# AN EXPLICIT VERSION OF CHEN'S THEOREM

## MATTEO BORDIGNON

We focus on Chen's theorem, proved in 1966 by Chen [5, 6]. We obtain the first completely explicit version of Chen's theorem and, in doing so, improve many mathematical tools that are needed for the task. We prove the following result.

THEOREM 1. *All even numbers bigger than* $\exp(36)$ *can be written as the sum of a prime and another integer that is the product of at most two primes.*

Here, it is interesting to note that while a lot of effort was put into making Vinogradov's proof of Goldbach's weak conjecture completely explicit, not much work was put into making Chen's theorem explicit, while arguably this result is an even better approximation of Goldbach's conjecture. The only attempt was made by Yamada [14], but some mistakes can be found in the proof. (See [14, (87) and (104)], where a log term appears to be missing. Also, no proof is given of the explicit version of the linear sieve that is used and this version is inconsistent with the versions in [11, 12]). We also show that the following result follows readily from Theorem 1.

THEOREM 2. *All even numbers bigger than* 2 *can be written as the sum of a prime and the product of at most* $\exp(33)$ *primes.*

For the proof of Theorem 1, we draw inspiration from the work by Nathanson [12] and Yamada [14]. We will now illustrate the most salient steps and results employed to obtain Theorem 1.

The proof of Chen's theorem is based on the linear sieve, proved by Jurkat and Richert [11] and Iwaniec [9], who were inspired by the work of Rosser [10]. We base our work on another version obtained by Nathanson in [12] from unpublished notes

by Iwaniec. Taking $S(A, P(z))$, a certain set of integers, sieved using $P(z)$, the product of certain primes, we find that for fixed $D \in \mathbb{R}^+$ and $s = \log D / \log z$,

$$S(A, P(z)) < (F(s) + o_K(s))X + R_Q,$$

whenever $s \geq 1$, and if $s \geq 2$, we have a lower bound

$$S(A, P(z)) > (f(s) - o_K(s))X - R_Q.$$

Here, $X$ and $R_Q$ are two specific sums and $F(s)$ and $f(s)$ two well-known functions, which are optimal. We can also note that it is simple to effectively bound the $X$ term.

To obtain an explicit version of Chen's theorem, we need an explicit version of the term $o_K(s)$. This was first obtained by Nathanson [12, Theorem 9.8]. He proved that we can take $o_K(s) = (K-1)e^{14-s}$. To optimise the lower bound on $N$ in Theorem 1, we need $o_K(s)$ to be as small as possible. We obtain this by using a more computational approach, compared with Nathanson's analytic one, and we thus reduce the term by a factor of 1000 (see [4]).

We are now left with bounding the error term $R_Q$ that is related to the prime number theorem for primes in arithmetic progressions (PNTPAP). To bound this term efficiently, we focus on improving the version of the PNTPAP for medium-sized $x$, and isolate the contribution of the possible Siegel zero, given by Yamada [13]. We obtain this result by proving an explicit version of the result by Goldston [7], to obtain a $\log x / \log \log x$ saving. We also improve the Bombieri–Vinogradov style theorem for nonexceptional moduli [13].

It remains to obtain a good explicit bound on the Siegel zero. To do so, we focus on improving the bound proven by Bennett *et al.* [1]. First we prove a general result that allows us to remove one of the two terms that appeared, in the previous results, in the upper bound of $L'(\sigma, \chi)$ (see [2]). We then introduce a different technique, following from a paper of Hua [8] on the average of Dirichlet characters, and compute better lower bounds for $L(1, \chi)$ to further improve the result for even characters (see [3]). As a consequence, we obtain

$$\beta_0 \leq 1 - \frac{100}{\sqrt{q} \log^2 q}. \tag{1}$$

The proof of Theorem 1 thus depends on the possible existence of an exceptional zero. We use some ideas from Yamada [13] to handle this. Letting $k_0$ be the modulus of the exceptional zero, we use two approaches that distinguish its size.

(1) If $k_0$ is 'big', we can bound the error term $R_Q$ using (1) as, in this case, we have $100/\sqrt{q} \log^2 q$ 'small'.
(2) If $k_0$ is 'small', we want to be sure that $k_0$ does not appear in the sum in $R_Q$ and this depends on the number of primes in $P(z)$. Here it is fundamental to use an inclusion–exclusion principle.

## References

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MATTEO BORDIGNON, Department of Science,
University of New South Wales Canberra, Campbell, ACT 2612, Australia
e-mail: matteobordignon91@gmail.com