

WHAT MODEL COMPANIONSHIP CAN SAY ABOUT THE CONTINUUM PROBLEM

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Abstract. We present recent results on the model companions of set theory, placing them in the context of a current debate in the philosophy of mathematics. We start by describing the dependence of the notion of model companionship on the signature, and then we analyze this dependence in the specific case of set theory. We argue that the most natural model companions of set theory describe (as the signature in which we axiomatize set theory varies) theories of H_{κ^+} , as κ ranges among the infinite cardinals. We also single out $2^{\aleph_0} = \aleph_2$ as the unique solution of the continuum problem which can (and does) belong to some model companion of set theory (enriched with large cardinal axioms). While doing so we bring to light that set theory enriched by large cardinal axioms in the range of supercompactness has as its model companion (with respect to its first order axiomatization in certain natural signatures) the theory of H_{\aleph_2} as given by a strong form of Woodin's axiom (*) (which holds assuming MM^{++}). Finally this model-theoretic approach to set-theoretic validities is explained and justified in terms of a form of maximality inspired by Hilbert's axiom of completeness.

§1. Introduction. There is no doubt that the continuum problem is one of the driving forces of set theory. The attempts to determine the cardinality of the continuum has accompanied the history of set theory, from its very beginning to the present day, and motivated many of its most significant advances. Cantor's definition of the Perfect Set Property was prompted by a partial solution to the continuum problem; more in general, the initial developments of descriptive set theory were also driven by an attempt to confirm the continuum hypothesis (CH: $2^{\aleph_0} = \aleph_1$) at least for the definable subsets of \mathbb{R} . In more recent times, Gödel's constructible universe and Cohen's method of forcing were devised to show the independence of CH from ZFC (the standard first order axiomatization of set theory).

The techniques developed by Gödel and Cohen clarified the intrinsic limitations of the axiomatic approach to the continuum problem, and profoundly influenced the subsequent development of set theory. As a matter of fact, after the 1960s, the many independence results obtained combining the methods of inner models and forcing

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partially shifted the main focus of set theory: set theorists progressively devoted more and more efforts to the study of the models of set theory, and only derivatively to the study of sets (understood as autonomous mathematical objects).

The present paper presents a new approach to the quest of a solution of the continuum problem and—more generally—to the search of new set theoretic truths; this approach stems from a model-theoretic perspective on set theory, and aims to clarify the philosophical import of the results in [38, 41]. As is standard, we recognize the intrinsic limitations of ZFC in capturing set-theoretic validities, but, to overcome these limitations, we bring forward the idea that a powerful criterion to validate a new axiom is given by the nice model theoretic properties of the first order presentations of set theory which include it.

The central notion we borrow from model theory and that guides the present approach is that of model companionship. This concept was developed by Abraham Robinson in the 1960s, and it is meant to capture, in an abstract setting, the closure properties that algebraically closed fields display with respect to commutative rings with no zero divisors. As algebraically closed fields contain solutions for all Diophantine equations, the models of the model companions of set theory will have solutions for all “simple” set theoretical problems.

A fundamental property of model companionship is its dependence on the signature.¹ Loosely speaking: there can be distinct first order axiomatizations of a mathematical theory T , one in signature σ and the other in signature τ , such that T admits a model companion when axiomatized according to σ , but does not when axiomatized according to τ .²

The dependence on the signature of model companionship motivates a detailed discussion aimed to single out the relevant signatures for set theory. We will start by clarifying some fundamental aspects of set-theoretical practice. In particular, we will explain why bounded formulae (i.e., Δ_0 -formulae) express set-theoretic concepts of low complexity and therefore should be included in any reasonably rich signature for set theory. We will then show how a careful choice of the signatures for set theory allows one to classify the complexity of set-theoretic concepts and to clarify the informal notion of “simplicity of a set-theoretic assertion.” Our analysis will provide precise criteria able to link logic complexity, forcing invariance, and simplicity of set-theoretic concepts. Once we have outlined clear criteria to select signatures, we will study set theory under different expansions of $\{\in\}$. By varying the signature, we will show that the corresponding model companions of set theory describe possible theories of the structures H_{κ^+} , for κ an infinite cardinal.³ Moreover, given an appropriate signature τ_κ for set theory (relative to a cardinal κ), the corresponding τ_κ -theory of H_{κ^+} will describe a structure which is closed-off with respect to specific constraints. Roughly speaking: much alike an algebraically closed field K contains solutions to all polynomial equations with coefficients in K , the τ_κ -structures realizing the model companion of set theory in signature τ_κ will make true all Σ_1 -properties for τ_κ (with the relevant parameters from H_{κ^+}) which are not “outward inconsistent.”

¹ See Section 4 for a precise formulation of this dependence.

² This has an analogy with what occurs for Birkhoff’s characterization of an algebraic variety in terms of a universal theory—the latter axiomatization may only be expressed in a signature different from the one in which the variety is initially axiomatized.

³ Recall that H_{κ^+} is the collection of sets whose transitive closure has size at most κ .

A precise syntactic description of this closing-off procedure will be given by the notion of absolute model companionship (AMC). We defer to Section 5.2 a thorough presentation of this notion, we can briefly describe it right away by saying that AMC describes those τ -theories S for which a model companion exists and is axiomatized by the Π_2 -sentences for τ which are consistent with the universal and existential fragment of any completion of S .⁴

Building on the properties of AMC, the sought solution to the continuum problem can be presented as follows:

- (i) $\neg\text{CH}$ and even a definable version⁵ θ of $2^{\aleph_0} = \aleph_2$ can be expressed by Π_2 -formulae in any signature for set theory which contains the Δ_0 -properties among its atomic formulae and has a parameter for ω_1 .
- (ii) For each infinite (and definable) cardinal κ there is at least one (actually several) signature(s) τ_κ containing the Δ_0 -properties among its atomic formulae, a constant to interpret the cardinal κ , and such that set theory axiomatized in the corresponding τ_κ admits an AMC.⁶
- (iii) Given any signature τ for set theory, containing the Δ_0 -properties among its atomic formulae, any first order τ -axiomatization T of set theory admitting an AMC S for τ will not have neither CH nor $\neg\theta$ among the τ -theorems of S , unless T models CH or $\neg\theta$ (furthermore for many choices of τ and T , even if T models CH and/or $\neg\theta$, S will not).
- (iv) There is at least one (recursive and natural⁷) signature τ^* for set theory which contains the Δ_0 -properties among its atomic formulae and such that all τ^* -axiomatizations T of any set theory enriched with large cardinal axioms⁸ admit an AMC S for τ^* that contains $\neg\text{CH}$ and θ among its axioms. Furthermore S describes the τ^* -theory of H_{\aleph_2} in models of strong forcing axioms.

Therefore, these results single out (a definable version of) $2^{\aleph_0} = \aleph_2$ as the unique solution of the continuum problem which can fall in at least one AMC of set theory enriched with large cardinal axioms (for example the AMC with respect to signature τ^* of item (iv) above).⁹

The present model-theoretic approach to set theoretic validities complements well standard strategies for producing axiomatic extensions of ZFC. Actually we see our approach as an implementation of Gödel's program. Given the stratification of V as the increasing union of its initial segments provided by the various H_{κ^+} (as κ ranges

⁴ AMC appears to be the correct notion of model companionship to apply to set theory. For example it is unknown to the authors whether there can be a signature for which the axiomatization of set theory in it admits a model companion which is not its AMC.

⁵ Theorem 7.8 and the discussion surrounding it will clarify what is meant here by the expression "a definable version of $2^{\aleph_0} = \aleph_2$."

⁶ Furthermore these AMCs describes theories of H_{κ^+} (in ways which we will unfold later on).

⁷ The discussion to follow makes clear that τ^* is a signature which naturally stems out from the set theoretic practice of the last 40 years.

⁸ For example T could have CH or $2^{\aleph_0} > \aleph_2$ among its axioms provided it extends ZFC+*there are class many supercompact cardinals*.

⁹ Furthermore the same argument we used for CH and θ applies mutatis-mutandis to any set theoretic statement ψ which can be formalized by a Π_2 -sentence in the signature τ^* of item (iv) and such that ψ is forcible over any model of set theory enriched by large cardinal axioms.

over the infinite cardinals), we can interpret the AMCs of set theory given by item (ii) as tentative descriptions of the theories of these H_{κ^+} , theories which coherently extend each other.¹⁰ Assuming this perspective, these τ_κ -theories of H_{κ^+} are gradually closing-off the universe of sets with respect to all “simple” set theoretical problems (with each τ_κ giving its own description of what is a “simple” set theoretic concept, and with each τ_κ making more and more set theoretic concepts becoming “simple” as κ increases). In this sense, the nice model-theoretic properties displayed by the theories of H_{κ^+} (when they describe possible model companions of set theory) realize a form of maximality that we dare to call Hilbertian completeness. As is well-known, Hilbert’s axiomatization of geometry expresses a form of maximality for geometrical objects, by stating that the domain of geometry could not be extended if it were to preserve the axioms of the *Grundlagen der Geometrie*. Similarly, the closing-off processes that each model companionship result carries along can (and will) be described as a way to assert the maximization of set-theoretical objects. Moreover, important axiomatic extensions of ZFC (considered candidates for the realization of Gödel’s program¹¹) like large cardinals and forcing axioms find, in the present approach, a further justification. As a matter of fact, the generic absoluteness results that we can obtain from large cardinal hypotheses are now of pivotal importance in detecting the suitable signature expansions able to allow the existence of a(n absolute) model companion for set theory. Another surprising fact which shows that model companionship results and forcing axioms complement each other is that the theory of H_{\aleph_2} in models of strong forcing axioms describes the (absolute) model companion of ZFC+*large cardinals* (as expressed in the natural signature τ^* of item (iv) above); conversely the existence of a model companion for the theory of V in this signature τ^* already implies that a strong form of bounded Martin’s maximum (which is one of the strongest forcing axioms) and Woodin’s axiom (*) both hold in V .

The paper is structured as follows: in Sections 2 and 3 we gently analyze the role that signatures can play in outlining the properties of a mathematical theory T (with a focus on how the morphisms between models of T can be used to detect the right signatures for T). In Section 4 we present a few standard notions from model theory (i.e., existentially closed structures, model completeness, and model companionship), which are discussed and generalized in Section 5, where we define the notions of partial Morleyization, absolute model companionship, and (absolute) model companionship spectrum. From Section 6 onwards we shift our attention to set theory. We first present precise mathematical criteria to detect what are the “simple” set-theoretical concepts. Then we analyze various signatures for set theory in Section 6.1. In Section 6.2 we show that the intended models of the model companions of set theory are structures

¹⁰ We will come back on this point with more precision (see (A) and the discussion following it, and Theorem 7.3); roughly the argument is that we can choose uniformly τ_λ for each ZFC-definable cardinal λ so that in a model (V, \in) of ZFC the τ_λ -theory of $H_{\lambda^+}^V$ is a model of the AMC of set theory in signature τ_λ and is naturally described in $H_{\kappa^+}^V$ for $\kappa > \lambda$ by looking at H_{λ^+} as a definable subset of H_{κ^+} .

¹¹ A clear justification of why large cardinal axioms are a partial realization of Gödel’s program can be found in [23]. An analysis of why (strong) forcing axioms can also be seen as a realization of this program can be found in [7] or in the introduction of [39]. See also [44–46] where it is proposed to realize Gödel’s program by introducing new axioms to describe the theories of H_{\aleph_1} and H_{\aleph_2} .

of the form H_{κ^+} , for κ an infinite cardinal. Finally, in Section 7 we present the main results regarding the model companions of set theory and discuss the information they convey on the cardinality of the continuum. We conclude, in Section 8, by comparing our approach with the various notions of maximality we find in the literature and with the debate on the justification of new axioms in set theory.

This paper aims to reach scholars with interests at the crossroads of philosophy and mathematics. We tried for this reason to minimize the mathematical and philosophical prerequisites needed to follow it. Those who made through this introduction should encounter no serious obstacles to read the remainder of this article. The model companionship results we discuss in the present paper are presented without proofs. The reader interested in them is referred to [38, 41].

Clearly the opinions expressed (and the errors occurring) in this article are the sole responsibility of the authors and do not extend in any way to those who had the patience to give some advice to improve our presentation.

§2. What is the right signature for a mathematical theory? One of the great successes of mathematical logic consists in providing an efficient formalization of mathematics: by means of first order logic it is possible to render mathematical theories the objects of a mathematical investigation. In this way logic is able to produce unexpected and non-trivial mathematical results as well as novel insights on a variety of mathematical fields. It is a matter of facts that there can be many distinct first order formalizations of a mathematical theory: varying the linguistic presentation of a theory (i.e., its *signature*) we obtain different axiomatic presentations of the same set of theorems. In order to appreciate the variety of possibilities we can encounter, let us consider the concrete case of group theory. This will help us to gently introduce one of the main themes of the present present: the role of the signature in detecting the properties of a theory formalized in it.

We can formalize group theory in first order logic using the signature $\{\cdot\}$, which simply consists of a binary function symbol and the following axioms:

$$\begin{aligned} &\forall x, y, z [(x \cdot y) \cdot z = x \cdot (y \cdot z)], \\ &\exists x \forall y (x \cdot y = y \wedge y \cdot x = y), \\ &\forall x \exists y \forall z [(x \cdot y) \cdot z = z \wedge (y \cdot x) \cdot z = z \wedge z \cdot (x \cdot y) = z \wedge z \cdot (y \cdot x) = z]. \end{aligned}$$

Notice that the third axiom, which expresses the existence of a multiplicative inverse, represents a rather complicated assertion, both from the point of view of its syntactic readability and of its Lévy complexity (being a Π_3 -sentence). The reason is that in this basic signature we lack a constant symbol to denote the neutral element of a group. Enriching the language to $\{\cdot, e\}$, with e a constant symbol, we can now formalize group theory with a simpler set of axioms:

$$\begin{aligned} &\forall x, y, z [(x \cdot y) \cdot z = x \cdot (y \cdot z)], \\ &\forall y (e \cdot y = y \wedge y \cdot e = y), \\ &\forall x \exists y [x \cdot y = e \wedge y \cdot x = e]. \end{aligned}$$

We increased readability and decreased (Lévy) complexity, as we are now dealing with a Π_2 -axiomatization. But we can do better. Indeed, if we consider the signature

$\{\cdot, e, {}^{-1}\}$, which further adds a unary operation symbol for the inverse operation, we can axiomatize group theory with a set of universal equations (Π_1 -sentences):

$$\begin{aligned} &\forall x, y, z [(x \cdot y) \cdot z = x \cdot (y \cdot z)], \\ &\forall x (x \cdot e = x \wedge e \cdot x = x), \\ &\forall x [x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e]. \end{aligned}$$

On the other hand, we could follow a completely different route and axiomatize group theory avoiding the use of function symbols. For example we can consider the signature $\{R, e\}$ consisting of a ternary relation symbol R and a constant symbol e and produce the following axiomatization for group theory:

$$\begin{aligned} &\forall x, y \exists! z R(x, y, z), \\ &\forall x, y, z, w, t [(R(x, y, w) \wedge R(y, z, t)) \rightarrow \exists u (R(x, t, u) \wedge R(w, z, u))], \\ &\forall y [R(e, y, y) \wedge R(y, e, y)], \\ &\forall x \exists y [R(x, y, e) \wedge R(y, x, e)]. \end{aligned}$$

At the cost of further complicating our axiomatization, we could even drop the use of the constant symbol e and formalize group theory in the signature $\{R\}$. The latter is clearly an artificial solution; moreover, the minimality of the signature does not help the perspicuity of the axiomatization.

Of all the above formalizations, the one mathematicians use more frequently is certainly the one in signature $\{\cdot, e, {}^{-1}\}$. On the contrary, to recognize that the system in signature $\{R, e\}$ is an axiomatization of group theory would surely require a bit of thought.

These considerations suggest that among the many possible signatures in which we can formalize a mathematical theory some are better than others. Our aim is to unfold criteria which allow us to detect the best signatures τ for the formalization of a mathematical theory T . To do so we appeal to two sets of arguments.¹²

(intCrit) On the one hand, we can select τ on the basis of specific considerations internal to T . For example, by checking the adherence of a τ -formalization of T to the standard informal practice (e.g., the signature $\{\cdot, e, {}^{-1}\}$ clearly gives the best presentation of group theory in terms of its basic operations).

(AbsCrit) On the other hand, we can give abstract criteria for the choice of τ based on the structural properties that the τ -axiomatization of T displays, disregarding any consideration on the adherence of the τ -axiomatization of T with the informal one.

It occurs for many T that the reasons internal to T for choosing a signature τ for its axiomatization cohere with the abstract criteria, and that the two approaches produce the same outcome. This is desirable as abstract criteria are easily expressible with a sufficient degree of mathematical precision: we only need to define precisely what structural properties are preferable; then for a given theory T we can select the signatures σ for which the σ -axiomatization of T has the preferred property. On the other hand it is less transparent how to give a precise mathematical formulation of

¹² (IntCrit) stands for “Internal Criterion,” (AbsCrit) stands for “Abstract Criterion.”

the reasons internal to T for choosing a signature. For example, which mathematical criteria allow us to recognize $\{\cdot, e\}$ as a better signature than $\{R, e\}$ for the formalization of group theory? For example both give Π_2 -axiomatizations,¹³ although it is clear that any mathematician would regard the $\{\cdot, e\}$ -axiomatization as more natural than the one given in $\{R, e\}$. Can we turn this qualitative preference into a precise mathematical criterion?

§3. Simple and complicated concepts for a mathematical theory. When dealing with a first order theory T in a signature τ , the τ -formulae naturally correspond to the concepts/notions T intends to formalize. The well-known Lévy complexity then gives a measure gauged by τ of the complexity of a concept/notion of T by analyzing the number and patterns of the quantifiers that appear in a τ -formula which should formalize it. The Lévy hierarchy of the τ -formulae with respect to T -equivalence stratifies the concepts/notions of T as follows: the basic notions are those formalized by a boolean combination of atomic τ -formulae. The complexity of a notion then increases according to the number of alternations of \forall, \exists quantifiers that a formula ϕ formalizing it displays when expressed in a prenex normal form. In assigning a Lévy complexity to a notion of T , we consider, among all the τ -formulae which are T -equivalent and formalize it, those in prenex normal form with the least number of quantifier alternations. In this way we can assign complexity Π_n, Σ_n or Δ_n to the notions of T that are expressible by means of τ -formulas. More precisely, $\Pi_0 = \Sigma_0 = \Delta_0$ is the complexity of notions formalized by boolean combinations of atomic formulae; Π_{n+1} represents the complexity of those notions formalized by a formula of type $\forall \vec{x} \psi(\vec{x}, \vec{y})$ with $\psi(\vec{x}, \vec{y}) \Sigma_n$; Σ_{n+1} represents the complexity of those notions formalized by a formula of type $\exists \vec{x} \phi(\vec{x}, \vec{y})$ with $\phi(\vec{x}, \vec{y}) \Pi_n$, and Δ_n represents the complexity of the notions whose complexity lies in $\Pi_n \cap \Sigma_n$. However, Levy complexity for a notion of T does not always match with the intuitive complexity we attribute in practice to it, and is clearly dependent on the signature τ we use to express it. For example the process of Morleyization (see [34, sec. 3.2] or Definition 5.5) produces a mathematically equivalent axiomatization T^* of a first order τ -theory T in a new signature $\tau^* \supseteq \tau$, such that every τ^* -formula is T^* -equivalent to an atomic formula; hence all notions of T become of the same Levy complexity when expressed in signature τ^* according to the mathematically equivalent theory T^* .

To better understand the relevance of the signature in describing the correct logical complexity of the notions of a theory, let us examine the case of the theory of *commutative semi-rings with no zero-divisors*. Consider the signature $\{+, \cdot, 0, 1\}$, which is standardly used to axiomatize rings and fields. In this signature we can provide the axioms of commutative semi-rings with no zero-divisors by means of the following universal sentences:

$$\begin{aligned} \forall x, y (x \cdot y = y \cdot x), \\ \forall x, y, z [(x \cdot y) \cdot z = x \cdot (y \cdot z)], \\ \forall x (x \cdot 1 = x \wedge 1 \cdot x = x), \end{aligned} \tag{1}$$

¹³ Note that $\forall \vec{x} \exists! y \psi(\vec{x}, y)$ for ψ quantifier-free is the Π_2 -sentence $\forall \vec{x} [\exists y \psi(\vec{x}, y) \wedge \forall w, z (\psi(\vec{x}, w) \wedge \psi(\vec{x}, z) \rightarrow w = z)]$.

$$\begin{aligned} &\forall x, y (x + y = y + x), \\ &\forall x, y, z [(x + y) + z = x + (y + z)], \\ &\forall y (x + 0 = x \wedge 0 + x = x), \\ &\forall x, y, z [(x + y) \cdot z = (x \cdot y) + (x \cdot z)], \\ &\forall x, y [x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)]. \end{aligned}$$

By further extending the above list we can obtain the theory of *commutative rings with no zero-divisors* (e.g., *integral domains*), adding the Π_2 -axiom

$$\forall x \exists y (x + y = 0), \tag{2}$$

and the theory of *fields* by further adding the Π_2 -axiom

$$\forall x [x \neq 0 \rightarrow \exists y (x \cdot y = 1)], \tag{3}$$

and finally the theory of *algebraically closed fields* by supplementing the above axioms with the following Π_2 -sentences¹⁴ for all $n \geq 1$

$$\forall x_0 \dots x_n \exists y \sum x_i \cdot y^i = 0. \tag{4}$$

A common feature of the above set of Π_2 -axioms is that they assert the existence of solutions to certain basic equations of the theory expressible in terms of sum, multiplication, 0, 1, i.e., the atomic formulae of $\{\cdot, +, 0, 1\}$. Indeed, any commutative semi-ring \mathcal{M} without zero divisors can be extended to an algebraically closed field simply by extending it with solutions to the basic polynomial equations with parameters in \mathcal{M} (and which may not exist in \mathcal{M}).

In the signature $\{+, \cdot, 0, 1\}$ we have as basic operations $+, \cdot$. Using this signature for integral domains we can also define the additive inverse, $-$, using the atomic formula $(x + z = 0)$, while for fields we can add the multiplicative inverse, $^{-1}$, using the boolean combination of atomic formulae $(x = 0 \wedge z = 0) \vee (x \neq 0 \wedge x \cdot z = 1)$. Notice that even when subtraction and division are partially defined on a semiring, we can still meaningfully interpret these two operations in it, just by adding two unary function symbols $-, ^{-1}$ for them and by adopting the convention that the corresponding inverse operations are trivially defined on the non invertible elements; this is captured for example by the axioms

$$\forall x [(\exists y (x \cdot y = 1) \wedge x \cdot x^{-1} = 1) \vee (\neg \exists y (x \cdot y = 1) \wedge x^{-1} = 0)] \tag{5}$$

to interpret $^{-1}$, and

$$\forall x [(\exists y (x + y = 0) \wedge x + (-x) = 0) \vee (\neg \exists y (x + y = 0) \wedge (-x) = 0)] \tag{6}$$

to interpret $-$.

It is clear that the class of $\{+, \cdot, 0, 1, -, ^{-1}\}$ -structures satisfying axioms (1), (5), and (6) are exactly the commutative semi-rings with no zero-divisors, however, expanding the signatures and introducing the new axioms which correctly interpret the new symbols, we perturbed the notion of morphism. To see this, notice that the inclusion

¹⁴ Notice that any model of axioms (1) and (4) is automatically a field.

of \mathbb{N} into \mathbb{Z} is a $\{+, \cdot, 0, 1\}$ -morphism but not a $\{+, \cdot, 0, 1, -,^{-1}\}$ -morphism as $-(2) = 0$ when computed in \mathbb{N} seen as a model of (1), (5), and (6), while $-(2) = -2$ when computed in \mathbb{Z} seen as a model of (1), (5), and (6). Similarly the inclusion of \mathbb{Z} into \mathbb{Q} is a $\{+, \cdot, 0, 1, -\}$ -morphism but not a $\{+, \cdot, 0, 1, -,^{-1}\}$ -morphism.

Moreover, we can observe that in the class of integral domains as formalized in signature $\{+, \cdot, 0, 1, -\}$ subtraction is axiomatized now by the universal sentence $\forall x [x + (-x) = 0]$ rather than the Π_2 -sentence (6), which defines it in the larger class of commutative semirings. Similarly, if we consider the class of $\{+, \cdot, 0, 1, -,^{-1}\}$ -structures which are fields, i.e., models of (1)–(3), then (5) becomes logically equivalent, modulo the other axioms, to $\forall x [x \neq 0 \rightarrow x \cdot x^{-1} = 1]$, which is a universal $\{+, \cdot, 0, 1, -,^{-1}\}$ -sentence. On the algebraic side note that the $\{\cdot, +, 0, 1\}$ -morphisms between integral domains naturally extend to $\{+, \cdot, 0, 1, -\}$ -morphisms, as the operation $x \mapsto -x$ is preserved by additive morphisms on rings. Similarly the $\{\cdot, +, 0, 1\}$ -morphisms between fields naturally extend to $\{+, \cdot, 0, 1, -,^{-1}\}$ -morphisms, as the operation $x \mapsto x^{-1}$ is preserved by additive and multiplicative morphisms between fields.

More generally, when a τ -theory T can define an operation by means of universal τ -equations, the τ -morphisms between τ -models of T preserve the operation. In our concrete example, $\{+, \cdot, 0, 1\}$ -morphisms between integral domains naturally extends to $\{+, \cdot, 0, 1, -\}$ -morphisms, while $\{+, \cdot, 0, 1\}$ -morphisms between fields naturally extends to $\{+, \cdot, 0, 1, -,^{-1}\}$ -morphisms exactly because the operations $-,^{-1}$ are defined by universal equations in the respective theories.

These examples suggest the following observation: when considering the first order axiomatization of a theory, we should carefully consider not only its class of models, but also the class of morphisms between these models. Dealing with the theory of commutative semi-rings with no zero-divisors, the notion of additive inverse has a complexity which exceeds that of $+$ and \cdot . We can detect this by noticing that the class of morphisms between these structures shrinks when we impose the preservation of this operation. On the other hand, in the context of commutative rings, the notion of additive inverse has the same complexity of $+$, \cdot . But, again, this is not the case for the notion of multiplicative inverse: we need to focus on the theory of field in order to regard it as a concept with the same complexity as addition and multiplication. We can sum up these considerations as follows:

- The signature $\{+, \cdot, 0, 1\}$ is suitable for commutative (semi)rings with no zero-divisors, and fields.
- The signature $\{+, \cdot, 0, 1, -\}$ is better suited for commutative rings with no zero-divisors, and fields.
- The signature $\{+, \cdot, 0, 1, -,^{-1}\}$ is better suited (only) for fields.

The algebraic case we discussed is instructive and guides us towards more general considerations on the interplay between the complexity of concepts/notions of a theory T , the first order language τ in which to express them, and the properties of the τ -morphisms between models of T . First of all, the Lévy hierarchy gives a potentially useful hierarchy by which we can stratify the complexity of concepts/notions of a theory; furthermore our discussion suggests that for a given a signature τ , the basic concepts of a τ -theory T are not only those expressed by (boolean combinations of) atomic τ -formulae, but also those expressed by universal τ -equations (e.g., the

additive inverse is a simple operation for ring theory, being axiomatized in signature $\{+, \cdot, 0, 1, -\}$ by universal quantification on an equation, while it is not for semi-ring theory as axiomatized in signature $\{+, \cdot, 0, 1, -\}$. The reason is that if $\mathcal{M} \sqsubseteq \mathcal{N}$ are τ -structures which model the same τ -theory T , the graph of a relation/operation $R^{\mathcal{M}}$ defined on \mathcal{M} as prescribed by the universal axiom ψ_R of T is exactly the restriction to \mathcal{M} of the graph of that same relation/operation $R^{\mathcal{N}}$ defined now on \mathcal{N} as prescribed by ψ_R (in our example, $-$ is preserved by $\{+, \cdot, 0, 1, -\}$ -morphisms of rings but not by $\{+, \cdot, 0, 1, -\}$ -morphisms of semi-rings).

Summing up, when dealing with a mathematical theory T for which we have a clear picture of its intended class of models, the choice of a signature τ in which to formalize T can be made on the basis of which morphism between models of T we would like to be the τ -morphisms between the τ -models of T , or, correspondingly, on the basis of which concepts of T we would like to be of low logical complexity in the Lévy hierarchy induced by τ on T . Once this decision is made, this affects the type of signature τ in which the theory can be formalized, what are the τ -morphisms between the T -models, and what are the universal τ -axioms of T . Therefore, we can get different stratifications of the complexity of concepts for a theory T by selecting the appropriate class of morphisms between its models and by selecting which concepts of T we require to be axiomatized by universal sentences.

§4. Existentially closed models, model completeness, and model companionship. It is now the time to introduce the notion of existentially closed model and of model companionship.¹⁵ These model-theoretic notions will be the structural properties of first order theories we will use to select good signatures for set theory. Before giving the precise mathematical definitions of these notions, let us start by introducing these concepts in the context of (semi-)ring theory.

A common trait of the set of axioms (2)–(4) is that they assert the existence of solutions to certain basic equations that are expressible by atomic formulae in the $\{+, \cdot, 0, 1\}$ -theory given by axiom (1). Indeed, any commutative (semi-)ring \mathcal{M} without zero divisors can be enlarged to an algebraically closed field simply by extending it with solutions to the basic polynomial equations with parameters in \mathcal{M} (which in principle may not exist in \mathcal{M}). Notice that the process of adding new solutions to a commutative (semi-)ring can be seen as closing it with respect to its basic operations. It is here that the choice of a signature τ for the formalization of a theory T becomes relevant: the richer τ is, the more complex are the concepts expressible by atomic τ -formulae (expressing the basic operations and relations) and universal τ -sentences (expressing the basic properties of the T -definable relations and operations). Therefore, the richer τ is, the more closed-off are the models of a universal τ -theory T that have solutions for the atomic τ -formulae with parameters in the model.

This closing-off process is one of the driving forces of mathematics. For example it is by adding new solutions to basic equations that commutative semi-rings with no zero-divisors (like \mathbb{N}) have been extended to commutative rings (like \mathbb{Z} , which contains all additive inverses), and then to fields (like \mathbb{Q} , which contains all multiplicative inverses),

¹⁵ Reference texts for existentially closed models, model completeness, and model companionship are [13] and the second author's notes [43].

and finally to algebraically closed fields (like \mathbb{C} , in which all diophantine equations have a solution).

Of course, the choice of the signature τ in which a theory T is formalized affects the outcome of this closing-off process. A basic request is that the τ -theory T' describing the validities in the closed-off structures, should agree with T at least with respect to the Π_1 -consequences of T in signature τ (as the latter express the properties of the basic concepts and operations of T according to τ). Indeed, only in this case we can say that the closing-off process has been performed with respect to the basic concepts and operations of T . The notion of existentially closed model defines precisely which structures are the outcome of this closing-off process.

DEFINITION 4.1 *Let τ be a first order signature and T be a τ -theory. A τ -structure \mathcal{M} is T -ec if:*

- *There is some $\mathcal{N}\tau$ -superstructure of \mathcal{M} which models T .*
- *\mathcal{M} is a Σ_1 -substructure of \mathcal{P} whenever \mathcal{P} is a τ -superstructure of \mathcal{M} which models T .*

A caveat is in order: while a mathematical theory T can be unambiguously defined regardless of any of its possible first order axiomatizations, to define unambiguously the notion of T -ec model we commit ourselves to choosing a specific signature τ in which to formalize T .

NOTATION 4.2 *Given a signature τ and a τ -theory T , T_{\forall}^{τ} is the set of universal τ -sentences which follows from T (similarly we define T_{\exists}^{τ} , $T_{\forall\exists}^{\tau}$, etc.).*

While the set of logical consequences of T is independent of the signature in which we axiomatize it, the set T_{\forall}^{τ} heavily depends on the choice of τ .

Notice that any T -ec τ -structure is a model of T_{\forall}^{τ} . A less trivial observation is the following.

FACT 4.3 [43, fact 8] *Given a τ -theory T , a τ -structure \mathcal{M} is T -ec if and only if it is T_{\forall}^{τ} -ec. Hence a τ -structure \mathcal{M} which is T -ec is also S -ec for any τ -theory S such that $S_{\forall}^{\tau} = T_{\forall}^{\tau}$.*

It is not hard to see that algebraically closed fields, in the signature $\sigma = \{+, \cdot, 0, 1\}$, are S -ec structures for S the σ -theory of commutative (semi-)rings with no zero-divisors. Consider the two σ -theories U and R , respectively, of fields and of integral domains that are not fields. Then, any σ -structure \mathcal{M} which is an algebraically closed field is automatically S -ec, U -ec, and R -ec at the same time (since $R_{\forall}^{\sigma} = S_{\forall}^{\sigma} = U_{\forall}^{\sigma}$).

In general, given a τ -theory T , the class of T -ec models might not be elementary. In the specific case of algebraically closed fields this is the case, since this class of structures is axiomatized by the $\{+, \cdot, 0, 1\}$ -theory ACF given by axioms (1) and (4); but this is a rather peculiar case, which also depends on the specific choice of the signature. On the contrary, if we consider the $\{\cdot, e, ^{-1}\}$ -theory T of groups, then the T -ec $\{\cdot, e, ^{-1}\}$ -structures do not form an elementary class¹⁶ [13, example 3.5.16]. Abraham Robinson

¹⁶ Moreover, none of the signatures for group theory that we presented in Section 2 have existentially closed models which give rise to an elementary class.

introduced the notion of model companionship to describe exactly when the collection of T -ec τ -models form an elementary class:

DEFINITION 4.4 *Let τ be a signature.*

- A τ -theory T is model complete¹⁷ if for any τ -structure \mathcal{M}

$$\mathcal{M} \models T \text{ if and only if } \mathcal{M} \text{ is } T_{\forall}^{\tau}\text{-ec.}$$

- A τ -theory T is the model companion of a τ -theory S if:
 - (i) S and T are jointly consistent: i.e., $T_{\forall}^{\tau} = S_{\forall}^{\tau}$, (or equivalently—by [41, Lemma 2.1.1] or [13, Remark 3.5.6(2)]—every model of T embeds in a model of S , and vice versa).
 - (ii) T is model complete.

It is useful to recall that model complete theories are axiomatized by their Π_2 -consequences (see [13, Proposition 3.5.10]), and that the model companion of a τ -theory T (if it exists) is unique (by [13, Proposition 3.5.13]).

Conforming to our previous notation of R being the $\{+, \cdot, 0, 1\}$ -theory of integral domains which are not fields, and U being the $\{+, \cdot, 0, 1\}$ -theory of fields, we can observe that no algebraically closed field is a model of R , and, conversely, no model of R is an algebraically closed field. As a matter of fact, the notion of model companionship for a τ -theory T does not necessarily isolate a proper subclass of the models of T , but rather the models that are closed-off with respect to the basic operations and relations definable in T by universal τ -sentences.

Given the connection between model companions and existentially closed structures, it should be no surprise that the notion of model companionship is dependent on the choice of the signature. For example, consider the language $\{+, \cdot, 0, 1,^{-1}\}$ and the theory S' which results from adding axiom (5) to axioms (1) and (2). We obtain that S' is still the theory of commutative rings with no zero-divisors, now formalized in the signature $\{+, \cdot, 0, 1,^{-1}\}$. However, the $\{+, \cdot, 0, 1,^{-1}\}$ -theory of algebraically closed fields ACF' given by axioms (1), (4), (5) is not the model companion of the $\{+, \cdot, 0, 1,^{-1}\}$ -theory S' . In order to see this, let \mathcal{M} be the field of complex numbers and let \mathcal{N} be $\mathbb{C}[X]$ (the ring of polynomials in complex coefficients and variable X), both seen as $\{+, \cdot, 0, 1,^{-1}\}$ -structures. Then it is the case that $\mathcal{M} \sqsubseteq \mathcal{N}$, but now \mathcal{N} models the Σ_1 -sentence for $\{+, \cdot, 0, 1,^{-1}\} \exists y (y \neq 0 \wedge y^{-1} = 0)$ (as witnessed by the polynomial X), while \mathcal{M} does not. In particular, \mathcal{M} is not S' -ec, as \mathcal{M} is not a Σ_1 -substructure of \mathcal{N} , which is a model of S' .

Given that the existence of a(n absolute) model companion for a theory T depends so much on the choice of the signature, we can use the former to select the latter:

Model-theoretic criterion for selecting signatures: Given a mathematical theory T and two signatures τ and σ in which T can be axiomatized, τ is preferable to σ for T if the τ -axiomatization of T admits a(n absolute) model companion, while its σ -axiomatization does not.

¹⁷ Usually model completeness is defined by requiring that the substructure relation coincides with the elementary substructure relation [13, definition on p. 186]. Our definition is equivalent in view of [13, proposition 3.5.15] combined with [41, lemma 2.1.1].

Clearly this criterion is in the mold of (AbsCrit). Let us see with a concrete example why criteria of this kind can cohere with others that are internal to a theory. Recall the example of integral domains: in Section 3 we argued on the basis of considerations internal to this theory that the signature $\{\cdot, +, 0, 1, -\}$ is better suited for it than $\{\cdot, +, 0, 1, -,^{-1}\}$. Let us apply the above abstract criteria to select which among the two signatures is best suited for the theory of integral domains. Once again the answer is $\{\cdot, +, 0, 1, -\}$, in this case because in this signature the theory of integral domains has a natural *and simple to compute* model companion (which is the $\{\cdot, +, 0, 1, -\}$ -theory of algebraically closed fields).¹⁸ It is actually the case (at least for concrete examples) that we can prove that some theory T have a(n absolute) model companion in a signature τ only when we outline reasons internal to T of why the universal τ -theory of T is catching certain central and basic concepts of T .

A category theoretic perspective can also be useful to justify this model-theoretic criterion. We already remarked that for a mathematical theory T , its semantic extension might be given not only by its class of models, but also by its intended class of *morphisms between these models*, i.e., by a certain category \mathcal{E}_T of models and morphisms (this is the standard approach with which ring theory is presented in a basic undergraduate course in algebra). By axiomatizing T in some signature τ we univocally describe the objects of \mathcal{E}_T as the τ -models of T , but the arrows of \mathcal{E}_T may not coincide with the τ -morphisms between τ -models of T (consider again the case of integral domains with the two signatures $\{\cdot, +, 0, 1, -\}$, $\{\cdot, +, 0, 1, -,^{-1}\}$). How do we detect the most natural family of morphisms between models of a mathematical theory T (i.e., the arrows of \mathcal{E}_T)? Can we even say that there is always a most natural family of morphisms between models of T (e.g., that \mathcal{E}_T can be meaningfully and univocally defined for any mathematical theory T)? Our model theoretic criterion to select signatures is a useful tool to address these questions.

§5. Partial Morleyizations, absolute model companionship, and the AMC-spectrum of a theory. We can now give a more formal treatment of the ideas presented in the previous sections, by introducing the central concepts of (partial) Morleyization and absolute model companionship.

5.1. Partial Morleyizations.

NOTATION 5.1 *Given a signature τ , let $\phi(x_0, \dots, x_n)$ be a τ -formula.*

We let:

- R_ϕ be a new $n + 1$ -ary relation symbols,
- f_ϕ be a new n -ary function symbols,¹⁹
- c_τ be a new constant symbol.

¹⁸ On the other hand (given our limited knowledge of algebra) a model companionship result for the theory of integral domains in signature $\{\cdot, +, 0, 1, -,^{-1}\}$ is a much harder problem (of which we don't know the answer); what we can say for sure is that the eventual model companion of integral domains for $\{\cdot, +, 0, 1, -,^{-1}\}$ won't be the theory of algebraically closed fields, and that the theory of integral domains does not have an absolute model companion in the above signature.

¹⁹ As usual we won't distinguish between 0-ary function symbols with constants.

We also let

$$AX_\phi^0 := \forall \vec{x} [\phi(\vec{x}) \leftrightarrow R_\phi(\vec{x})],$$

$$AX_\phi^1 := \forall x_1, \dots, x_n \left[(\exists! y \phi(y, x_1, \dots, x_n) \rightarrow \phi(f_\phi(x_1, \dots, x_n), x_1, \dots, x_n)) \wedge (\neg \exists! y \phi(y, x_1, \dots, x_n) \rightarrow f_\phi(x_1, \dots, x_n) = c_\tau) \right]$$

for $\phi(x_0, \dots, x_n)$ having at least two free variables, and

$$AX_\phi^1 := [(\exists! y \phi(y)) \rightarrow \phi(f_\phi)] \wedge [(\neg \exists! y \phi(y)) \rightarrow c_\tau = f_\phi]$$

for $\phi(x)$ having exactly one free variable.

Let Form_τ denotes the set of τ -formulae. For $A \subseteq \text{Form}_\tau \times 2$:

- τ_A is the signature obtained by adding to τ all relation symbols R_ϕ , for $(\phi, 0) \in A$, and all function symbols f_ϕ , for $(\phi, 1) \in A$ (together with the special symbol c_τ if at least one $(\phi, 1)$ is in A).
- $T_{\tau,A}$ is the τ_A -theory having as axioms the sentences AX_ϕ^i for $(\phi, i) \in A$.

We let $\tau^* = \tau_C$ for $C = \text{Form}_\tau \times \{0\}$ and T^* be the τ^* -theory $T_{\tau,C}$.

Roughly speaking the axioms AX_ϕ^i are meant to formalize the idea that the (potentially complicated) concept formalized by ϕ according to τ becomes an elementary concept if we formalize instead it using $\tau_{\{(\phi,i)\}}$. More precisely, given a τ -theory S :

- AX_ϕ^0 transforms the concept of S formalized by the τ -formula ϕ (which could have a high Levy complexity according to τ) in a concept formalized by an atomic $\tau \cup \{R_\phi\}$ -formula according to $S + AX_\phi^0$.
- For any τ -theory S , and for $C = \text{Form}_\tau \times \{0\}$, $S^* = S + T_{\tau,C}$ is a $\tau^* = \tau_C$ -theory admitting quantifier elimination (the Morleyization of T).²⁰
- AX_ϕ^1 performs a delicate Skolemization process. The standard Skolemization introduces for each formula ϕ a Skolem term f_ϕ together with an axiom stating $\forall \vec{x} [\exists y \phi(\vec{x}, y) \leftrightarrow \phi(\vec{x}, f_\phi(\vec{x}))]$. However, this has serious drawbacks. Consider the following situation: we have a τ -structure \mathcal{M} which is such that for all $b \in \mathcal{M} \exists y \phi(b, y)$ holds, and also such that for some $a \in \mathcal{M}$, we have that $\phi(a, y)$ has more than one solution in \mathcal{M} . In this case we can find distinct functions $h_0, h_1 : \mathcal{M} \rightarrow \mathcal{M}$ such that $\forall x [\exists y \phi(x, y) \leftrightarrow \phi(x, f_\phi(x))]$ is satisfied in the expansions of \mathcal{M} to a $\tau \cup \{f_\phi\}$ -structure which interprets f_ϕ by either one of the h_i . Which one of the two expansions should we consider the natural Skolemization of \mathcal{M} for the formula ϕ ? AX_ϕ^1 rules out this source of ambiguity by saying that when $\phi(a, y)$ has more than one solution in \mathcal{M} , then $f_\phi^{\mathcal{M}}$ does not select any of them and instead behaves trivially by assigning to a the value $c_\tau^{\mathcal{M}}$. In this way, given a formula $\psi(\vec{x}, y)$, once we choose the interpretation of c_τ , any τ -structure has a unique expansion to a $\tau \cup \{f_\psi, c_\tau\}$ -structure which is a model of AX_ψ^1 and has that interpretation of c_τ .

²⁰ See [34, sec. 3.2, pp. 31–32].

Notice that with respect to the signatures expanding $\{\cdot, +, 0, 1\}$ discussed in the previous section, axiom (5) is (equivalent to) AX_ϕ^1 for $\phi(x, y)$ being $x \cdot y = 1$ (assuming c_τ is interpreted by 0). Similarly axiom (6) is (equivalent to) AX_ψ^1 for $\psi(x, y)$ being $x + y = 0$. In what follows we want to analyze what happens when the Morleyization process is performed on arbitrary subsets of $\text{Form}_\tau \times 2$.

5.2. Absolute model companionship. The following properties will bring us to introduce the notion of absolute model companionship.

- A τ -structure \mathcal{M} is T -ec if and only if it is $T_{\forall\exists}^\tau$ -ec (by [43, fact 8]).
- If T is *complete*, a T -ec structure \mathcal{M} realizes any Π_2 -sentence which is *consistent* with $T_{\forall\exists}^\tau$ (by [43, fact 12]).
- If a τ -theory S is the model companion of a τ -theory T , S is axiomatized by its Π_2 -consequences for signature τ (by Robinson’s test, e.g., [43, lemma 14]).

These results are outlining “syntactic properties” of T -ec models which reflect the “semantic fact” that T -ec models are the “closed-off” structures for T ; furthermore they outline that the axiomatization of the model companion of a *complete* theory can be detected by separately solving a consistency task for each Π_2 -sentence for τ . One might therefore wonder whether these syntactic properties can give another equivalent characterization of model companionship, i.e., if the model companion S of a τ -theory T (whenever it exists) can be axiomatized by the family of all Π_2 -sentences for τ which holds in some model of the universal and existential fragment of T . This is not always the case; it holds when T is a *complete* model companionable theory, however there is a standard counterexample: the $\{\cdot, +, 0, 1\}$ -theory Fields_0 of fields of characteristic 0, given by axioms (1)–(3) and the infinitely many axioms granting that the characteristic of its models is 0 (note that Fields_0 is not a complete theory) has as its model companion ACF_0 , which adds axiom (4) to Fields_0 (note that ACF_0 is a complete theory). Indeed, the sentence $\forall x \neg(x^2 + 1 = 0)$ is a Π_2 -sentence (actually Π_1) which is not in ACF_0 but holds in \mathbb{Q} , hence is consistent with the universal and existential fragment of Fields_0 .

Absolute model companionship rules out counterexamples of this kind to the above tentative characterization of model companionship.

NOTATION 5.2 Given a τ -theory T , $T_{\forall\exists}^\tau$ denotes the boolean combinations of universal τ -sentences which follow from τ .

Note that $T_{\forall\exists}^\tau$ may convey more information than $T_{\forall}^\tau \cup T_{\exists}^\tau$, as there could be $\psi \vee \phi$ (with ψ universal and ϕ existential) which is in the former but does not follow from the latter.

DEFINITION 5.3 [41, definition 2.2.4, lemma 2.3.4] Given a τ -theory T , we say that a τ -sentence ψ is strongly $T_{\forall\exists}^\tau$ -consistent for τ whenever $\psi + R_{\forall\exists}^\tau$ is consistent for any τ -theory R which extends T .

A τ -theory S is the absolute model companion (AMC) of a τ -theory T if it is the model companion of T and it is axiomatized by the Π_2 -sentences (for τ) which are strongly $T_{\forall\exists}^\tau$ -consistent for τ .

Equivalently (by [43, lemma 22]) T is the AMC of S if:

- $T_{\forall\exists}^\tau = S_{\forall\exists}^\tau$,
- T is model complete.

Note that a model complete τ -theory T is the model companion of T_{\forall}^τ and the AMC of $T_{\forall\exists}^\tau$.

A τ -theory S can have an AMC if, to some extent, it has already maximized the family of existential sentences which are consistent with its universal consequences. Let us elaborate more on this point. Given a τ -theory T , the family of strongly $T_{\forall\exists}^\tau$ -consistent Π_2 -sentences for τ describes a fragment of the Π_2 -sentences which hold in a T -ec τ -structure (see [41, fact 2.2.5]). An AMC describes the set-up in which the choice of τ is such that the family of strongly $T_{\forall\exists}^\tau$ -consistent Π_2 -sentences axiomatizes a model complete theory. This leads to another peculiar aspect separating model-companionship from absolute model companionship:

LEMMA 5.4 [41, lemma 1.5] *Assume S, S' are τ -theories such that S' is the AMC of S . Then for all $T \supseteq S$, $S' + T_\forall$ is the AMC of T .*

On the contrary this property fails for model companionship: the universal fragment of the theory of $(\mathbb{Q}, +, \cdot, 0, 1)$ is inconsistent with the $\{+, \cdot, 0, 1\}$ -theory of algebraically closed fields. As we already mentioned, complete first order τ -theories are model companionable if and only if they admit an AMC. Indeed, for a complete theory S , the information conveyed by S_\forall^τ and by $S_{\forall\exists}^\tau$ is the same. On the other hand we have already seen that there are non-complete τ -theories admitting a model companion but not an AMC, e.g., the $\{+, \cdot, 0, 1\}$ -theory of fields. Let us briefly analyze more in details what is the obstruction for the $\{+, \cdot, 0, 1\}$ -theory of fields of characteristic 0 to have the $\{+, \cdot, 0, 1\}$ -theory of algebraically closed fields of characteristic 0 as its AMC. The problem is that Diophantine equations can be expressed by atomic formulae of $\{+, \cdot, 0, 1\}$ and not all of them have rational solutions. On the other hand the existence of solutions for these equations does not contradict the field axioms. Note that the smallest existentially closed $\{+, \cdot, 0, 1\}$ -model \mathcal{M} for the theory of fields is given by the algebraically closed field consisting exactly of the solutions of diophantine equations. Its standard construction builds it as a direct limit of Galois extensions of the rationals each adding new solutions to polynomial equations with coefficients in \mathbb{N} but without rational solutions. By performing this construction we can preserve the field axioms and the characteristic 0 (which holds in \mathbb{Q} as well as in all the Galois extensions of \mathbb{Q} under consideration), but at each stage we may invalidate some universal sentence that is true in \mathbb{Q} (i.e., a universal sentence asserting that a certain diophantine equation does not have rational solutions). In particular, the closing-off process performed in signature $\sigma = \{+, \cdot, 0, 1\}$ which brings from the theory of fields T to that of algebraically closed fields cannot be described only by the Π_2 -sentences ψ which are strongly $T_{\forall\exists}^\tau$ -consistent for σ . Indeed, there are other Π_2 -sentences which need to be realized in an algebraically closed field and which are not strongly $T_{\forall\exists}^\tau$ -consistent for σ (i.e., the existential statements asserting the existence of solutions for diophantine equations). In particular, the $\{+, \cdot, 0, 1\}$ -theory of algebraically closed fields is the AMC only of the theory of integral domains which extends the field of algebraic numbers (i.e., which have solutions to all polynomial equations expressible in the signature $\{+, \cdot, 0, 1\}$).

On the other hand we will argue that in the case of set theory, AMC faithfully describes the closing-off process of models of set theory with respect to basic set-theoretic operations.

5.3. The (A)MC-spectra of a first order theory. Model theory has been extremely successful in classifying the complexity of a mathematical theory according to its structural properties, and has produced a variety of properties and criteria to separate the (so-called) tame mathematical theories from the (so-called) untame ones (e.g.,

o -minimality, stability, simplicity, NIP), see [8] for a philosophical appraisal of this theme. Typically a mathematical theory is considered hard to classify (and thus untame) if it can code in itself first order arithmetic. In this respect the \in -theory ZFC is untame.

We have already observed that most mathematical theories admit many different first order axiomatizations in (almost) as many distinct signatures. A common characteristic of tameness properties such as o -minimality, stability, simplicity, and NIP is that they are *signature invariant*. More precisely, if we take a τ -theory T and we consider its Morleyization T^* in the signature τ^* (see Notation 5.1), T is o -minimal (stable, simple, and NIP) if and only if so is T^* . In contrast, this is not the case for Robinson’s notion of model companionship, as we have already observed for the theory of integral domains. We instantiate the model-theoretic criterion to select signatures formulated in the last part of Section 4 by means of the following:

DEFINITION 5.5 *Let T be a τ -theory.*

- Its AMC-spectrum ($\text{spec}_{\text{AMC}}(T, \tau)$) is given by the sets $A \subseteq \text{Form}_\tau \times 2$ such that $T + T_{\tau,A}$ has an AMC (which we denote by $\text{AMC}(T, A)$).
- Its MC-spectrum ($\text{spec}_{\text{MC}}(T, \tau)$) is given by the sets $A \subseteq \text{Form}_\tau \times 2$ such that $T + T_{\tau,A}$ has a model companion (which we denote by $\text{MC}(T, A)$).

Clearly $\text{spec}_{\text{MC}}(T, \tau)$ is a superset of $\text{spec}_{\text{AMC}}(T, \tau)$ and the two can be distinct. Observe that $C = \text{Form}_\tau \times \{0\}$ is always in the AMC-spectrum of a theory T , as $T + T_{\tau,C}$ admits quantifier elimination and therefore it is model complete and its own AMC in signature $\tau_C = \tau^*$. Moreover, \emptyset is in the (A)MC-spectrum of T if and only if T has an (absolute) model companion. For $\sigma = \{+, \cdot, 0, 1\}$ and the various σ -theories considered in Section 3 we have that:

- \emptyset is in $\text{spec}_{\text{MC}}(T, \sigma) \setminus \text{spec}_{\text{AMC}}(T, \sigma)$ for T the σ -theory of commutative (semi)rings with no 0-divisors or the theory of fields.
- For ϕ the σ -formula defining the additive inverse, $\{(\phi, 1)\}$ is in $\text{spec}_{\text{MC}}(T, \sigma)$ if T is the theory of commutative rings with no 0-divisors or the theory of fields, while $\{(\phi, 1)\}$ is not in $\text{spec}_{\text{MC}}(S, \sigma)$ if S is the σ -theory of commutative semirings with no 0-divisors.
- For ψ the σ -formula defining the multiplicative inverse, $\{(\psi, 1)\}$ is in $\text{spec}_{\text{MC}}(T, \sigma)$ if T is the theory of fields, while $\{(\psi, 1)\}$ is not in $\text{spec}_{\text{MC}}(S, \sigma)$ if S is the σ -theory of commutative (semi)rings with no 0-divisors.
- It can be shown that $\text{spec}_{\text{AMC}}(T, \sigma) = \emptyset$ for T the σ -theory of commutative (semi)rings with no 0-divisors or the theory of fields.

A category theoretic perspective on the notion of (A)MC spectrum can again be enlightening: given a τ -theory T , consider the category \mathcal{C}_T given by the elementary class of its τ -models as objects and the τ -morphisms (e.g., maps which preserve the *atomic* τ -formulae) between them as arrows. By taking a subset A of $\text{Form}_\tau \times 2$ we can pass to the category $\mathcal{C}_{T+T_{\tau,A}}$ whose objects are τ_A -models of $T + T_{\tau,A}$ and whose arrows are the τ_A -morphisms (in accordance with Notation 5.1). There is a natural identification of the objects of \mathcal{C}_T and those of $\mathcal{C}_{T+T_{\tau,A}}$, but the arrows of $\mathcal{C}_{T+T_{\tau,A}}$ are now a possibly much narrower subfamily of the arrows of \mathcal{C}_T . Important structural information on a τ -theory T is given (at least from a categorical point of view) by classifying which sets A produce an elementary class of τ_A -models for which the $T + T_{\tau,A}$ -ec models constitute themselves an elementary class. The model companionship spectrum of T

gives exactly this structural information on the arrows of \mathcal{C}_T . Similar considerations apply for the AMC-spectrum of a theory. On the other hand simplicity, stability, and NIP provide fundamental structural informations on the class of objects of \mathcal{C}_T , but it is not transparent whether they also convey information on its class of arrows.

§6. The (A)MC-spectra of set theory. From now on we focus our attention on set theory. Our main goal is to study the model companions of set theory and the relevant signatures for which they exist. We will not only provide a new set of arguments in favour of $2^{\aleph_0} = \aleph_2$, but will also present a new method to detect set theoretic validities. The realist-minded readers can consider the following part of the paper as an attempt to produce a complete (first order) axiomatization of set theory. Although a realist stance towards mathematics is not needed to understand and prove the results we are going to present, it is within a realist agenda that we can better appreciate their meaning. We can make this perspective more explicit by outlining a principle and a goal that will motivate the present model-theoretic approach.

Semantic realism: The universe of all sets (V, \in) is sufficiently well-defined so that its \in -theory is a well-defined (complete and non-recursive) first order theory. Therefore, any set theoretical statement axiomatizable in first order logic receives, in (V, \in) , a well-defined truth value.

Hilbertian completeness: (V, \in) should offer a *complete* picture of what exists in set theory. That is to say, (V, \in) should contain any set whose existence is not in direct contradiction with already recognized truths of set theory (e.g., ZFC).²¹

In the context of a realist conception of sets, a consistent formalization of Hilbertian completeness would come along with a notion of *maximality* for set theory. Notice that the quest for maximality is what motivates many set-theoretical investigations; for this reason we believe it is appropriate to analyze the role that model companionship can play in its investigation; however, we defer this task to the the last section of this paper. Meanwhile, we can propose to implement Hilbertian completeness by means of a more concrete (but still partially informal) principle clearly inspired by it.

(IEC) **Informal Existential Completeness** If $P(x, Y)$ is a “simple” property formalizable in the \in -signature in parameter some set Y of V and $\exists x P(x, Y)$ is consistent with the basic principles of set theory, then in V a witness of this existential statement should exist.

Our task now consists in spelling out a precise mathematical definition of which set theoretic properties are to be regarded as “simple” and which sets Y of V we can accept as parameters for the property P (also attention must be paid to singling out what we exactly mean by “consistent with the basic principles of set theory” to avoid trivial counterexamples to (IEC)). Towards this end we can take advantage of our previous discussion about signatures and AMC.

²¹ Taken at face value, this seems like an impossible task. As a matter of fact, we can consistently add to ZFC the statement that there is a bijection between 2^{\aleph_0} and \aleph_1 , but also the statement that the continuum has size \aleph_2 . Clearly, the two statements are jointly inconsistent. For this reason, we interpret Hilbertian completeness as a motivating goal that needs further refinement in order to be consistently implemented.

(a) We determine that a property of sets P is “simple” by analyzing the peculiarity of set theory as a mathematical theory. After choosing $A \subseteq \text{Form}_{\{\in\}} \times 2$ and performing a partial Morleyization with respect to A , we can check whether $P(x, y)$ can be formalized by a quantifier-free formula in signature $\{\in\}_A$. If this is the case we consider $P(x, y)$ simple (with respect to A). We clearly need criteria for the choice of A . Indeed, if $A = \text{Form}_{\{\in\}} \times 2$, then any \in -formalizable set theoretic property becomes simple, since it is expressible by an atomic formula of $\{\in\}_A$. In order to avoid trivialities, we use set theoretic practice and forcing as a guiding tool to select the appropriate sets A and, consequently, the simple set theoretic properties. We regard a property $P(\bar{x})$ as simple if one or more of the following conditions are met:

- (i) Its \in -formalization can be expressed by a Δ_0 -formula.
- (ii) Its meaning is invariant across forcing extensions: i.e., $P(\bar{a})$ holds in V if and only if it holds in any generic extension of V .²²
- (iii) The basic truths about $P(\bar{x})$ are forcing invariant, which roughly amounts to require that, whenever $\psi(\bar{x})$ is an \in -formula formalizing $P(\bar{x})$ and $\phi(\bar{y})$ is a boolean combination of Δ_0 -formulae and ψ , then the truth value of $\forall \bar{y} \phi(\bar{y})$ cannot be changed by forcing.²³

Notice that the second and the third conditions are implied by the first. There are properties $P(x)$ which are not expressible using Δ_0 -formulae but for which nonetheless the second or third (or both) conditions apply; for example, the second condition applies to the provably Δ_1 -properties for ZFC^- and (assuming large cardinals) to the property “ x belongs to a certain universally Baire set B ”, while the third holds (again assuming large cardinals) for $P(x)$ being “ x is a stationary subset of ω_1 ” (see Theorem 7.4 and the comments following it). We can give separate motivations for why each of the above condition is a simplicity criterion: (i) is motivated by set theoretic practice (we expand on this in Section 6.1). (ii) and (iii) are simplicity criteria as they state that we cannot use forcing to change the basic truths regarding properties satisfying them. We consider forcing invariance a legitimate simplicity check, since forcing is the most efficient method to produce independence results and (by the results we aim to expose in this paper) also the only method at all, at least for a large class of statements; we expand more on this in Section 6.3. Furthermore, notice that assuming large cardinals many set theoretic concepts become forcing invariant (this is the content of generic absoluteness results). In this respect if one is eager to accept large cardinal axioms as correct set theoretical principles (as

²² By generic extension of V we mean the standard way of coding a forcing extension of the universe of set within V using, for example, a Boolean-valued construction.

²³ For example let $P(x)$ be the property x is a stationary subset of ω_1 , then $\exists x [P(x) \wedge P(\omega_1 \setminus x)]$ is a ZFC-theorem, hence its truth value cannot be changed by forcing. This is true regardless of the fact that whenever G is V -generic for a forcing collapsing ω_1^V to become countable, the witnesses of the truth of $\exists x [P(x) \wedge P(\omega_1 \setminus x)]$ in V and that of $\exists x [P(x) \wedge P(\omega_1 \setminus x)]$ in $V[G]$ cannot be the same set. Note that in an axiomatization of set theory in a signature τ which has stationarity on ω_1 formalized by a unary predicate P and has atomic predicates interpreting the Δ_0 -formulae, the above can be formalized by the Σ_1 -sentence for $\tau \exists x \exists y [P(x) \wedge P(y) \wedge R(x, y)]$ with $R(x, y)$ being the atomic τ -formula corresponding to the Δ_0 -property ($y = \cup x \setminus x$).

- we do), it is then natural to regard as “simple” all set theoretic concepts and properties which are made forcing invariant by large cardinals.
- (b) Rather than focusing on the particularly delicate task of establishing which sets Y do exist in V and which do not (in order to argue whether $\exists xP(x, Y)$ can be consistent or valid), we just investigate if the Π_2 -sentence $\forall x\exists y P(x, y)$ is consistent with the basic principles of set theory. This task can be realised by checking whether the sentence $\forall x\exists y P(x, y)$ belongs to $AMC(T, A)$, for T some \in -theory extending $ZFC + \text{large cardinals}$, and $A \subseteq \text{Form}_{\{\in\}} \times 2$ such that $P(x, y)$ is formalizable by a quantifier-free formula in signature $\{\in\}_A$. Moreover (assuming (IEC)), if $P(x, y)$ satisfies any of the above conditions for simplicity listed in (a), we can regard a Π_2 -sentence $\forall x\exists y P(x, y)$ which is consistent with the basic principles of set theory as a sentence expressing a true property of V . Indeed, by its being Π_2 , such a sentence would assert that the (suitable fragment of the) universe of sets in which it holds is closed-off with respect to the simple operations and relations described by P .

Now that we have provided insights on which mathematical criteria we use to detect “simplicity” and that we have clarified what we mean by “consistent with the basic principles of set theory,” we can give a first general outline of the main results on the model companions of set theory.

- (A) The (A)MC-spectrum of set theory can be used to characterize theories of the various H_{κ^+} as κ ranges over the infinite cardinals. More precisely, for any signature τ that extends $\{\in\}$ and includes all Δ_0 -formulae among its atomic formulae, if the τ -theory of sets admits a model companion, this model companion is a model of Extensionality, Foundation, Union, it is closed under rudimentary functions or Gödel operations, and satisfies the replacement axiom for a very large family of Σ_1 -formulae for τ . Furthermore, for any extensions T of ZFC and any infinite cardinal κ definable in T we can (uniformly in κ) cook up at least one signature τ_κ (extending $\{\in\}$ and including all Δ_0 -formulae among its atomic formulae) such that the τ_κ -theory of $H_{\kappa^+}^{\mathcal{M}}$ common to all \mathcal{M} which model T is the AMC of T . Moreover these τ_λ -theories of H_{λ^+} cohere; more precisely for $\lambda < \kappa$ both definable in T we can further assume that in any model \mathcal{M} of T the τ_λ -theory of $H_{\lambda^+}^{\mathcal{M}}$ is correctly computed in the structure $(H_{\kappa^+}^{\mathcal{M}}, \tau_\kappa^{\mathcal{M}})$, being $(H_{\lambda^+}^{\mathcal{M}}, \tau_\lambda^{\mathcal{M}})$ a definable structure in $(H_{\kappa^+}^{\mathcal{M}}, \tau_\kappa^{\mathcal{M}})$ (see Theorem 7.3 and Notation 6.3 for details).
- (B) There is (more than one, and even recursive ones) $B \subseteq \text{Form}_{\{\in\}} \times 2$ such that:
- $T_{\{\in\}, B}$ makes any Δ_0 -formula logically equivalent to some $\{\in\}_B$ -atomic formula, and \in_B has a constant symbol which is interpreted as $\omega_1^{\mathcal{M}}$ in any $\{\in\}_B$ -model \mathcal{M} of $ZFC + T_{\{\in\}, B}$.
 - All quantifier-free formulae of $\{\in\}_B$ express simple properties according to simplicity criterion (a) (assuming large cardinals). Furthermore the signature \in_B introduces relations, constants, and function symbols which denote *natural* and central concepts in the set-theoretical practice of the last 40 years.²⁴

²⁴ The reader can find in [42] a detailed introductory account of why the signatures \in_B under consideration are *natural*. A signature \in_B which works can be obtained enriching

- Whenever S is any extension of the $\{\in\}$ -theory $ZFC + \text{there are class many supercompact cardinals}$,²⁵ B in $\text{spec}_{\text{AMC}}(S, \{\in\})$.
 - Whenever S is as in the previous item, the corresponding theory $\text{AMC}(S, B)$ describes a theory of H_{ω_2} , and is axiomatized by the Π_2 -sentences (in the signature $\{\in\}_B$) which hold in the H_{ω_2} of a model of strong forcing axioms.²⁶ Among these Π_2 -sentences there are $\neg\text{CH}$ and a definable version θ of $2^{\aleph_0} = \aleph_2$.
- (C) For²⁷ any theory R extending ZFC such that $R + \neg\text{CH}$ is consistent, CH cannot be in $\text{AMC}(R, A)$ for any A in $\text{spec}_{\text{AMC}}(R, \{\in\})$ such that the Δ_0 -formulae are ($T_{\{\in\}_A}$ -equivalent to) atomic $\{\in\}_A$ -formulae.
- (D) For any²⁸ S, B as in (B), any Π_2 -sentence ψ of $\{\in\}_B$ is in $\text{AMC}(S, B)$ if and only if S proves that $\psi^{H_{\omega_2}}$ is forcible.
- (E) The model completeness results for the theory of H_{\aleph_2} of (B) mesh perfectly with certain forms of strong forcing axioms; for example one can prove that if one is in a model (V, \in) of S where the $\{\in\}_B$ -theory of $H_{\aleph_2}^V$ is model complete (hence the AMC of the \in_B -theory of V), then a strong form of bounded Martin’s maximum holds.

The resulting picture of V that emerges from (A) is that of a cumulative hierarchy whose initial segments (the H_{κ^+} ’s) are closed off with respect to more and more complex set-theoretic operations and produce a coherent step-by-step approximations of the theory of V given by the various theories of H_{λ^+} as λ ranges over the infinite cardinals.

In this way, set theory is treated algebraically like field theory: once we choose the cardinal κ and a corresponding signature τ_κ , the initial segment H_{κ^+} of the universe V has a theory which behaves, with respect to ZFC_{τ_κ} (e.g., ZFC enriched with the replacement axiom for τ_κ -formulae, see 6.1 for details), as algebraically closed fields behave with respect to field theory as axiomatized in $\{\cdot, +, 0, 1\}$.

Moreover, by (B) above, this analogy between set theory and algebra is exemplified for the cardinal \aleph_1 by a signature \in_B which renders the theory of H_{\aleph_2} the result of a closing-off process with respect to simple set-theoretical concepts (according to our simplicity criteria (a)), and to the standard set theoretic practice of the last 40 years.²⁹ Consequently, the algebraic completeness displayed by H_{\aleph_2} , as the model companion of set theory with respect to \in_B , satisfies the conditions described in (IEC)

the \in -signature with predicate symbols for the Δ_0 -formulae, function symbols for the Gödel (and/or rudimentary) operations, constants for ω and ω_1 , a predicate symbol for the non-stationary ideal, and predicate symbols for the family of sets of reals definable in $L(\text{Ord}^\omega)$ by an \in -formula without parameters; this family will give a recursive list of predicate symbols for universally Baire sets (assuming sufficiently many large cardinals exist). See also Definition 6.1 and/or [41, notations 3.1.1, 5.1, 5.3, definition 5.2].

²⁵ See Theorem 7.4 for a precise formulation of the (as of now) optimal large cardinal assumptions.

²⁶ For example strong forms of Woodin’s axiom $(*)$ or MM^{++} .

²⁷ In this item (C) we can replace CH by an \in -sentence θ which is formalized by a Π_2 -sentence for \in_B and is such that $\text{ZFC} + \theta$ decides $2^{\aleph_0} = \aleph_2$ (see Section 7.5 for details).

²⁸ Item (D) applies to the Π_2 -sentences $\neg\text{CH}$ and θ considered in (C).

²⁹ Universally Baire sets and the non-stationary ideal are regarded as tame concepts (at least in the presence of large cardinals), and Δ_0 -properties are certainly considered elementary set-theoretic concepts (see Section 6.1).

and substantiates the Hilbertian completeness encoded by it. Furthermore, (C) shows that the study of this model companion of set theory gives trustworthy information on the value of the continuum, while (D) suggests an explanation of the incredible success of the method of forcing (and of forcing axioms) in providing a consistent solution to problems formalizable at the level of H_{\aleph_2} as Π_2 -sentences for \in_B . Finally condition (E) shows that the present model-theoretic approach and the use of forcing axioms are two sides of the same coin. This, in turn, can be seen as a further justification for extending the axioms of set theory with forcing axioms and, consequently, a justification of Gödel’s program at least at the level of (a large fragment of) third order arithmetic.

6.1. What is the right signature for set theory? The standard axioms of ZFC in the \in -signature are clearly sufficient to provide a first order axiomatization of set theory. However a closer inspection reveals that many simple set-theoretic concepts are not formalized by simple \in -formulae.

Consider, for example, the notion of ordered pair. While we informally write $x = \langle y, z \rangle$ to mean that x is the ordered pair with first component y and second component z , in set theoretic terms this statement hides a non-trivial coding of the concept of ordered pair (for example) by means of Kuratowski’s definition: $x = \{\{y\}, \{y, z\}\}$. A proper definition of the concept of ordered pair in the \in -signature can then be given by the following \in -formula:

$$\begin{aligned} \exists t \exists u [\forall w (w \in x \leftrightarrow w = t \vee w = u) \\ \wedge \forall v (v \in t \leftrightarrow v = y) \wedge \forall v (v \in u \leftrightarrow v = y \vee v = z)]. \end{aligned}$$

It is clear that the meaning of this \in -formula is hardly recognizable with a rapid glance (unlike $x = \langle y, z \rangle$). Moreover, from a purely logical perspective, its Lévy complexity is already Σ_2 . This clashes with our understanding that the concept of ordered pair is simple. Indeed, we do not regard the notion of ordered pair as a complex concept, contrary to other more complicated and theoretically loaded ones like that of uncountability, or many of the properties of the continuum (such as its correct place in the hierarchy of uncountable cardinals). In a similar vein other very basic notions such as being a function, a binary relation, or the domain or the range of a function are formalized by rather complicated \in -formulae, both from the point of view of their readability and of their Lévy complexity.

The standard solution adopted in set theory textbooks³⁰ is to regard as basic all those \in -formulae in which the quantifiers are bounded to range over the elements of some set, that is the Δ_0 -formulae. In order to make these observations precise we need to be extremely cautious on our notational conventions.

NOTATION 6.1 For any $A \subseteq \text{Form}_{\{\in\}} \times 2$ we write \in_A rather than $\{\in\}_A$, and we let T_{\in_A} be the \in_A -theory

$$T_{\{\in\}_A} + \forall x [(\forall y y \notin x) \leftrightarrow c_{\{\in\}} = x],$$

where the theory $T_{\{\in\}_A}$ (according to Notation 5.1 for $\{\in\}$ and A) is reinforced by an axiom asserting that the interpretation of the constant symbol $c_{\{\in\}}$ is the empty set.

³⁰ See, for example, [25, chap. IV, definition 3.5] or [22, definition 12.9].

We use the abbreviations \in_{Δ_0} and T_{Δ_0} to denote what, according to the above conventions, should rather be slight extensions of³¹ $\in_{\Delta_0 \times \{0\}}$ and $T_{\in_{\Delta_0 \times \{0\}}}$.

For any $\tau \supseteq \in_{\Delta_0}$ we also let ZFC_τ denote $ZFC + T_{\Delta_0}$ enriched with the replacement axiom³² for all τ -formulae, and ZFC_τ^- denote ZFC_τ without the powerset axiom. ZFC_{Δ_0} denotes ZFC_τ for τ being \in_{Δ_0} , and similarly we define $ZFC_{\Delta_0}^-$.

The reason why set-theoretic practice regards the concepts expressed by quantifier-free formulae of \in_{Δ_0} as “simple” is the fact that the truth value of these formulae is invariant among transitive models of large enough fragments of ZF (e.g., [25, corollary IV.3.6]), and thus also forcing invariant (e.g., [22, lemma 14.21]). Furthermore \in_{Δ_0} is a signature which allows one to formalize many fundamental set theoretic concepts using formulae whose Lévy complexity is in accordance to our intuitive understanding (e.g., [22, chap. 13, lemma 13.10]).

For example consider the following notions and their logical complexity:

- $(x \text{ is a cardinal})$ is the Π_1 -formula (for \in_{Δ_0})

$$(x \text{ is an ordinal}) \wedge \forall f [(f \text{ is a function} \wedge \text{dom}(f) \in x) \rightarrow \text{ran}(f) \neq x].$$

- $(x \text{ is } \aleph_1)$ is the boolean combination of Σ_1 -formulae

$$(x \text{ is a cardinal}) \wedge (\omega \in x) \wedge \wedge \exists F [(F : \omega \times x \rightarrow x) \wedge \forall \alpha \in x (F \upharpoonright \omega \times \{\alpha\} \text{ is a surjection on } \alpha)].$$

- CH is the Σ_2 -sentence

$$\exists f [(f \text{ is a function} \wedge \text{dom}(f) \text{ is } \aleph_1) \wedge \forall r \subseteq \omega (r \in \text{ran}(f))]$$

and \neg CH is the boolean combination of Π_2 -sentences³³

$$\exists x (x \text{ is } \aleph_1) \wedge \forall f [(\text{dom}(f) \text{ is } \aleph_1 \wedge f \text{ is a function}) \rightarrow \exists r \subseteq \omega (r \notin \text{ran}(f))].$$

- $(x \text{ is } \aleph_2)$ is the Σ_2 -formula

$$(x \text{ is a cardinal}) \wedge$$

$$\wedge \exists F \exists y [(y \text{ is } \aleph_1) \wedge (y \in x) \wedge (F : y \times x \rightarrow x) \wedge \forall \alpha \in x (F \upharpoonright y \times \{\alpha\} \text{ is a surjection on } \alpha)].$$

³¹ The precise definition of \in_{Δ_0} and T_{Δ_0} can be found in [41, notation 3.1.1]. They are obtained by enriching $\in_{\Delta_0 \times \{0\}}$ and $T_{\in_{\Delta_0 \times \{0\}}}$ with a constant symbol for ω and function symbols for the Gödel operations (as defined in [22, definition 13.6]) and axioms to interpret them correctly.

³² We use the following strong form of the replacement axiom: whenever $R \subseteq X \times V$ is a definable class with domain X being a set, there is a uniformizing function $f : X \rightarrow V$ with f a set and $R(x, f(x))$ holding for all $x \in X$. This strong form of replacement holds in \in -models (V, \in) of ZFC and is inherited by the substructures of the form (H_λ, \in) for λ a regular cardinal of V . The resulting weak set theory one obtains when dropping the powerset axiom does not have the pathological models which can have other formalizations of set theory without powerset axiom where replacement is expressed by a weaker axiom schema (see, for example, [16]).

³³ We let \neg CH include the Σ_2 -sentence $\exists x (x \text{ is } \aleph_1)$, for otherwise its failure could be witnessed by the assertion that there is no uncountable cardinal, a statement which holds true in H_{ω_1} , regardless of whether CH or its negation is true in the corresponding universe of sets.

- $2^{\aleph_0} > \aleph_2$ is the boolean combination of Π_2 -sentences

$$\exists x (x \text{ is } \aleph_2) \wedge \forall f [(f \text{ is a function} \wedge \text{dom}(f) \text{ is } \aleph_2) \rightarrow \exists r (r \subseteq \omega \wedge r \notin \text{ran}(f))].$$
- $2^{\aleph_0} \leq \aleph_2$ is the Σ_2 -sentence

$$\exists f [(f \text{ is a function}) \wedge \text{dom}(f) \text{ is } \aleph_2 \wedge \forall r (r \subseteq \omega \rightarrow r \in \text{ran}(f))].$$

Let us also introduce the notation we will use to handle the substructure relation over expanded signatures. The following conventions supplement Notation 5.1.

NOTATION 6.2 *Given τ -structures \mathcal{M}, \mathcal{N} , $\mathcal{M} \sqsubseteq \mathcal{N}$ denotes the substructure relation, $\mathcal{M} \prec_n \mathcal{N}$ denotes the Σ_n -substructure relation, and $\mathcal{M} \prec \mathcal{N}$ denotes the elementary substructure relation.*

NOTATION 6.3 *Let $\tau \supseteq \in_{\Delta_0} \cup \{\kappa\}$ be a signature with κ a constant symbol, (M, τ^M) a τ -structure, and B a subset of $\text{Form}_\tau \times 2$. Then, (M, τ_B^M) is the unique extension of (M, τ) defined in accordance with Notations 5.1 which satisfies $T_{\tau, B}$. In particular, (M, τ_B^M) is a shorthand for $(M, S^M : S \in \tau_B)$. If (N, τ^N) is a substructure of (M, τ^M) we also write (N, τ_B^M) as a shorthand for $(N, S^M \upharpoonright N : S \in \tau_B)$.*

6.2. Lévy absoluteness and the possible AMCs of set theory. Recall that for an infinite cardinal λ , H_λ is the initial transitive fragment of V given by those sets whose transitive closure has size less than λ and is by itself a set in V (see [25, sec. IV.6]). We have the following results.

Stratification of V : The universe of sets V can be stratified as the union, along the class of infinite cardinals κ , of the sets H_{κ^+} .

Standard theory of H_λ : [25, theorem IV.6.5] For any uncountable cardinal λ , $(H_\lambda, \in_{\Delta_0}^V)$ is a model of $\text{ZFC}_{\Delta_0}^-$ (recall Notation 6.1).

Strong Lévy absoluteness: [38, lemma 5.3] For all $A_i \subseteq \mathcal{P}(\kappa)^{n_i}$ for $i = 1, \dots, k$,

$$(H_{\kappa^+}^V, \in_{\Delta_0}^V, A_1, \dots, A_k) \prec_1 (V, \in_{\Delta_0}^V, A_1, \dots, A_k).$$

Second order characterization of H_{κ^+} : Whenever \mathcal{M} is an $\in_{\Delta_0} \cup \{\kappa\}$ -model of $\text{ZFC}_{\Delta_0} + \kappa$ is an infinite cardinal $(H_{\kappa^+}^{\mathcal{M}}, \in_{\Delta_0}^{\mathcal{M}}, \kappa^{\mathcal{M}})$ is a model of the Π_2 -sentence for $\in_{\Delta_0} \cup \{\kappa\}$

$$\forall x \exists f (f : \kappa \rightarrow x \text{ is a surjection}). \tag{7}$$

Furthermore $H_{\kappa^+}^V$ can be described as the unique set $M \in V$ such that:

- $\kappa \in M$ and M is transitive;
- $(M, \in_{\Delta_0}^V)$ satisfies (7) and $\text{ZFC}_{\Delta_0}^-$;
- M has the (second order property) that for all $a \in M$ and $b \subseteq a$, $b \in M$ as well.

In particular, the theory of each H_{κ^+} offers a Σ_1 -approximation of the theory of V with respect to a signatures τ_κ which extends $\in_{\Delta_0} \cup \{\kappa\}$ and contains predicates for subsets of $\mathcal{P}(\kappa)^n$ for some $n \in \mathbb{N}$. Consequently, if we want to study the Σ_1 -properties of the reals (second order arithmetic), or the powerset of the reals (third order arithmetic) it is sufficient to study them within the theory of some H_{κ^+} for κ a large enough regular cardinal (and for most purposes H_{\aleph_1} suffices for second order arithmetic, and H_{\aleph_2} for third-order arithmetic). Moreover, by Lévy absoluteness, H_{κ^+}

and V agree on the computation of many simple set theoretic operations and relations: for example the operations and relations formalizable by means of Δ_0 -formulae and Δ_0 -definable Skolem functions.

The above results entail that whenever T is a τ -theory that extends $ZFC_{\Delta_0} + \kappa$ is a cardinal, τ is a signature that contains $\in_{\Delta_0} \cup \{\kappa\}$ and is contained in

$$\in_{\Delta_0} \cup \{A : A \subseteq \mathcal{P}(\kappa)^n, n \in \mathbb{N}\},$$

and (V, τ^V) models T , then

$$(H_{\kappa^+}^V, \tau^V) \prec_1 (V, \tau^V).$$

Hence (H_{κ^+}, τ^V) witnesses that (7) is consistent with the universal fragment of the τ -theory of V . In particular, if T has an AMC for τ , say S , then (7) must be in S , since it is a Π_2 -sentence for τ which is strongly $T_{\forall\exists}^\tau$ -consistent.

We can now collect all these observations and conclude the following.

- (1) The structure of V is uniquely determined by the structure of the various H_{κ^+} as κ ranges among the infinite cardinals.
- (2) Given an infinite cardinal κ , consider a signature τ extending \in_{Δ_0} only with predicates for subsets of $\mathcal{P}(\kappa)$ and a constant symbol for κ . Then, if some $T \supseteq ZFC_\tau$ has an AMC, this AMC looks like a theory of H_{κ^+} and is axiomatized by the Π_2 -sentences for τ which are strongly $T_{\forall\exists}^\tau$ -consistent.

Now assume T is some \in -theory $T \supseteq ZFC + \text{large cardinals}$ and $A \subseteq \text{Form}_\in \times 2$ is in $\text{spec}_{\text{AMC}}(T, \{\in\})$. Assume further that:

- For some constant symbol κ of \in_A

$$ZFC^- + \forall X \exists f (f : \kappa \rightarrow X \text{ is a surjection})$$

is in $\text{AMC}(T, A)$.

- Every atomic formula of \in_A satisfies at least one of the simplicity criteria set forth in (a).

Then (on the basis of (IEC)) we should conclude that $\text{AMC}(T, A)$ describes the correct theory of H_{κ^+} , since criterion (B) is satisfied by all axioms of $\text{AMC}(T, A)$.

6.3. Existentially closed fragments of the set theoretic universe versus AMC. We conclude this section by giving a non-exhaustive list of results showing that, for certain infinite cardinals κ , the corresponding H_{κ^+} already witnesses some important existential closure properties (e.g., Lévy absoluteness, Shoenfield’s absoluteness, BMM, and BMM^{++}) and its theory has some traits which are peculiar of model complete theories (e.g., Woodin’s absoluteness, MM^{+++}). The reader does not need to be familiar with these results, as they only serve as motivation to bring our focus on existentially closed models for set theory.³⁴

Lévy absoluteness: [38, lemma 5.3] Whenever λ is a regular uncountable cardinal,

$$(H_\lambda, \in_{\Delta_0}^V) \prec_1 (V, \in_{\Delta_0}^V).$$

³⁴ For further examples of existentially closed models of set theory, see [37].

Shoenfield’s absoluteness: (see [40, lemma 1.2] for the apparently weaker formulation we give here.) Whenever G is V -generic for some forcing notion in V ,

$$(H_{\omega_1}, \in_{\Delta_0}^V) \prec_1 (V[G], \in_{\Delta_0}^{V[G]}).$$

Woodin’s absoluteness: (see [40, lemma 3.2] for the weak form of Woodin’s result we give here.) Whenever G is V -generic for some forcing notion in V (and there are class many Woodin cardinals in V),

$$(H_{\omega_1}^V, \in_{\Delta_0}^V) \prec (H_{\omega_1}^{V[G]}, \in_{\Delta_0}^{V[G]}).$$

Bounded Martin’s Maximum: (BMM) [6] Whenever G is V -generic for some stationary set preserving forcing notion in V ,

$$(H_{\omega_2}, \in_{\Delta_0}^V) \prec_1 (V[G], \in_{\Delta_0}^{V[G]}).$$

BMM⁺⁺: [47, definition 10.91] Whenever G is V -generic for some stationary set preserving forcing notion in V ,

$$(H_{\omega_2}, \in_{\Delta_0}^V, \mathbf{NS}_{\omega_1}^V) \prec_1 (V[G], \in_{\Delta_0}^{V[G]}, \mathbf{NS}_{\omega_1}^{V[G]}),$$

where \mathbf{NS}_{ω_1} is a unary predicate symbol interpreted by the non-stationary ideal on ω_1 .

Bounded category forcing axioms, MM^{+++} , $\text{RA}_\omega(\text{SSP})$ [4, 5, 39] Whenever V and $V[G]$ are models of MM^{+++} ($\text{RA}_\omega(\text{SSP})$, $\text{BCFA}(\text{SSP})$) and G is V -generic for some stationary set preserving forcing notion in V ,

$$(H_{\omega_2}^V, \in_{\Delta_0}^V) \prec (H_{\omega_2}^{V[G]}, \in_{\Delta_0}^{V[G]}).$$

Woodin’s axiom: (*) (see [26, definition 7.5], or [3] for the formulation given here.) Whenever there are class many Woodin cardinals, \mathbf{NS}_{ω_1} is a precipitous ideal, and G is V -generic for some stationary set preserving forcing notion in V

$$\begin{aligned} & (H_{\omega_2}, \in_{\Delta_0}, \mathbf{NS}_{\omega_1}, A : A \in \mathcal{P}(\mathcal{P}(\omega))^{L(\mathbb{R})}) \\ & \prec_1 (V[G], \in_{\Delta_0}^{V[G]}, \mathbf{NS}_{\omega_1}^{V[G]}, A^{V[G]} : A \in \mathcal{P}(\mathcal{P}(\omega))^{L(\mathbb{R})}) \end{aligned}$$

(where $A^{V[G]}$ is the canonical interpretation in $L(\mathbb{R})^{V[G]}$ of the definition of A in $L(\mathbb{R})$).

§7. Main results. We can now present results culminating the discussion of the previous sections. The proofs can be found in [41].

NOTATION 7.1 For $A \subseteq \text{Form}_\in \times 2$, ZFC_A denotes the \in_A -theory ZFC_{\in_A} defined in Notation 6.1. Similarly we define ZFC_A^- .

DEFINITION 7.2 Let $T \supseteq \text{ZFC}^-$ be an \in -theory. κ is a T -definable cardinal if for some \in -formula $\phi_\kappa(x)$, T proves:

- $\exists! x \phi_\kappa(x)$ and
- $\forall x [\phi_\kappa(x) \rightarrow (x \text{ is a cardinal})].$
- κ is the constant f_{ϕ_κ} existing in the signature $\in_{\{(\phi_\kappa, 1)\}}$.

7.1. The AMC of set theory are theories of H_{κ^+} . The first result shows that the AMC spectrum of set theory isolates a rich set of theories which produce models of ZFC^- , that is, structures which behave like an H_λ for some regular uncountable cardinal λ .

THEOREM 7.3 *Let R be an \in -theory extending ZFC.*

- (i) *Assume $A \in \text{spec}_{MC}(R, \in)$ and $\in_A \supseteq \in_{\Delta_0}$. Then any model of $MC(R, A)$ models extensionality, foundation, choice, is closed under Gödel operations, and satisfies the replacement axiom for any Σ_1 -formula $\psi(x, y)$ of \in_{Δ_0} such that $R + T_{\in, A}$ models $\forall x \exists y \psi(x, y)$.*
- (ii) *Assume $A \in \text{spec}_{MC}(R, \in)$, $\in_A \supseteq \in_{\Delta_0}$, and λ is an R -definable cardinal represented by a constant symbol of \in_A and such that³⁵*

$$(H_{\lambda^+}^{\mathcal{M}}, \in_A^{\mathcal{M}}) \prec_1 \mathcal{M},$$

whenever \mathcal{M} models $R + T_{\in, A}$. Then $\forall x \exists f (f : \lambda \rightarrow x \text{ is a surjection})$ is in $MC(R, A)$.

- (iii) *Assume λ is an R -definable cardinal.³⁶ Then there exists $A_\lambda \in \text{spec}_{AMC}(R, \in)$ with \in_{A_λ} containing \in_{Δ_0} such that $AMC(R, A_\lambda)$ is given by the \in_{A_λ} -theory common to the structures $H_{\lambda^+}^{\mathcal{M}}$ as \mathcal{M} ranges among the \in_{A_λ} -models of $R + T_{\in, A_\lambda}$.*

7.2. Forcibility versus absolute model companionship. The following is the major result relating AMC to forcibility and to forcing axioms:³⁷

THEOREM 7.4 *Let S be the \in -theory*

$$ZFC + \text{there exist class many supercompact cardinals.}$$

Then there is a recursive set³⁸ $B \in \text{spec}_{AMC}(S, \in)$ with \in_B containing \in_{Δ_0} , and such that for any Π_2 -sentence ψ for \in_B and any \in -theory $R \supseteq S$ the following are equivalent:

- (a) $\psi \in AMC(R, B)$;
- (b) $(R + T_{\in, B})_{\forall \forall \exists}^{\in_B} + S + MM^{++} + T_{\in, B}$ proves³⁹ $\psi^{H_{\omega_2}}$;
- (c) R proves that $\psi^{H_{\omega_2}}$ is forcible⁴⁰ by a stationary set preserving forcing;
- (d) R proves that $\psi^{H_{\omega_2}}$ is forcible by some forcing;
- (e) For any $R' \supseteq R$, $\psi + (R' + T_{\in, B})_{\forall \forall \exists}^{\in_B}$ is consistent.

³⁵ $H_{\lambda^+}^{\mathcal{M}}$ denotes the substructure of \mathcal{M} whose extension is given by the formula defining H_{λ^+} in the model (using the parameter λ).

³⁶ Furthermore we can uniformly choose the sets A_λ so that in any model \mathcal{M} of R and for κ, λ definable R -cardinals such that \mathcal{M} models $\kappa > \lambda$, $(H_{\lambda^+}^{\mathcal{M}}, \in_{A_\lambda}^{\mathcal{M}})$ is a definable structure in $(H_{\kappa^+}^{\mathcal{M}}, \in_{A_\kappa}^{\mathcal{M}})$ (more precisely, for each finite $\sigma \subseteq \in_{A_\lambda}$, $(H_{\lambda^+}^{\mathcal{M}}, \sigma^{\mathcal{M}})$ is definable in $(H_{\kappa^+}^{\mathcal{M}}, \in_{A_\kappa}^{\mathcal{M}})$).

³⁷ The reader unaware of what MM^{++} or a stationary set preserving forcing is can skip the second and third items of the theorem.

³⁸ \in_B can be for example $\in_{NS_{\omega_1, \mathcal{A}}}$ as defined in [41, notation 5.3] where \mathcal{A} consists of the sets of reals lightface definable in $L(\text{Ord}^\omega)$; see also (B) on page (B).

³⁹ Here and elsewhere we write ψ^N to denote the relativization of ψ to a definable class (or set) N ; see [25, definition IV.2.1] for details. Recall that for a τ -theory $SS_{\forall \forall \exists}^\tau$ denotes the boolean combinations of universal τ -sentences which follow from S .

⁴⁰ Here and in the next item we mean that the \in -formula θ which is $T_{\in, B}$ -equivalent to ψ is such that $\theta^{H_{\omega_2}}$ is forcible by the appropriate forcing.

Furthermore for any θ which is a boolean combination of Π_1 -sentences for \in_B and any (V, \in) model of S , TFAE:

- (A) (V, \in_B^V) models θ ;
- (B) (V, \in_B^V) models that some forcing notion P forces θ ;
- (C) (V, \in_B^V) models that all forcing notions P force θ .

The equivalence of (a) and (b) shows that the \in_B -theory of H_{\aleph_2} is model complete in models of strong forcing axioms; we give a partial converse in Theorem 7.9.

The second part of the theorem shows that forcing cannot change the Π_1 -fragment of the theory of V in signature $\in_B \supseteq \in_{\Delta_0}$. Moreover, a direct consequence of the above result is that if (V, \in) is a model of S and S^* is the \in_B -theory of its unique extension to a model of $T_{\in, B}$, we get that a Π_2 -sentence ψ for \in_B is consistent with the universal fragment of S^* if and only if $\psi^{H_{\omega_2}}$ is forcible over V .

7.3. The AMC-spectrum of set theory and the continuum problem. The study of the AMC spectrum of set theory places \aleph_2 in a very special position among the possible values of the continuum.

THEOREM 7.5 *The following holds:*

- (1) Let $R \supseteq \text{ZFC}$ be an \in -theory. Assume $A \in \text{spec}_{\text{AMC}}(R, \in)$ is such that \in_A contains \in_{Δ_0} and $\neg\text{CH} + (R + T_{\in, A})_{\forall\exists}^{\in_A}$ is consistent. Then $\text{CH} \notin \text{AMC}(R, A)$.
- (2) For the signature \in_B and the theory S of Theorem 7.4 $\neg\text{CH}$ is in $\text{AMC}(R, B)$ for any \in -theory $R \supseteq S$.

Let us briefly argue why CH cannot be in $\text{AMC}(R, A)$ whenever $R + \neg\text{CH}$ is consistent.

Proof We use the following peculiar property of AMC (which can fail for model companionship⁴¹):

FACT 7.6 [41, proposition 2.2.8, definition 2.2.4, lemma 2.3.4] *Let T, S be τ -theories with T the AMC of S , and ψ be a Π_2 -sentence such that $S + \psi$ is consistent. Then there is a model of $T + \psi$. □*

Recall that CH is the conjunction of two Σ_2 -sentences (for \in_{Δ_0}): ψ_0 stating “the first uncountable cardinal exists” and ψ_1 stating “there is a surjective function with domain the first uncountable cardinal and range the reals,” while $\neg\text{CH}$ is the conjunction of ψ_0 with the Π_2 -sentence $\neg\psi_1$.

Now if $\psi_0 \notin \text{AMC}(R, A)$, $\text{CH} \notin \text{AMC}(R, A)$ as well. Otherwise $R + \neg\text{CH}$ is consistent by assumption, hence so is $R + T_{\in, A} + \psi_0 \wedge \neg\psi_1$, since the latter is a conservative extension of the former. By the above, we can find a model \mathcal{M} of $\text{AMC}(R, A) + \neg\psi_1$. This model witnesses that $\text{AMC}(R, A)$ cannot prove CH , even when $\psi_0 \in \text{AMC}(R, A)$.

Let us also argue why $\neg\text{CH}$ falls in the AMC of $R + T_{\in, B}$ for the set $B \subseteq \text{Form}_{\in} \times 2$ and the theory R of (2) above. Towards this aim we appeal to the equivalence between (A) and (C) of Theorem 7.4 (which gives a precise instantiation of simplicity criterion (iii)). First of all notice that an \in -theory R is complete if and only if so is $R + T_{\in, B}$. Let R_0 be some \in -completion of $R \supseteq S$ and (V, \in) a model of R_0 . We now note that

⁴¹ For example no algebraically closed field can be a model of the Π_2 -sentence (actually Π_1) $\forall x (\neg x^2 + 1 = 0)$ (which is consistent with the theory of fields, as it holds in \mathbb{Q}).

\in_B has a constant κ to denote ω_1 , $\text{AMC}(S, B)$ satisfies the $\in_{\Delta_0} \cup \{\kappa\}$ -sentence “ κ is the first uncountable cardinal”, $\text{AMC}(R_0, B) = \text{AMC}(S, B) + (R_0 + T_{\in_B})_{\forall\forall\exists}$ (by Lemma 5.4). Let B be a cba such that $\llbracket \neg\text{CH} \rrbracket_B = 1_B$ holds in (V, \in) .⁴² Let G be any ultrafilter on B . Then

$$V^B/G \models \forall f (f \text{ is a function with domain } \kappa \rightarrow \exists r \subseteq \omega \text{ which is not in its range})$$

(by [22, lemma 14.14]) and V^B/G also models the universal and existential \in_B -fragment of $R + T_{\in_B}$ (by (A) \Leftrightarrow (C) of Theorem 7.4, and [22, lemma 14.14]). In particular, the Π_2 -conjunct of $\neg\text{CH}$ is consistent with the universal and existential fragment of any completion of $S + T_{\in_B}$, hence it belongs to the AMC of $S + T_{\in_B}$, while the Σ_2 -conjunct of $\neg\text{CH}$ is in the AMC of $S + T_{\in_B}$ since the axiom Ax^1_ψ for ψ defining the first uncountable cardinal is in T_{\in_B} .

In particular, we see that forcing becomes a powerful tool to prove that a Π_2 -sentence formalizable in \in_B is in the AMC of $S + T_{\in_B}$. Indeed, it suffices to prove over S that this sentence is forcible. More generally minimal variants of the above argument give:

REMARK 7.7 *Let S be the theory of Theorem 7.4, and $R_0 \supseteq \text{ZFC}$ be such that $R_0 \cup S$ is consistent. Consider an \in -sentence ψ such that ψ as formulated in signature \in_B is $R_0 + T_{\in_B}$ -equivalent to a Π_2 -sentence for \in_B such that R_0 proves that $\psi^{\aleph_{\aleph_2}}$ is forcible. Then:*

- $\psi \in \text{AMC}(R, B)$ for any $R \supseteq S \cup R_0$;
- $\neg\psi \notin \text{AMC}(T, A)$ for any A and \in -theory $T \supseteq R_0$ such that \in_A contains all the symbols of \in_B appearing in ψ , and $(T + T_{\in_A})_{\forall\forall\exists} + \psi$ is consistent.

This shows that the argument described for CH is far more general and can be used to single out many other set theoretic properties. We can apply it for example to $2^{\aleph_0} > \aleph_2$: Moore introduced in [32] a Π_2 -sentence θ_{Moore} for \in_{Δ_0} to show the existence of a definable well order of the reals in type ω_2 in models of the bounded proper forcing axiom.⁴³

THEOREM 7.8 *There is a Π_2 -sentence θ_{Moore} for \in_{Δ_0} such that the following holds:*

- (1) θ_{Moore} is independent of $S + T_{\Delta_0}$, where S is the \in -theory of Theorem 7.4.
- (2) $\text{ZFC}^-_{\Delta_0} + \exists x (x \text{ is } \aleph_1) + \theta_{\text{Moore}}$ proves that there exists a well-ordering of⁴⁴ $\mathcal{P}(\omega)$ in type ω_2 .
- (3) $\text{ZFC}^-_{\Delta_0} + \exists x (x \text{ is } \aleph_2) + \theta_{\text{Moore}}$ proves⁴⁵ that $2^{\aleph_0} = \aleph_2$.

⁴² For example the boolean completion of the partial order originally devised by Cohen. See for details [22, theorem 14.32]

⁴³ We use here the definable well-ordering of the reals in type ω_2 existing in models of bounded forcing axioms isolated by Moore, but Theorem 7.8 could be proved replacing θ_{Moore} with any other coding device which produce the same effects, for example those introduced in [12, 35], or the sentence ψ_{AC} of Woodin as in [26, sec. 6].

⁴⁴ More precisely, there is a $\text{ZFC}^-_{\Delta_0}$ -provably Δ_1 -property $\psi(x, y, z)$ such that in any model \mathcal{M} of the mentioned theory there is a parameter $d \in \mathcal{M}$ such that $\psi(x, y, d)$ defines an injection of $\mathcal{P}(\omega)$ of the model with the class of ordinals of size at most ω_1 of the model.

⁴⁵ Recall that $2^{\aleph_0} = \omega_2$ is the conjunction of the Σ_2 -sentences \aleph_2 exists and there is a bijection of \aleph_2 onto the reals, in particular, we need the assumption $\exists x (x \text{ is } \aleph_2)$ to hold in a fragment of ZFC which is able to make sense of the sentence $2^{\aleph_0} = \omega_2$. This sentence fails for trivial reasons in a model of \aleph_2 does not exist even if the model has a definable well order of the

- (4) For S and \in_B the theory and signature considered in Theorem 7.4, $\exists x (x \text{ is } \aleph_1)$, θ_{Moore} are both in $\text{AMC}(R, B)$ for any \in -theory R extending S .
- (5) If R extends ZFC, $A \in \text{spec}_{\text{AMC}}(R, \in)$ is such that \in_A contains \in_{Δ_0} , $\exists x (x \text{ is } \aleph_2) \in \text{AMC}(R, A)$ and

$$\theta_{\text{Moore}} + (R + T_{\in_A})_{\forall\exists}^{\in_A} + \text{ZFC}$$

is consistent, then $2^{\aleph_0} > \aleph_2$ is not in $\text{AMC}(R, A)$.

The two theorems single out $2^{\aleph_0} = \aleph_2$ among all possible solutions of the continuum problem.⁴⁶ For any \in -theory R extending ZFC+large cardinals there is at least one $B \in \text{spec}_{\text{AMC}}(R, \in)$ with $\neg\text{CH}$ (actually with a definable version of $2^{\aleph_0} = \aleph_2$) in $\text{AMC}(R, B)$, and this occurs even if $R \models \text{CH}$ or $R \models 2^{\aleph_0} > \aleph_2$. On the other hand for any \in -theory R extending ZFC, if CH is independent of R , then CH is never in $\text{AMC}(R, A)$ for any $A \in \text{spec}_{\text{AMC}}(R, \in)$ (if \in_A contains \in_{Δ_0}) and similarly if θ_{Moore} is independent of R , $2^{\aleph_0} > \aleph_2$ is never in $\text{AMC}(R, A)$ for any $A \in \text{spec}_{\text{AMC}}(R, \in)$ (if \in_A contains \in_{Δ_0}).⁴⁷

7.4. Model completeness for the theory of H_{\aleph_2} versus bounded forcing axioms and Woodin’s axiom (*). Finally we can show that for certain signatures τ and a model (V, \in) of ZFC+large cardinals, a model completeness result for the τ -theory of $H_{\aleph_2}^V$ is equivalent to assert that V satisfies a strong form of Woodin’s axiom (*). The reader interested in the (so far) optimal result is referred to [41, theorem 5.6]. Recall the axioms BMM, BMM⁺⁺, (*) presented in Section 6.3. An immediate consequence of the definition of \in_B and of the invariance of the universal \in_B -theory of V across forcing extensions is the following result.⁴⁸

THEOREM 7.9 Assume S and B are as in Theorem 7.4 and (V, \in) is a model of S such that the \in_B -theory of $H_{\aleph_2}^V$ is model complete. Then (V, \in) models BMM⁺⁺.

Assume furthermore that NS_{ω_1} is precipitous in V . Then (V, \in) models (*).

Note that NS_{ω_1} is a precipitous ideal in V , if V models Martin’s maximum (of which BMM is a weakening). The theorem follows almost immediately from Theorem 7.4 by the observation that if G is V -generic for a stationary set preserving forcing and (V, \in) models S

$$(H_{\omega_2}, \in_B^V) \sqsubseteq (V[G], \in_B^{V[G]}).$$

§8. On maximality and justification. In this last section we focus on two philosophical aspects of the present approach. First, we address the following question: which notion of maximality is displayed by the model companions of ZFC+large cardinals? Then, we study the role of large cardinals and strong forcing axioms in

continuum in type \aleph_2 . This is the reason why this item of the theorem is distinct from the previous one.

⁴⁶ These results bring to light the role of forcing axioms in deciding the value of the continuum; an overview of the proofs of $2^{\aleph_0} = \aleph_2$ given by forcing axioms is given for example in [33].

⁴⁷ Furthermore the last part of Theorem 7.4 shows that CH, $2^{\aleph_0} = \aleph_2$, $2^{\aleph_0} > \aleph_2$, θ_{Moore} are all boolean combination of Π_2 -sentences in the signature \in_{Δ_0} which cannot be expressed by boolean combination of Π_1 -sentences for the signature $\in_B \supseteq \in_{\Delta_0}$ in models of S (with S and B as in Theorem 7.4), as their truth value can be changed by forcing.

⁴⁸ \in_B extends \in_{Δ_0} with a predicate for the non-stationary, and contains a predicate for a (lightface definable in $L(\text{Ord}^{\omega})$) set of reals which is Wadge above any set of reals in $L(\mathbb{R})$.

providing model completeness for the theory of H_{\aleph_2} . The role these axioms play in obtaining these model companionship results suggests a second question: is there a form of justification of set-theoretical axioms that emerges from the present approach? In the remainder of this section we address these two questions.

8.1. Maximality. Maximality principles play an important role in the study of the axiomatic extensions of ZFC. Maximality conveys the idea that the universe of sets is as full as it could be, while maximality principles are (first order) statements which try to capture this idea axiomatically.⁴⁹

There is no unique way to realize maximality in set theory, and its various forms depend on what in fact is maximized. Indeed, the process of maximization can be applied to objects, or possibilities.⁵⁰

- With respect to objects we find two standard forms of maximization: height maximality (expressing maximality for the lengths of the series of ordinals) and width maximality (expressing the maximality of the power-set operation). The former is successfully exemplified by large cardinals axioms [24], while the latter represents a cluster of notions often connected to the justification of forcing axioms, generic absoluteness principles [7], and forms of horizontal reflection like the Inner Model Hypothesis [1].
- For what concerns the maximization of possibilities, we find modal principles of forcing like the so-called Maximality Principle [18] (expressing the idea that everything that is possibly necessary is true) or resurrection axioms [4, 5, 20] (expressing the idea that if something is true, then it is necessarily possible). This form of maximality is closely related to forms of generic absoluteness, but it differs from it conceptually. Indeed, the justification of these modal principles often relays on potentialist views of set theory [27, 28] that are not always compatible with the actualist ones that motivate generic absoluteness principles.⁵¹

Now, *what form of maximality is displayed by the notion of model companionship?* Can it be compared to any of the above forms of set-theoretical maximality?

⁴⁹ See [21] for a complete and informative survey on the topic of maximality principles in set theory.

⁵⁰ To be faithful to the literature, we should also mention another form: the maximization of domains. This is a strategy inspired by Hilbert's (second order) axiom of completeness that was (almost informally) presented in the following terms: "The elements (points, straight lines, and planes) of geometry form a system of things that, compatibly with the other axioms, can not be extended; i.e., it is not possible to add to the system of points, straight lines, and planes another system of things in such a way that in the resulting system all the axioms I–IV and V.I are satisfied." Although this axiom was meant to provide the completeness of the real line (and consequently of the real plane), it is standard to interpret its (somewhat vague) formulation as stating, in a second order way, the maximality of the domain of interpretation of Hilbert's axioms of geometry. Contrary to the other forms of maximality, this approach did not give rise to new well-studied set theoretical principles. However, a first order axiomatic realization of this approach is given by McGee in terms of his Completeness Principle [31]. McGee's principle, like Hilbert's axiom, is sufficient to prove categoricity for the class of pure sets (in a context of a set theory with Urelemente).

⁵¹ Notice that this incompatibility only concerns the philosophical justification of these principles, but does not affect their mathematical coherence with generic absoluteness principles (which can be justified on the basis of width or height maximality).

A first step toward answering the above questions consists in analyzing whether the maximality provided by models companions concerns primarily the underlying ontology (thus suggesting a form of maximization of objects), or the class of sentences that are considered valid (as it seems the case when considering generic absoluteness principles or modal principles of forcing). In other words, does the maximality provided by model companions concerns primarily the ontology of sets, or the family of set-theoretic truths? This question does not have a clear-cut answer. As a matter of fact, we believe that model companionship results for set theory display a multifaceted character related to both forms of maximality.

For what concerns height maximality, this notion is successfully realized by large cardinal axioms. While the model companionship results for set theory do not seem to enforce this form of maximality, they can nonetheless be used to justify it (we argue on this point in the next section). On the other hand, for what concerns width maximality, notice that bounded forcing axioms (and similar principles meant to express forms of horizontal maximality) can be reformulated in terms of generic absoluteness properties for H_{ω_2} . Many of the results we presented in the previous section outlines that such generic absoluteness results come in pairs with the model completeness for the theory of H_{\aleph_2} with respect to natural signatures for this structure (e.g., Theorem 7.9 or [41, theorem 5.6]). Note that the standard formulation of bounded forcing axioms, such as BMM, BMM⁺⁺, (*), yield forms of absoluteness for the theory of H_{\aleph_2} , which are (apparently) weaker than the model completeness result given by Theorem 7.4: they only assert that (for the appropriate fragment τ of \in_B) H_{\aleph_2} is Σ_1 -elementary for τ only with respect to certain forcing extensions of V (which are τ -models of the universal τ -theory of V by the second part of Theorem 7.4). On the contrary, model completeness of the \in_B -theory of H_{\aleph_2} yields Σ_1 -elementarity for the much larger class of *all* \in_B -superstructures which satisfy the same universal \in_B -theory.

For what concerns the modal principles of forcing, we can argue as for width maximality. As a matter of fact, the Maximality Principle and resurrection axioms, as exposed in [4, 5, 18, 20] are, by all means, principles of generic absoluteness for the theory of H_{\aleph_2} (at least in the presence of mild forms of bounded forcing axioms). Indeed, for many of these principles their natural formulation asserts that the theory of H_{\aleph_2} is given by those Π_2 -sentences φ (for an appropriate fragment of \in_B) for which $\varphi \rightarrow \Box \Diamond \varphi$ holds in the (Kripke frame given by the) generic multiverse composed of initial segments of forcing extensions of V . Therefore, they describe the theory of H_{\aleph_2} with respect to the generic multiverse much like infinitely generic structures (introduced by Robinson in connection with the notion of infinite forcing) describe the theory of the existentially closed models for a first order theory. Leveraging on the model completeness results for \in_B we can actually identify the theory of H_{\aleph_2} in models of strong forcing axioms with that given by the infinitely generic structures, in signature \in_B , for the class of models of ZFC+*large cardinals*. Hence the absoluteness displayed by the model companions of set theory, relative to \in_B , largely encompasses that of the modal principles of forcing, since the latter are infinitely generic (at least on the face of their definition) only with respect to the generic multiverse and not with respect to the elementary class of the \in_B -theory of ZFC_B+*large cardinals*.⁵²

⁵² See [37] for the details on this comparison between infinite forcing and modal principles of forcing.

Finally, what is the connection between the maximization given by model companionship results and the (somewhat vague) form of completeness that is encoded in the principle that we called Hilbertian Completeness? Although our aim, similar to Hilbert's, is to implement the idea that the universe (V, \in) contains *all possible* sets, our formal way to make this idea concrete differs significantly from Hilbert's approach. Instead of focusing directly on the objects, we maximize existential sentences using our principle of Informal Existential Completeness (IEC). Thus, maximality of the objects receives in our approach a syntactic twist, being formulated in terms of maximality of solutions to simple problems (according to a syntactic measure of simplicity given by the selection of a specific first order signature).

Summing up, it seems that the form of maximality provided by (absolute) model companionship does not precisely fit with any of the standard forms of set-theoretical maximality. Nonetheless there are very strong similarities with generic absoluteness principles and certain types of forcing axioms,⁵³ and conceptual similarities with Hilbert's idea of completeness. We see the notion of model companionship as a useful tool towards a mathematical formalization of the somewhat vague ideas of width maximality and Hilbertian completeness. Our opinion is that the model companions of set theory yield a form of maximality that we may call *algebraic maximality*. The inspiration clearly comes from algebraically closed fields. We consider algebraic maximization of a theory T as a two steps process: first, one determines which are the basic, simple, concepts of the theory of interest; then, once the language is extended to a signature τ with symbols for these concepts, one analyzes the T -ec τ -models. T is *algebraically maximal* for τ in case the τ -theory of the T -ec τ -models is model complete and also when its T -ec τ -models can be seen as "parts of" the models of T . This can be formally implemented as follows:

DEFINITION 8.1 *Let T be a τ -theory and A be a subset of $\text{Form}_\tau \times 2$. T is algebraically maximal relative to A , if $A \in \text{spec}_{\text{AMC}}(T, \tau)$, and there is a τ -formula $\psi_A(x)$ such that for any $\mathcal{M} = (M, \tau^M)$ which models T :⁵⁴*

- $(\psi_A(\mathcal{M}), \tau_A^M)$ realizes $\text{AMC}(T, A)$,
- $(\psi_A(\mathcal{M}), \tau_A^M)$ is a τ_A -substructure of (M, τ_A^M) .

In terms of algebraic maximality we briefly review the results of Section 7. First of all notice that Theorem 7.3 states that for any extension T of ZFC, algebraic maximality can be obtained for T by sets A_κ such that the AMC of T in signature \in_{A_κ} describes the theory of H_{κ^+} in models of T , as κ ranges among the T -definable cardinals.⁵⁵ Moreover, the first part of Theorem 7.4 states that algebraic maximality, relative to B , holds for the theory $\text{ZFC} + \text{large cardinals} + \text{forcing axioms}$, while its second part states that \in_B satisfies the simplicity criteria set forth in (IEC) of Section 6. Finally,

⁵³ To the extent that a strong form of Woodin's axiom $(*)$ can be equivalently formulated as the assertion that for a certain signature τ the τ -theory of H_{\aleph_2} is the AMC of the τ -theory of V .

⁵⁴ $\psi_A(\mathcal{M})$ is the extension of $\psi_A(x)$ in \mathcal{M} , and τ_A^M is the unique interpretation of the predicates of τ_A in M which can make (M, τ_A^M) an expansion of \mathcal{M} which is a τ_A -model of $T_{\tau, A}$.

⁵⁵ Notice that we do not assert that the set theoretic concepts made elementary by the different A_κ 's can be considered simple according to any "reasonable" simplicity criterion for κ . We give separately below arguments showing that for $\kappa = \aleph_0$ or $\kappa = \aleph_1$, A_κ can be chosen to be "simple" according to our simplicity criteria. As of now, we do not have anything to say about the "simplicity" of \in_{A_κ} for any $\kappa > \aleph_1$.

we have that Theorem 7.9 (and its optimal strengthening given by [41, theorem 5.6]) shows that algebraic maximality for the \in_B of Theorem 7.4 entails (or is equivalent to) certain strong forcing axioms.

In conclusion we can argue that Theorems 7.4 and 7.9 show that algebraic maximality for set theory, relative to \in_B , describes the *correct* axioms for the theory of H_{\aleph_2} , as algebraic maximality for \in_B realizes (IEC) for this fragment of the universe. This naturally lead us to discuss the second topic of this section: justification.

8.2. Justification. Let us now turn to the topic of justification. When it comes to justifying new set-theoretical principles, the proposed reasons normally fall into two main categories: *intrinsic* or *extrinsic*. These forms of justification have been developed within the so-called Gödel's program:⁵⁶ a step-by-step extension of ZFC, aimed to coherently complete our picture of the universe of sets. In this sense, the present approach is clearly in the wake of Gödel's program. Indeed, the axiomatization of the model companions of set theory should, eventually, provide a description of the theory of V as the stratification of the theories of the various H_{κ^+} . Furthermore, the essential role that large cardinals and strong forms of forcing axioms play in the individuation of these model companionship results suggests a justification of the truth of these axioms in view of the nice model-theoretic properties for the theories of H_{\aleph_1} or H_{\aleph_2} they entail.

In order to better argue for this form of justification of large cardinals and forcing axioms, let us briefly revise what we mean by intrinsic and extrinsic reasons.

- A form of justification is considered intrinsic when it is based on intrinsic features of the concept of set or, derivatively (given the foundational role of set theory), on notions that are key to the whole edifice of mathematics and logic. In this sense, a new axiom is justified when it captures an essential aspect of the concept of set or when it formalizes notions that are fundamental for mathematics and logic. This form of justification is utterly conceptual, since it rests on the theoretical priority of the notion of set over its formalization. Consequently, an extension of ZFC is well justified, when it faithfully represents the correct concept of set (e.g., the iterative conception [11] or the quasi-combinatorial one [10]).
- A form of justification is considered extrinsic when we are forced to accept an axiom by the abundance of its desirable consequences; even in the absence of intrinsic reasons. In this case an axiom is justified even if it does not seem *prima facie* to capture any relevant aspect of the concept of set. This second form of justification is clearly meant to overcome the limits of intrinsic reasons and to account for an experimental methodology in the context of the foundations of mathematics. Extrinsic justifications are pragmatic in nature, since they rely on the fruitfulness of an axiom in solving new problems, shortening proofs of already known results, and unifying substantial bodies of theory. The form of justification put forward by extrinsic reasons is akin to an inference to the best explanation within mathematics. Therefore, an extrinsically justified axiom is judged by its consequence and not by its meaning.

Intrinsic and extrinsic reasons have been extensively studied in the philosophy of set theory [29, 30] and they have been widely applied, in recent debates, for the justifications

⁵⁶ Presented by Gödel in his seminal paper [17].

of competing programs [2, 23]. These two forms of justification have also been criticized for their opacity in offering clear criteria of application and for the lack of demarcation between intrinsic and extrinsic reasons [9]. Moreover, as it happened in the debate on the analytic-synthetic distinction, there is also no shortage of contributions that reject the problem of justification at its very base. Following a naturalistic account of mathematics, authors like Hamkins (or before him Cohen [14]) propose to dismiss the problem of justification, together with the problem of independence, by defending the liberty of mathematicians to study different universes of set theory and by declaring the study of the variation of truth values among the models of ZFC to be all that mathematically matters for the study of independence [19].

Now, *in which sense the nice model-theoretic properties of model companions are able to justify large cardinals and strong forcing axioms?* Is the role of the latter set-theoretical principles in the individuation of model companions of set theory able to provide an intrinsic or an extrinsic justification for them?

A first complication that we face in addressing these questions is that the present approach does not only deal with axiomatic extensions of ZFC, but also with linguistic ones: the introduction of new signatures for set theory. This peculiar aspect is responsible for a justification of large cardinals and strong forcing axioms that contains elements of both intrinsic and extrinsic arguments. Indeed, on the one hand strong forcing axioms provide a maximization of the Π_2 -statements realized over H_{\aleph_2} and, thus, have tremendously abundant consequences on third order arithmetic (and on this ground they can be extrinsically justified). On the other hand, because of the possibility to provide (generic) absoluteness results, large cardinal axioms determine the “simple” concepts that need to be included in the new signatures for set theory.⁵⁷ Therefore, large cardinals and strong forcing axioms help us understanding what are the basic concepts on which third order arithmetic should be based (and for this reason they can be intrinsically justified).⁵⁸

Another aspect of the present approach (that places the justification of new axioms somewhat outside the standard practice) is the focus on models, instead of sets (like intrinsic reasons), or problems (like extrinsic ones). As a matter of fact, large cardinals and strong forcing axioms play an important role in the construction of model companions of set theory, by detecting the natural signatures for which one has nice model-theoretic properties for (the models of) set theory. Consequently, the explicit link these axioms establish between the choice of a signature and model companionship results for set theory can also be seen as an indirect effect of these axioms in capturing certain natural features of the universe of sets. In this sense, the present approach

⁵⁷ The difficulty in choosing whether forcing axioms can be justified on intrinsic or extrinsic grounds is not new. See [36] for a discussion of the connection between forcing axioms and their intrinsic justification in terms of the clarification of the notion of arbitrary set.

⁵⁸ To be more precise, the justification of large cardinals that we are proposing only concerns those axioms of infinity that appear in the model companionship results for third order arithmetic (i.e., large cardinals in the range of supercompactness). In this sense this claim is weaker than it seems. The situation is similar to what happens when we justify Woodin cardinals on the basis of the generic absoluteness results for $L(\mathbb{R})$ (we thank one of the referees for pointing out this analogy). Our argument for justifying large cardinals on the basis of the model companionship results, however, gives an explicit and mathematically sound formulation of the intrinsic justification of large cardinals rooted on the—somewhat vague—assertion that these axioms render “simple” many “natural” set theoretic concepts.

makes evident the role that certain additional axioms of set theory have in clarifying the model theory of set theory, but also, and derivatively, the role that model-theoretic properties have in selecting what could be the natural theories for initial segments of the universe of sets. It is with respect to this double role (selecting simple concepts and providing natural theories) that our approach rooted in model companionship argues for the justifications of new axioms of set theory.

Because of the limits of the standard intrinsic–extrinsic dichotomy to capture the form of justification of this model-theoretic approach, we may call this new form of justification *algebraic*, since it rests on the model-theoretic properties of algebraically maximal theories. It is a form of justification that has both intrinsic and extrinsic elements and that heavily relies on the model-theoretic form of maximality that we described in the previous section.

In order to exemplify this algebraic form of justification, let us see how the present approach advances Gödel’s program in providing well-justified extensions of ZFC. This particular implementation of Gödel’s program is devised to provide new well-justified axioms of larger and larger initial segments of the universe V , concretely, by providing axioms able to make the theories of H_{κ^+} (as κ ranges among the infinite cardinal) the model companions of set theory for the “correct” signature aimed at describing the elementary properties of $\mathcal{P}(\kappa)$. In this way the theory of V is gradually completed by adding new principles whose justification rest on the nice model-theoretic properties they provide for the theories of the different H_{κ^+} . Let us see how this justification process can be implemented.

We start, of course, from ZFC and notice that this theory is algebraically maximal relative to \in_A , which is the signature obtained by adding to \in_{Δ_0} predicates for the lightface definable subsets of $\mathcal{P}(\omega)$ in the Chang model $L(\text{Ord}^\omega)$.⁵⁹ In this case its existentially complete models describe the theory of H_{ω_1} , and—because H_{ω_1} is a definable sub-model of V —we have that all condition of algebraic maximality are fulfilled. By [38, theorem 5.2], this theory is the AMC of any extension of ZFC for \in_A .

However the set theoretic concepts expressed by atomic \in_A -formulae or by the universal \in_A -theory of $ZFC + T_{\in_A}$ are not simple according to any of our criteria (put forward in (a) of Section 6). For example $\forall x (x \subseteq \omega \rightarrow x \in L)$ is a Π_1 -sentence for \in_A which holds in the ZFC model given by the constructible universe L and fails in $L[x]$ where x is a Cohen real over L . On the other hand, assuming as our base set theory $ZFC + \text{there are class many Woodin cardinals which are a limit of Woodin cardinals}$, the \in_A -axiomatization of this set theory satisfies our simplicity criteria. Henceforth, we should accept large cardinal axioms—among other reasons—also because they render simple (according to our criteria) the notions codified by \in_A . Consequently, on the base of this argument one can claim that the AMC of set theory enriched with large cardinal axioms, in the signature \in_A describes the correct theory of H_{\aleph_1} , and thus realizes the first step of our proposed way to implement Gödel’s program.

A similar argument can be applied to the theory of H_{\aleph_2} , by appealing to Theorem 7.4. In this case the extension of ZFC by large cardinal axioms is able to make the signature \in_B , in which Theorem 7.4 is expressed, simple according to our criteria of simplicity.

⁵⁹ Equivalently one can obtain \in_A from the signature \in_B of Theorem 7.4 removing the constant symbol for ω_1 and the predicate symbol for the non-stationary ideal on ω_1 .

Furthermore the nice model-theoretic properties of the theory of H_{\aleph_2} described by the AMC of $\text{ZFC} + \text{large cardinals}$ relative to \in_B renders $\text{ZFC} + \text{large cardinal axioms} + \text{strong forms of bounded forcing axioms}$ algebraically maximal with respect to the signature \in_B . We can thus justify forcing axioms and large cardinals as they provide (by means of algebraic maximality relative to \in_B) the correct picture of the theory of H_{\aleph_2} .

Now, starting with base theory $\text{ZFC} + \text{large cardinals} + \text{strong forms of bounded forcing axioms}$, we should search for criteria to detect which definable subsets of $\mathcal{P}(\omega_2)$ describe simple set theoretic notions, and search for the AMC of $\text{ZFC} + \text{large cardinals} + \text{strong forcing axioms}$ relative to the signature expanded by these concepts. If and when this task is achieved,⁶⁰ we can argue that the corresponding AMC provides the correct description of H_{\aleph_3} and that the relevant axioms should be added to the axioms of set theory.

In a nutshell this is the general strategy: to climb the cardinal hierarchy in order to implement algebraic maximality for set theory with respect to signatures that describe the properties of $\mathcal{P}(\kappa)$ for larger and larger cardinals κ . This will come in pairs with set-theoretic axioms that fulfill (possibly weaker and weaker) criteria of simplicity for these signatures⁶¹ (which are algebraically maximal for more and more of their initial segments). This strategy has been successful so far with respect to $H_{\aleph_1}, H_{\aleph_2}$. Theorem 7.3 shows that no obstruction to its implementation for other cardinals is in sight.⁶²

We conclude by noting that there is another sense in which these model companionship results, the justification they offer to new axioms, and the solution they yield for the continuum problem are utterly Hilbertian. This is the autonomy they provide for mathematics with respect to its foundations. As a matter of fact, the use of the notion of model companionship to fix the theory of second and third order arithmetic (and to enforce algebraic maximality) does not introduce any element that is foreign to mathematical practice. The sought solution to the continuum problem is a solution obtained with formal tools and that is justified on a purely mathematical ground. In this sense the results presented in this paper seem to realize in full the possibility of an autonomous foundation for mathematics, as the one sought by Hilbert [15].

⁶⁰ To be honest, we currently do not have any clue on how this task could be performed.

⁶¹ For example \in_A does not satisfy the simplicity criterion (ii) of page 17 relative to ZFC , while it does relative to $\text{ZFC} + \text{large cardinals}$; henceforth as our theories increase in logical strength, more and more concepts could become simple according to the criteria set forth in page 17. Note that \in_{Δ_0} would have been recognized as a natural signature codifying by atomic formulae simple set-theoretic concepts even before the discovery of forcing and the introduction of large cardinal axioms; on the other hand, \in_A would have not been considered natural before the generic absoluteness results for second order arithmetic and the clear picture large cardinals give of the theory of universally Baire sets. As it occurred for forcing invariance, it might be the case that, as we accept as valid set-theoretic truths certain set-theoretic principles, other criteria of simplicity may emerge.

⁶² When the concepts making the theory of H_{κ^+} an AMC of set theory are simple, one should expect this AMC to be “empirically” complete, i.e., to decide most “relevant” set-theoretic problems whose solution is correctly computed in H_{κ^+} . Again this “empirical completeness” holds for the AMCs describing the theories of $H_{\aleph_1}, H_{\aleph_2}$.

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BIBLIOGRAPHY

- [1] Antos, C., Barton, N., & Friedman, S.-D. (2021). Universism and extensions of V. *Review of Symbolic Logic*, **14**(1), 112–154.
- [2] Arrigoni, T., & Friedman, S.-D. (2013). The hyperuniverse program. *Bulletin of Symbolic Logic*, **19**(1), 77–96.
- [3] Asperó, D., & Schindler, R. (2021). Martin's maximum⁺⁺ implies Woodin's axiom (*). *Annals of Mathematics. Second Series*, **193**(3), 793–835.
- [4] Asperó, D., & Viale, M. (2022). Incompatible bounded category forcing axioms. *Journal of Mathematical Logic*, **22**(2), Paper no. 2250006, 76 pages.
- [5] Audrito, G., & Viale, M. (2017). Absoluteness via resurrection. *Journal of Mathematical Logic*, **17**(2), 1750005, 36 pages.
- [6] Bagaria, J. (2000). Bounded forcing axioms as principles of generic absoluteness. *Archive for Mathematical Logic*, **39**(6), 393–401.
- [7] ———. (2005). Natural axioms of set theory and the continuum problem. In Hájek, P., Valdés-Villanueva, L., and Westertåhl, D., editors. *Proceedings of the 12th International Congress of Logic, Methodology, and Philosophy of Science*. London: King's College London Publications, pp. 43–64.
- [8] Baldwin, J. T. (2018). *Model Theory and the Philosophy of Mathematical Practice. Formalization without Foundationalism*. Cambridge: Cambridge University Press.
- [9] Barton, N., Ternullo, C., & Venturi, G. (2020). On forms of justification in set theory. *Australasian Journal of Logic*, **17**(4), 158–200.
- [10] Bernays, P. (1935). Sur le platonisme dans les mathématiques. *L'Enseignement Mathématique*, **34**, 52–69.
- [11] Boolos, G. (1971). The iterative conception of set. *Journal of Philosophy*, **68**(8), 215–231.
- [12] Caicedo, A. E., & Veličković, B. (2006). The bounded proper forcing axiom and well orderings of the reals. *Mathematical Research Letters*, **13**(2–3), 393–408.
- [13] Chen Chang, C., & Keisler, H. J. (1990). *Model Theory* (third edition). Studies in Logic and the Foundations of Mathematics, Vol. 73. Amsterdam: North-Holland.
- [14] Cohen, P. (2002). The discovery of forcing. *Rocky Mountain Journal of Mathematics*, **32**(4), 1071–1100.
- [15] Franks, C. (2009). *The Autonomy of Mathematical Knowledge: Hilbert's Program Revisited*. Cambridge: Cambridge University Press.
- [16] Gitman, V., Hamkins, J. D., & Johnstone, T. A. (2016). What is the theory ZFC without power set? *Mathematical Logic Quarterly*, **62**(4–5), 391–406.

- [17] Gödel, K. (1947). What is Cantor's continuum problem? *American Mathematical Monthly*, **54**, 515–525; errata, **55**, 151.
- [18] Hamkins, J. D. (2003). A simple maximality principle. *Journal of Symbolic Logic*, **68**(2), 527–550.
- [19] ———. (2012). The set-theoretic multiverse. *Review of Symbolic Logic*, **5**, 416–449.
- [20] Hamkins, J. D., & Johnstone, T. A. (2014). Resurrection axioms and uplifting cardinals. *Archive for Mathematical Logic*, **53**(3–4), 463–485.
- [21] Incurvati, L. (2017). Maximality principles in set theory. *Philosophia Mathematica*, **25**(2), 159–193.
- [22] Jech, T. (2003). *Set Theory* (The third millennium edition, revised and expanded). Springer Monographs in Mathematics. Berlin: Springer.
- [23] Koellner, P. (2006). On the question of absolute undecidability. *Philosophia Mathematica*, **14**(2), 153–188.
- [24] ———. (2010). Independence and large cardinals. In *Stanford Encyclopedia of Philosophy*.
- [25] Kunen, K. (1980). *Set Theory: An introduction to Independence Proofs*. Studies in Logic and the Foundations of Mathematics, Vol. 102. Amsterdam: North-Holland.
- [26] Larson, P. B. (2010). Forcing over models of determinacy. In Foreman, M. and Kanamori A., editors. *Handbook of Set Theory*. Dordrecht: Springer, pp. 2121–2177.
- [27] Linnebo, Ø. (2013). The potential hierarchy of sets. *Review of Symbolic Logic*, **6**(2), 205–228.
- [28] Linnebo, Ø., & Shapiro, S. (2019). Actual and potential infinity. *Noûs*, **53**(1), 160–191.
- [29] Maddy, P. (1997). *Naturalism in Mathematics*. Oxford: Oxford University Press.
- [30] ———. (2007). *Second Philosophy: A Naturalistic Method*. Oxford: Oxford University Press.
- [31] McGee, V. (1992). Two problems with tarski's theory of consequence. *Proceedings of the Aristotelian Society*, **92**(1), 273–292.
- [32] Moore, J. T. (2005). Set mapping reflection. *Journal of Mathematical Logic*, **5**(1), 87–97.
- [33] ———. (2017). What makes the continuum \aleph_2 . *Contemporary Mathematics*, **690**, 259–287.
- [34] Tent, K., & Ziegler, M. (2012). *A Course in Model Theory*. Cambridge: Cambridge University Press.
- [35] Todorćević, S. (2002). Generic absoluteness and the continuum. *Mathematical Research Letters*, **9**(4), 465–471.
- [36] Venturi, G. (2019). Genericity and arbitrariness. *Logique et Analyse*, **248**, 435–452.
- [37] ———. (2020). Infinite forcing and the generic multiverse. *Studia Logica*, **108**(2), 277–290.
- [38] Venturi, G., & Viale, M. (2023). Second order arithmetic as the model companion of set theory. *Archive for Mathematical Logic*, **62**(1–2), 29–53.
- [39] Viale, M. (2016). Category forcings, MM^{+++} , and generic absoluteness for the theory of strong forcing axioms. *Journal of the American Mathematical Society*, **29**(3), 675–728.

- [40] ———. (2016) Martin's maximum revisited. *Archive for Mathematical Logic*, **55**(1–2), 295–317.
- [41] ———. (2021). Absolute model companionship, forcibility, and the continuum problem. Preprint, [arXiv:2109.02285](https://arxiv.org/abs/2109.02285).
- [42] ———. (2023). Strong forcing axioms and the continuum problem [after Asperó, Schindler]. Exposé 1201. In *Séminaire Bourbaki. Volume 2022/2023. Exposés 1197–1211*. Paris: Société Mathématique de France (SMF).
- [43] ———. (2023). Notes on model completeness, model companionship, Kaiser hulls. Available on author's webpage.
- [44] Woodin, W. H. (2001). The continuum hypothesis. II. *Notices of the American Mathematical Society*, **48**(7), 681–690.
- [45] ———. (2001). The continuum hypothesis. Part I. *Notices of the American Mathematical Society*, **48**(6), 567–576.
- [46] ———. (2005). The continuum hypothesis. In R. Cori, A. Razborov, S. Todorčević, and C. Wood, editors. *Logic Colloquium 2000*. Lecture Notes in Logic, Vol. 19. Urbana, IL: Association for Symbolic Logic, pp. 143–197.
- [47] ———. (2010). *The Axiom of Determinacy, Forcing Axioms, and the Non-stationary Ideal* (revised edition). De Gruyter Series in Logic and its Applications, Vol. 1. Berlin: Walter de Gruyter.

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