AN EXAMPLE INVOLVING A NON-REGULAR $\mathscr{D}$-CLASS IN A SEMIGROUP

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1. Introduction and summary

We answer in the negative (by considering an example) the problem posed in exercise 6 of § 2.3, page 62 [1], namely: If a $\mathscr{D}$-class D of a semigroup S is a subsemigroup of S, then is D necessarily bisimple? For any semigroup T, we let $\mathscr{L}_T$, $\mathscr{R}_T$, $\mathscr{H}_T$, $\mathscr{D}_T$ and $\mathscr{J}_T$ denote Green's relations on T.

2. The example

Denote the set of real numbers by R and put $R^+ = \{x \in R \colon x > 0\}$. Consider the following sets of 2×2 matrices over R:

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} : a, b \in R^+ \right\} ;$$

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} : b \in R^+ \right\} ; \text{ and } S = K \cup G.$$

Under matrix multiplication, S is a semigroup, K is an ideal of S, and G is a subgroup of S; we note that K is included among the examples in exercises 8, 9, 10 of § 2.1 and exercise 7 of § 5.4 [1].

Take now any elements a, b, c, d in R^+. Using easy calculations, we may show that

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} K = \left\{ \begin{pmatrix} 1 & 0 \\ a+x & y \end{pmatrix} : x, y \in R^+ \right\},$$

$$K \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ z & w \end{pmatrix} \in K : z/w > a/b \right\},$$

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} S = \left\{ \begin{pmatrix} 1 & 0 \\ a+x & y \end{pmatrix} : y \in R^+, x \in R \text{ and } x \geqq 0 \right\},$$

and

$$S \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ z & w \end{pmatrix} \in K : z/w \geqq a/b \right\}.$$

It follows that $\mathscr{L}_K = \mathscr{R}_K = \mathscr{D}_K = \iota_K$, the identity relation on K, while

$$\begin{pmatrix}1 & 0\\ a & b\end{pmatrix} \mathscr{R}_S \begin{pmatrix}1 & 0\\ c & d\end{pmatrix} \text{ if and only if } a = c, \text{ and}$$

$$\begin{pmatrix}1 & 0\\ a & b\end{pmatrix} \mathscr{L}_S \begin{pmatrix}1 & 0\\ c & d\end{pmatrix} \text{ if and only if } a/b = c/d.$$

It follows that

$$\begin{pmatrix}1 & 0\\ a & b\end{pmatrix} \mathscr{R}_S \begin{pmatrix}1 & 0\\ a & ad/c\end{pmatrix} \mathscr{L}_S \begin{pmatrix}1 & 0\\ c & d\end{pmatrix}.$$

Clearly now $\mathscr{D}_S \supseteq (K \times K) \cup (G \times G)$, and $\mathscr{D}_S \neq S \times S$ since K is an ideal of S. It follows that K is a $\mathscr{D}_S$-class, and is a subsemigroup of S, but is 'far from' being bisimple.

REMARK 1. For any $\mathscr{L}_S$-class, L say, and $\mathscr{R}_S$-class, R say, both contained in the $\mathscr{D}_S$-class K, we have $LR = K$, c.f. exercise 2 of § 2.3, page 61 [1].

REMARK 2. Since

$$\mathscr{D}_S \subseteq \mathscr{J}_S \subseteq (K \times K) \cup (G \times G) = \mathscr{D}_S,$$

we see that K is the kernel of S. No principal left ideal of K is also a left ideal of S, c.f. the example given by Clark [2].

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups* (Math. Surveys, number 7, Amer. Math. Soc., Vol. I 1961).

[2] W. E. Clark, 'Remarks on the kernel of a matrix semigroup', *Czechoslovak Math. J.* 15 (1965) 305—310.

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