi'\right| &= \left|\int_0^t \iota_{X_f}\left(\bar{\Phi}^s\right)^*\exp(Tf)\omega ds\right| \\ &\le \int_0^t \left|\iota_{X_f}\left(\bar{\Phi}^s\right)^*\exp(Tf)\omega\right| ds \\ &\le \int_0^t \left|\left(\bar{\Phi}^s\right)^*\exp(Tf)\omega\right| ds \\ &\le C\exp\left(\left(b^{-1}-a\right)t\right) \quad \left(\text{since on } U_c',\ \left(\bar{\Phi}^t\right)^*\omega \le C\exp(-at)\right).\end{aligned} \tag{7.9}$$

We claim that $\exp(-Tf)\psi' \in L^2\Lambda^*(M)$. If granted, then $d\psi' = \exp(Tf)\omega$, and hence $d_{Tf}\exp(-Tf)\psi' = \omega$ – that is, ω is trivial in $H^*\left(\Omega^\bullet_{(2)}, d_{Tf}\right)$.

Now we prove the claim. It suffices to prove that $\int_{U_{c'}\cup U'_{c'}} |\exp(-Tf)\psi'|^2\, dvol < \infty$. Let $K_{c'} = f^{-1}\{-c'\}$ and $K_c' = f^{-1}\{c\}$, and endow them with induced metrics. Define a diffeomorphism $\Psi_{c'}: K_{c'} \times (0,\infty) \mapsto U_{c'}$ as follows:

$$\Psi_{c'}(p,t) = \bar{\Phi}^t(p).$$

Similarly, we can define a diffeomorphism $\Psi_c': K_c' \times (0,\infty) \mapsto U_c'$.

On $U_{c'}$, $|\psi'| = |\omega'|$, and hence for $p \in K_{c'}$,

$$\begin{aligned}&\left|\left(\bar{\Phi}^t\right)^*(\exp(-Tf)\psi')(p)\right| = \left|\left(\bar{\Phi}^t\right)^*(\exp(-Tf)(p)\int_0^\infty \left(\bar{\Phi}^s\right)^*\left(\exp(Tf)\iota_{X_f}\omega(p)\right) ds\right| \\ &= \exp\left(Tc' + b^{-1}t\right)\int_0^\infty \left(\left(\exp\left(-Tc' - b^{-1}(s+t)\right)\left(\bar{\Phi}^{s+t}\right)^*\iota_{X_f}\omega(p)\right) ds\right) \\ &\le \int_0^\infty \left(\exp\left(-b^{-1}s\right)\left|\left(\bar{\Phi}^{s+t}\right)^*\iota_{X_f}\omega(p)\right|\right) ds \\ &\overset{(a)}{\le} C\exp(-at)\int_0^\infty \exp\left(-\left(a+b^{-1}\right)s\right) ds \\ &\le C'\exp(-at),\end{aligned} \tag{7.10}$$

where inequality (a) follows from the fact that

$$\left(\bar{\Phi}^{s+t}\right)^*\omega \le C\exp(-a(s+t)).$$

Then

$$\begin{aligned}\int_{U_{c'}} |\exp(-Tf)\psi'|^2\, dvol &= \int_0^\infty \int_{K_{c'}} \left(\Psi_c'\right)^*\left(|\exp(-Tf)\omega'|^2\, dvol_{K_{c'}}dt\right) \\ &\le C'\int_0^\infty \int_{K_{c'}} \exp(-2at)\left(\Psi_c'\right)^*\left(dvol_{K_{c'}}dt\right) \quad (\text{by equation (7.10)}) \\ &\le C\int_0^\infty \int_{K_{c'}} \exp(-2a't) dvol_{K_{c'}}dt < \infty \quad (\text{compare Lemma 4.5}),\end{aligned}$$

where a' is some positive number which is smaller than a.

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For $p \in U_c'$, we have

$$\left|\left(\bar{\Phi}^t\right)^*(\exp(-Tf)\psi')(p)\right| = \left|\left(\exp\left(-Tc - b^{-1}t\right)\left(\bar{\Phi}^t\right)^*\psi'\right)(p)\right|$$
$$\leq C\exp(-at) \quad \text{(by equation (7.9))}.$$

Similarly, we have $\int_{U_c'}|\exp(-Tf)\psi'|^2\, dvol < \infty$.

Now we show that $\mathcal{L}$ is surjective. We claim that any cohomology class $\xi \in H^j(M,U_c)$ can be represented by a smooth closed j-form ϕ so that $\phi|_{U_c} = 0$. Moreover, on U, $\iota_{X_f}\phi = 0$ and $\left(\bar{\Phi}^t\right)^*\phi$ does not depend on t for large t. Given the claim, it follows that $\exp(-Tf)\phi \in L^2\Lambda^*(M)$ via a similar argument as before. Hence we can find $\nu \in \ker\Box_{Tf}$ such that $\nu - \exp(-Tf)\phi$ is exact. Therefore, we can find $\psi \in \mathrm{Im}(\delta_{Tf})$ such that $\nu - \exp(-Tf)\phi = d_{Tf}\psi = (d_{Tf} + \delta_{Tf})\psi$. As a result, $\Box_{Tf}\psi = 0$ on U_c, and hence ψ is of exponential decay in U_c, which implies that $\mathcal{L}(\psi)$ is well defined. Now $\mathcal{L}(\nu) - \mathcal{L}(\exp(-Tf)\phi) = d_C\mathcal{L}(\psi)$, yielding $\mathcal{L}(\nu) \in \xi$ – that is, $\mathcal{L}$ is surjective.

Thus, it suffices to prove the claim. Indeed, one can first realise ξ by a closed form ϕ on M with $d\phi = 0$ and $\phi|_{U_c} = 0$. Now let $\eta : M \mapsto \mathbb{R}$ denote a smooth function which is identically 0 on K and identically 1 on $f^{-1}((S+1,\infty)\cup(-\infty,-I-1))$. The form

$$\phi'(p) = \phi(p) - d\left(\eta\int_{bT(-f(p)+I/2+S/2)}^{0}\left(\bar{\Phi}^s\right)^*\iota_{X_f}\phi\, ds\right)$$

is cohomologous to ϕ and satisfies the additional conditions as claimed.

Appendix A. Decomposition of L^2 space

In this section, we investigate the decomposition (1.1). For this purpose we first have to understand the Friedrichs extension of $\Delta_{H,f}$. Here we assume that all operators considered in this section are closable, as are our d_{Tf} and δ_{Tf} (compare [20, Theorem VIII.1]).

A.1. Review on Friedrichs extension

Let A be a nonnegative, symmetric (unbounded) operator on a Hilbert space $\mathcal{H}$, with $\mathrm{Dom}(\mathrm{A}) = V$ – that is,

$$(\mathrm{A}\,\alpha,\beta)_{\mathcal{H}} = (\alpha,\mathrm{A}\,\beta)_{\mathcal{H}}, \quad \forall\alpha,\beta \in V; \qquad (\mathrm{A}\,\alpha,\alpha)_{\mathcal{H}} \geq 0.$$

Define a norm $\|\cdot\|_{V_1}$ on V by

$$\|\alpha\|_{V_1}^2 = (\alpha,\alpha)_{\mathcal{H}} + (\alpha,\mathrm{A}\,\alpha)_{\mathcal{H}}.$$

Let V_1 be the completion of V under $\|\cdot\|_{V_1}$. Then for any $\beta \in \mathcal{H}$, one can construct a bounded linear functional L_β on V_1 as follows:

$$L_\beta(\phi) = (\phi,\beta)_{\mathcal{H}}, \quad \phi \in V_1. \tag{A.1}$$

Since $|(\phi,\beta)_{\mathcal{H}}| \leq \|\phi\|_{\mathcal{H}}\|\beta\|_{\mathcal{H}} \leq \|\phi\|_{\mathcal{H}}\|\beta\|_{V_1}$, L_β is indeed bounded functional on V_1. By Riesz representation, there exists $\gamma \in V_1$ such that $(\phi,\gamma)_{V_1} = (\phi,\beta)_{\mathcal{H}}$.

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Set $\mathrm{B} : \mathcal{H} \to V_1$, $\beta \mapsto \gamma$; then B is bounded and injective. Taking $\square = \mathrm{B}^{-1} - \mathrm{I}$, where I is the identity (inclusion) map, $\square$ is the Friedrichs extension of A, with $\mathrm{Dom}(\square) = \mathrm{Im}(\mathrm{B})$.

Remark A.1. *From the construction of the Friedrichs extension* $\square$ *of* A, *we can see that* $Dom(\square) = \mathrm{Im}\left((\mathrm{I}+\square)^{-1}\right)$.

Let T, S be two unbounded operators on the Hilbert space $\mathcal{H}$ such that

1.
$$V = \mathrm{Dom}(\mathrm{T}) = \mathrm{Dom}(\mathrm{S}), \quad \mathrm{T}V \subset V, \text{ and}$$

2. S is a formal adjoint of T : $\forall \alpha, \beta \in V$:
$$(\mathrm{T}\alpha, \beta)_{\mathcal{H}} = (\alpha, \mathrm{S}\beta)_{\mathcal{H}}.$$

Let $\|\cdot\|_W$ be the norm on V given by
$$\|\alpha\|_W^2 = (\alpha,\alpha)_{\mathcal{H}} + (\mathrm{T}\alpha, \mathrm{T}\alpha)_{\mathcal{H}}, \quad \alpha \in V,$$
and W be the completion of V under the norm $\|\cdot\|_W$. Then we can extend T to $\bar{\mathrm{T}}_{\min}$ with $\mathrm{Dom}\left(\bar{\mathrm{T}}_{\min}\right) = W$.

Let $\bar{\mathrm{S}}_{\max}$ be the closure of S with $\mathrm{Dom}\left(\bar{\mathrm{S}}_{\max}\right) = \{\alpha \in \mathcal{H} : |(\alpha, \mathrm{T}\phi)_{\mathcal{H}}| \le M_\alpha \|\phi\|_{\mathcal{H}},\ \forall \phi \in V\}$. Namely, for any $\alpha \in \mathrm{Dom}\left(\bar{\mathrm{S}}_{\max}\right)$, since V is dense in $\mathcal{H}$, by Riesz representation, there exists a unique $\nu \in \mathcal{H}$ such that $(\nu,\phi)_H = (\alpha, \mathrm{T}\phi)$. Now define $\bar{\mathrm{S}}_{\max}(\alpha) = \nu$.

Since $\mathrm{T}V \subset V$, $\mathrm{S}\mathrm{T}$ is symmetric and nonnegative, with $\mathrm{Dom}(ST) = V$.

Proposition A.2. *The Friedrichs extension* Δ *of* $\mathrm{S}\mathrm{T}$ *is just* $\bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min}$.

Proof. Since $\mathrm{T}V \subset V$, we see that V_1 constructed in equation (A.1) is the same as W. Indeed, for any $\phi, \psi \in V$, we have
$$(\psi,\phi)_{\mathcal{H}} + (\mathrm{T}\psi, \mathrm{T}\phi)_{\mathcal{H}} = (\psi,\phi)_{\mathcal{H}} + (\mathrm{S}\mathrm{T}\psi,\phi)_{\mathcal{H}}.$$
Hence we have
$$\mathrm{Dom}(\Delta) = \left\{\alpha \in W : \alpha = (I+\Delta)^{-1} f,\ f \in \mathcal{H}\right\},$$
$$\mathrm{Dom}\left(\bar{S}_{\max}\bar{T}_{\min}\right) = \left\{\alpha \in W : \bar{\mathrm{T}}_{\min}\alpha \in \mathrm{Dom}\left(\bar{\mathrm{S}}_{\max}\right)\right\}.$$
We now divide our discussion into two cases.

(a) We first prove that $\mathrm{Dom}\bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min} \subset \mathrm{Dom}(\Delta)$, and for all $\alpha \in \mathrm{Dom}\left(\bar{\mathrm{S}}_{\max}\right)$, $\bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min}\alpha = \Delta\alpha$.

For any $\alpha \in \mathrm{Dom}\bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min}$, let
$$\beta = \alpha + \bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min}\alpha. \tag{A.2}$$
Then for any $\phi \in W$, we have
$$\begin{aligned}(\alpha,\phi)_W &= \lim_{n\to\infty}(\alpha,\phi_n)_W \\ &= \lim_{n\to\infty}(\alpha,\phi_n)_{\mathcal{H}} + \left(\bar{\mathrm{T}}_{\min}\alpha, \mathrm{T}\phi_n\right)_{\mathcal{H}}\end{aligned}$$

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$$= \lim_{n\to\infty} (\alpha,\phi_n)_{\mathcal{H}} + \left(\bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min}\alpha,\phi_n\right)_{\mathcal{H}} \quad \left(\text{since } \phi_n \in V,\ \bar{\mathrm{T}}_{min}\alpha \in \mathrm{Dom}\left(\bar{\mathrm{S}}_{\max}\right)\right)$$
$$= \lim_{n\to\infty} \left(\alpha + \bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min}\alpha,\phi_n\right)_{\mathcal{H}} = \left(\alpha + \bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min}\alpha,\phi\right)_{\mathcal{H}}$$
$$= (\beta,\phi)_{\mathcal{H}}, \tag{A.3}$$

where $\phi_n \in V$ and $\phi_n \to \phi$ in the norm $\|\cdot\|_W$. By the construction of the Friedrichs extension and equation (A.3), we deduce that $\alpha \in (I+\Delta)^{-1}\mathcal{H}$ and $(I+\Delta)\alpha = \beta$. Comparing with equation (A.2), we obtain $\bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min}\alpha = \Delta\alpha$.

(b) We next show that $\mathrm{Dom}(\Delta) \subset \mathrm{Dom}\left(\bar{\mathrm{S}}_{\max}\bar{\mathrm{T}}_{\min}\right)$.

Take any $\alpha \in \mathrm{Dom}(\Delta) \subset W$. We can find $f \in \mathcal{H}$ such that $\alpha = (I+\Delta)^{-1}f$. We now just need to show that $\bar{\mathrm{T}}_{\min}\alpha \in \mathrm{Dom}\left(\bar{\mathrm{S}}_{\max}\right)$. For this, it suffices to prove that for all $g \in V$, $\left|\left(\bar{\mathrm{T}}_{\min}\alpha, \mathrm{T}g\right)_{\mathcal{H}}\right| \le M\|g\|_{\mathcal{H}}$ for some $M > 0$.

In fact, by standard functional calculus,

$$\left|\left(\bar{\mathrm{T}}_{\min}\alpha, \mathrm{T}g\right)_{\mathcal{H}}\right| = |(\alpha, ST g)_{\mathcal{H}}| \quad \left(\text{via } \alpha_n \in V,\ \alpha_n \to \alpha \in \|\cdot\|_W\right)$$
$$= \left|\left((I+\Delta)^{-1}f, \Delta g\right)_{\mathcal{H}}\right|$$
$$= \left|\left(f, (I+\Delta)^{-1}\Delta g\right)_{\mathcal{H}}\right|$$
$$\le M\|g\|_{\mathcal{H}}.$$

□

A.2. The Friedrichs extension of $\Delta_{H,f}$

By Proposition A.2, the Friedichs extension $\square_f$ of $\Delta_{H,f}$ is

$$\left(\overline{d_f + \delta_f}\right)_{\max} \left(\overline{d_f + \delta_f}\right)_{\min}.$$

If 0 is an eigenvalue of $\square_f$ with finite multiplicity, we have the decomposition

$$L^2\Lambda^*(M) = \ker \square_f \oplus \mathrm{Im}\left(\overline{d_f + \delta_f}\right)_{\max}. \tag{A.4}$$

Could we say more about decomposition (A.4)?

Proposition A.3. *Let* T, S *be two unbounded operators on a Hilbert space* $\mathcal{H}$ *such that the following are true:*

1.
$$V = \mathrm{Dom}(\mathrm{T}) = \mathrm{Dom}(\mathrm{S}), \quad \mathrm{T}V \subset V.$$

2. $\mathrm{Im}(T)$ *is orthogonal to* $\mathrm{Im}(S)$, *and*
$$(\mathrm{T}\alpha,\beta)_{\mathcal{H}} = (\alpha, \mathrm{S}\beta)_{\mathcal{H}}.$$

3. $T+S$ *is essential self-adjoint – that is,* $\left(\overline{T+S}\right)_{\min} = \left(\overline{T+S}\right)_{\max}$.

Then

$$\overline{T+S} = \bar{T}_{\min}|_{\mathrm{Dom}\bar{S}_{\min}\cap\mathrm{Dom}\bar{T}_{\min}} + \bar{S}_{\min}|_{\mathrm{Dom}\bar{S}_{\min}\cap\mathrm{Dom}\bar{T}_{\min}}$$
$$= \bar{T}_{\max}|_{\mathrm{Dom}\bar{S}_{\max}\cap\mathrm{Dom}\bar{T}_{\max}} + \bar{S}_{\max}|_{\mathrm{Dom}\bar{S}_{\max}\cap\mathrm{Dom}\bar{T}_{\max}}.$$

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Proof. Since $\mathrm{Dom}\overline{(T+S)}_{\min}$ is the closure of V under the metric

$$(\phi,\phi)_{\mathcal{H}}+((T+S)\phi,(T+S)\phi)_{\mathcal{H}}=(\phi,\phi)_{\mathcal{H}}+(T\phi,T\phi)_{\mathcal{H}}+(S\phi,S\phi)_{\mathcal{H}}, \qquad (**)$$

we have $\mathrm{Dom}\overline{(T+S)}_{\min}\subset \mathrm{Dom}\bar{S}_{\min}\cap\mathrm{Dom}\bar{T}_{\min}$. Also, for any $\phi\in\mathrm{Dom}(T+S)_{\min}$,

$$\overline{(T+S)}_{\min}\phi=\lim_{n\to\infty}(T+S)\phi_n=\lim_{n\to\infty}T\phi_n+S\phi_n=\mathrm{T}_{\min}\phi+\mathrm{S}_{\min}\phi,$$

where $\phi_n\in V\to\phi$ in the metric (**).

For each $\phi\in\mathrm{Dom}\bar{S}_{\max}\cap\mathrm{Dom}\bar{T}_{\max}$, $\psi\in V$, we have

$$\begin{aligned}&(\phi,(T+S)\psi)_{\mathcal{H}}=(\phi,T\psi)_{\mathcal{H}}+(\phi,S\psi)_{\mathcal{H}}\\&=\left(\bar{\mathrm{T}}_{\max}\phi,\psi\right)_{\mathcal{H}}+\left(\bar{\mathrm{S}}_{\max}\phi,\psi\right)_{\mathcal{H}}\\&\le C\|\psi\|_{\mathcal{H}}.\end{aligned}$$

Therefore $\phi\in\mathrm{Dom}\left(\overline{(T+S)}_{\max}\right)$ and $\overline{(T+S)}_{\max}\phi=\bar{\mathrm{T}}_{\max}\phi+\bar{\mathrm{S}}_{\max}\phi$, which means that $\mathrm{Dom}\bar{S}_{\min}\cap\mathrm{Dom}\bar{T}_{\min}\subset\mathrm{Dom}\left(\overline{(T+S)}_{\max}\right)$. □

Our Theorem 2.3 – the Kodaira decomposition for the Witten decomposition – follows from equation (A.4), Theorem 2.1 and Proposition A.3.

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