THE DECOMPOSITION OF GRAPHS INTO A FINITE NUMBER OF PATHS

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1. Introduction. In **(3)** Ore poses two problems concerning the decomposition of graphs into edge-disjoint paths. The first is to find the conditions on a graph so that it can be decomposed into a finite number k of edge-disjoint, two-way infinite paths and no fewer. In **(2)** Nash-Williams solves this problem. The results of **(2)** are used here to solve the second problem, to find conditions on a graph so that it can be decomposed into a finite number k of edge-disjoint paths (finite, one-way infinite, and two-way infinite) and no fewer. Lemmas 2 and 3 below establish the result, and are essentially corollaries of Lemmas 8 and 13 respectively of **(2)**. Erdös, Grünwald, and Vàzsonyi state the necessary and sufficient conditions for a graph to be decomposable into a finite number of edge-disjoint paths in **(1)**, but they do not discuss the number of paths. Let t, s, and f be respectively the numbers of two-way infinite, one-way infinite, and finite paths in a decomposition of a graph into a minimal finite number of edge-disjoint paths. Conditions are found in Section 4 of this paper so that t, s, and f are uniquely determined by the graph.

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2. Definitions and notation. For a graph G, $V(G)$ and $E(G)$ denote respectively the set of vertices and the set of edges of G. We define

$$|G| = |V(G)| + |E(G)|,$$

where $|V(G)|$ is the cardinality of $V(G)$, and similarly for $|E(G)|$. If $\{H_\alpha\}$ is a collection of graphs, then their *sum* is the graph with vertices $\cup_\alpha V(H_\alpha)$ and edges $\cup_\alpha E(H_\alpha)$. The additive notation $H_1 + H_2 + \ldots + H_n$ is used for finite collections. If H and K are two graphs, then $H - K$ denotes the subgraph of H which has for edges $E(H) - E(K)$ and for vertices any vertex which is either an end-point of one of these edges or is in $V(H) - V(K)$. $H - E(K)$ denotes the subgraph of H with vertices $V(H)$ and edges $E(H) - E(K)$. Two graphs are called *disjoint* if they have no common vertex, and *edge-disjoint* if they have no common edge. If F is a subgraph of the graph G, then $\rho_G(F)$ denotes the number of edges of G with one end-point in $V(F)$ and the other in $V(G) - V(F)$. A vertex v of the subgraph F is an *even*, *odd*, or *infinite vertex of* F according to the number of end-points of edges of F at v, where *loops*, edges with both end-points at v, are counted twice.

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A *path sequence* in a graph G is a sequence $\ldots, v_i, e_i, v_{i+1}, \ldots$ of edges and vertices alternately with each edge preceded by one of its end-points and succeeded by the other, no edge appearing more than once. A vertex which either has no preceding edge or has no succeeding edge is an *end* of the sequence. A path sequence is *finite*, *one-way infinite*, or *two-way infinite* respectively if it has two ends, exactly one end, or no ends. A *path* is a subgraph of G consisting of the edges and vertices of some path sequence. The path is then said to be *derived* from the sequence. A path is finite, one-way infinite, or two-way infinite according to the path sequence from which it is derived. A path may be derivable from many path sequences, and thus, for example, might be both one-way and two-way infinite. However, when a path P is introduced without a corresponding sequence being specified, it will be understood that one of the sequences from which P is derivable is associated with it. If it is introduced, say, as a one-way infinite path, then it will be understood that the path sequence associated with it is one-way infinite. A similar convention will hold for finite and two-way infinite paths. The *path ends* (or simply ends) of a path are the ends of the associated path sequence. A finite path always has two ends, and when these are the same, the path is said to be *closed*. The *length* of a finite path is the number of edges in it. A *decomposition* of a graph G is a collection of mutually edge-disjoint paths such that their sum is equal to G. For a graph with a finite decomposition, that is, a decomposition into a finite number of paths, a *minimal decomposition* is a decomposition with the smallest possible number of paths. Two vertices of a graph are said to be *connected* if there is a finite path which includes them both. A *connected* graph is a graph with every pair of vertices connected. Connectedness is an equivalence relation on $V(G)$. Each equivalence class, together with all the edges of G with any of its members for end-points, is a connected subgraph of G, called a *connected component* of G, or just a component.

An *l-splitting* of the graph G is a collection of l disjoint infinite subgraphs $H_1, \ldots, H_l$ (i.e. $|H_i|$ is infinite) such that $G = H_1 + H_2 + \ldots + H_l + H$, where H is a finite subgraph of G, called a *completion* of the splitting. G is *l-limited* if there is an l-splitting of G but no $(l+1)$-splitting. G is *limited* if it is l-limited for some l. The collection of all subgraphs of G can be divided into equivalence classes **(2)**, where H and K are in the same class if and only if $V(H) \triangle V(K)$ and $E(H) \triangle E(K)$ are both finite, $\triangle$ being the symmetric difference. It is shown in **(2)** that if G is l-limited, there are l distinct equivalence classes such that if $H_1, \ldots, H_l$ is any l-splitting of G, then one of the H_i is in each of these classes. These classes are called the *wings* of G.

A graph will be called an (m, q)-graph if, for m, q non-negative integers, there are exactly m odd vertices of G and q finite components without odd vertices, called here *Euler* components, and $0 < |G| \leqslant \aleph_0$. Note that the *relevant* graphs of **(2)** are the $(0, 0)$-graphs.

$L(v, v')$ denotes the graph consisting of two vertices, v and v', $v \neq v'$, a single edge joining them, and a countably infinite number of loops at v'. If G

is a graph containing the vertices $v_1, \ldots, v_k$ (not necessarily distinct), then $G(v_1, \ldots, v_k)$ denotes the graph obtained from G by adjoining the graphs $L(v_j, v_j')$, $j = 1, \ldots, k$, with the conditions that none of the v_j' are in $V(G)$ and that all of the v_j' are distinct. The subgraph of $L(v_j, v_j')$ consisting of v_j, v_j', and the single edge between them is denoted by E_j.

3. Minimal decompositions.

LEMMA 1. *Let $v_1, \ldots, v_k$ be vertices (not necessarily distinct) of a graph G. Then $G(v_1, \ldots, v_k)$ is $(l + k)$-limited if and only if G is l-limited.*

It will suffice to show the result for $k = 1$, since induction to arbitrary k follows easily if we note that

$$G(v_1, \ldots, v_k) = G(v_1, \ldots, v_{k-1})(v_k). \tag{1}$$

If H is a completion of a t-splitting $H_1, \ldots, H_t$ of G, then $H + E_1$ is a completion of a $(t + 1)$-splitting of $G(v_1)$, namely $H_1, \ldots, H_t, L(v_1, v_1') - E_1$. On the other hand, if $K_1, \ldots, K_{t+1}$ is a $(t + 1)$-splitting of $G(v_1)$ with completion K, then $L(v_1, v_1')$ must have all but a finite number of edges in one of the K_i, say K_{t+1}, since the K_i are disjoint. The remaining edges of $L(v_1, v_1')$ must be in K. Then $(K - L(v_1, v_1')) + (v_1)$ is a completion of the t-splitting $K_1, \ldots, K_{t-1}, K_t + (K_{t+1} - L(v_1, v_1'))$ of G. Thus a t-splitting of G exists if and only if a $(t + 1)$-splitting of $G(v_1)$ exists. Then $G(v_1)$ is $(l + 1)$-limited if and only if G is l-limited. This is the desired result.

It will be convenient to note that by iteration of the construction in this lemma, we see that $H_1, \ldots, H_l$ is an l-splitting of G if and only if

$$H_1, \ldots, H_l, K_1, \ldots, K_k \text{ is an } (l + k)\text{-splitting of } G(v_1, \ldots, v_k), \tag{2}$$

where $K_i = L(v_i, v_i') - E_i$.

Let G be an l-limited (m, q)-graph. (As noted in **(1)**, the requirement that there be a finite number of Euler components is superfluous for limited graphs. However, the redundant notation will be retained for convenience.) Let $H_1', \ldots, H_l'$ be an l-splitting of G with completion H'. Let F be the subgraph of G consisting of all the Euler components of G, all the edges with an end-point at an odd vertex of G, and all vertices which are end-points of these edges. Let $H_i = H_i' - F$. Then $H_1, \ldots, H_l$ is an l-splitting of G with completion $H' + F$. This splitting has the property that none of the H_i contains any odd vertex of G or any vertex of one of the Euler components. Such a splitting will be called a *restricted* l-splitting of the (m, q)-graph G. We see that for any given l-splitting with completion H', there exists a restricted l-splitting such that some completion contains H'.

Let the odd vertices of G be $v_1, \ldots, v_m$, and the Euler components $C_1, \ldots, C_q$. Now form $G' = G(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+2q})$, where the vertices v_{m+2h} are chosen from the C_h respectively, $h = 1, \ldots, q$, and, in order not to introduce any odd vertices, the v_{m+2h-1} are chosen by letting $v_{m+2h-1} = v_{m+2h}$. Then G'

is a (0, 0)-graph, and by Lemma 1, G' is $(l + m + 2q)$-limited. Let $K_1, \ldots, K_l$ be another restricted l-splitting of G. We can assume that K_i and H_i are in the same wing of G, $i = 1, \ldots, l$. From (2) we see that the H_i can be made part of an $(l + m + 2q)$-splitting of G', and thus that the H_i and K_i belong to the wings of G'. Since H_i and K_i belong to the same wing of G, they belong to the same wing of G', and similarly all the H_i belong to distinct wings of G'. By **(2**, Lemmas 4 and 5**)**

$$(3) \qquad \rho_{G'}(H_i) \equiv \rho_{G'}(K_i) \pmod 2.$$

Now since $H_1, \ldots, H_l$ is a restricted l-splitting of G, we have

$$(4) \qquad \rho_G(H_i) = \rho_{G'}(H_i).$$

Similarly,

$$(5) \qquad \rho_G(K_i) = \rho_{G'}(K_i).$$

Then

$$(6) \qquad \rho_G(H_i) \equiv \rho_G(K_i) \pmod 2.$$

We can thus define a wing of G to be *even* or *odd* depending on whether $\rho_G(H)$ is even or odd, where H is any subgraph in the wing which is part of a restricted l-splitting of G. This definition agrees with that in **(2)** for (0, 0)-graphs. Let α be the number of even wings of G, and β the number of odd wings. Then define $p(G) = \alpha + \frac{1}{2}(\beta + m + 2q)$. This also agrees with the definition in **(2)** for (0, 0)-graphs. We note that the $L(v_j, v_j')$ are in odd wings of G' from (2), and using (4) we get

$$(7) \qquad p(G) = p(G').$$

LEMMA 2. *If G is decomposable into a finite number, k, of paths, then G is limited and it is an (m, q)-graph. Also, $k \geqslant p(G)$.*

Let $P_1, \ldots, P_k$ be the edge-disjoint paths of a decomposition, and let $v_1, \ldots, v_n$ be their path ends, where each vertex is listed as many times as it appears as a path end among the P_r (counting it twice each time it appears as the ends of a closed path, in particular). Let $G^* = G(v_1, \ldots, v_n)$. G^* is decomposable into k two-way infinite paths. For if P_1, say, is a finite path, with ends v_1 and v_2, then $L(v_1, v_1') + P_1 + L(v_2, v_2')$ is certainly a two-way infinite path. A similar result holds for the P_r which are one-way infinite paths. In this way all the v_j are exhausted, and since the P_r are edge-disjoint, the two-way infinite paths thus formed are edge-disjoint. By **(2**, Lemma 8**)** G^* is limited, say $(l + n)$-limited, and thus by Lemma 1, G is l-limited. G is certainly an (m, q)-graph since odd vertices can only occur at path ends, and since there are no more than k components all together. Thus the first part of the lemma is established.

Let $H_1, \ldots, H_l$ be a restricted l-splitting of G. Let a_i be the number of the v_j which are in H_i (counting multiplicities), $i = 1, \ldots, l$. Then

$$(8) \qquad \rho_{G^*}(H_i) = \rho_G(H_i) + a_i.$$

By (2) the H_i are also part of an $(l+n)$-splitting of G^*. Since G^* is a $(0, 0)$-graph, this must be a restricted $(l + \mathrm{n})$-splitting. Thus, by definition, the wing of G^* to which H_i belongs is even or odd according to the parity of $\rho_{G^*}(H_i)$. Let g be the number of H_i which are in even wings of G but in odd wings of G^*, and let h be the number of H_i which are in odd wings of G but in even wings of G^*. Then by (8) $g+h$ is just the number of a_i which are odd. Thus

$$\sum_{i=1}^{l} a_i \geqslant g + h. \tag{9}$$

Now each odd vertex of G must be a path end for at least one of the paths, and each Euler component of G must have a finite path in it, and thus two path ends. This accounts for $m + 2q$ path ends, all distinct from those in the H_i since the H_i form a restricted l-splitting. Thus

$$n \geqslant \sum_{i=1}^{l} a_i + m + 2q. \tag{10}$$

From (2) we see that the wings of G^* contain either one of the $L(v_j, v_j')$ or one of the H_i. The wings containing the $L(v_j, v_j')$ are odd; there are n of these. Let α^* and β^* be the numbers of even and odd wings of G^* respectively, and α and β be the corresponding numbers for G. Then we have

$$\alpha^* = \alpha + (h - g); \qquad \beta^* - n = \beta + (g - h). \tag{11}$$

Since G^* is decomposable into k two-way infinite paths, (**2**, Lemma 8) applies, and we have

$$k \geqslant \alpha^* + \tfrac{1}{2}\beta^*. \tag{12}$$

Applying (11) and (10), we get

$$\begin{aligned} k &\geqslant \alpha + (h-g) + \tfrac{1}{2}(\beta + n + g - h) = \alpha + \tfrac{1}{2}(\beta + h - g + n) \\ &\geqslant \alpha + \frac{1}{2}\left(\beta + h - g + m + 2q + \sum_{i=1}^{l} a_i\right). \end{aligned} \tag{13}$$

Finally, using (9), we obtain

$$\begin{aligned} k &\geqslant \alpha + \tfrac{1}{2}(\beta + h - g + m + 2q + g + h) = \alpha + \tfrac{1}{2}(\beta + m + 2q) + h \\ &\geqslant \alpha + \tfrac{1}{2}(\beta + m + 2q) = p(G). \end{aligned} \tag{14}$$

This establishes the last part of the lemma.

LEMMA 3. *A limited (m, q)-graph is decomposable into $p(G)$ paths.*

Let G be a limited (m, q)-graph. Form G' as before. G' is a limited $(0, 0)$-graph, and thus by (**2**, Lemma 13) G' can be decomposed into $p(G')$ two-way infinite paths.

Suppose $P_1, \ldots, P_r$ are the paths of the decomposition which have edges in $L(v_1, v_1')$. Then all but one of them, say P_1, must lie entirely in $L(v_1, v_1')$. Also $P_1 - L(v_1, v_1')$ is a path. Thus $G' - L(v_1, v_1')$ can be decomposed into

$p(G') - r + 1$ paths. Repeating this argument for each of the $L(v_j, v_j')$ shows that G can be decomposed into k paths, where $k \leqslant p(G')$. But $p(G') = p(G)$ by (7). Thus $k \leqslant p(G)$, and by Lemma 2, $k = p(G)$.

THEOREM 1. *A graph G can be decomposed into a finite number of paths if and only if it is a limited (m, q)-graph. In this case the minimum number of paths into which it can be decomposed is $p(G)$.*

4. Uniqueness. In this section we consider G to be a connected, l-limited (m, q)-graph. Suppose $P_1, \ldots, P_k$ is a minimal decomposition of G. Let t, s, f be respectively the numbers of two-way infinite, one-way infinite, and finite paths among the P_i. We have

$$t + s + f = k = p(G) = \alpha + \tfrac{1}{2}(\beta + m) + q. \tag{15}$$

In general, t, s, and f are not uniquely determined by G, but depend upon the choice of the decomposition and upon the choice of the sequences from which each P_i is derived. Under certain conditions, however, they are uniquely determined by G. These are established below.

LEMMA 4. *There is a restricted l-splitting $H_1, \ldots, H_l$ of G such that some completion H is connected, each H_i is connected, and each H_i contains either one or no odd vertex of H, according to whether the wing to which it belongs is odd or even respectively.*

Such a restricted l-splitting of G will be called a *proper* l-splitting.

Let $K_1, \ldots, K_l$ be a restricted l-splitting of G with completion K. Since G is connected, there is a finite subgraph of G which is connected and which contains K. Call it K'. Now the construction proceeds just as in (**2**, Lemma 12). Suppose C is a component of $G - K'$ which contains two odd vertices of K', v_1 and v_2. Then there is a path P in C with v_1 and v_2 for ends. Then v_1 and v_2 are even in $K' + P$. Hence $K' + P$ has two fewer odd vertices than K'. Repeating this procedure reduces the number of odd vertices further. Since the number of odd vertices is finite in K', this process must stop after some finite number of steps. Let H' be the subgraph thus obtained from K' after the last step. No component of $G - H'$ has more than one odd vertex of H' in it. Let F be the sum of the finite components of $G - H'$. Since G is connected and H' is finite, the number of components in F is finite, and thus F is finite. Also, if $H = H' + F$, then H is connected, since H' is. Let $H_i = K_i - H$, $i = 1, \ldots, l$. The components of $G - H$ are all infinite, and thus, since G is l-limited, they are just the H_i. Each H_i is disjoint from F, and thus the only odd vertices of H which it contains are odd vertices of H', of which there is at most one by the definition of H'. $H_1, \ldots, H_l$ is a restricted l-splitting of G with completion H since H contains K. $\rho_G(H_i)$ is even or odd according to whether H_i contains no odd vertex of H or one odd vertex of H. Thus the wing of G to which H_i belongs is even or odd according to whether H_i contains no odd vertex of H or one odd vertex of H. Each H_i is connected since it is a component of $G - H$. Hence $H_1, \ldots, H_l$ is the desired proper l-splitting of G.

Let H be a subgraph of G which is in one of the wings W of G. Suppose K is another member of W. If H has an infinite vertex v, then, since H and K can only differ in a finite number of edges, v is also an infinite vertex of K. We can thus say that the wing W *contains the infinite vertex* v if v is an infinite vertex of any member of W.

LEMMA 5. *If any even wing of G contains an infinite vertex, then neither t nor s is uniquely determined.*

Let W be an even wing of G, and suppose that W contains an infinite vertex v. If $p(G) = 1$, then $\alpha = 1$, $\beta = 0$, $m = 0$. Then **(2**, Lemmas 11 and 10**)** imply, respectively, that G can be considered either as a one-way infinite path or as a two-way infinite path. Thus t and s are not uniquely determined. So assume that $p(G) > 1$. Let $H_1, \ldots, H_l$ be a proper l-splitting of G with, say, H_1 in W. Let H be a completion of the splitting as in Lemma 4. Then any vertex common to H and H_1 must be even in H. Since it is a restricted splitting, none of these common vertices can be odd in G, and thus they are not odd in H_1. All of the vertices of H_1 must then be either even or infinite. Now $G - H_1$ is an $(l - 1)$-limited (m, q)-graph, and $p(G - H_1) = p(G) - 1$. Let $G - H_1$ be decomposed into $p(G) - 1$ paths with the corresponding numbers t', s', and f'. Since H_1 is in W, the vertex v is an infinite vertex of H_1. Then **(2**, Lemmas 10 and 11**)** apply to H_1, and we can consider H_1 to be either a two-way infinite or a one-way infinite path. Then G can be decomposed into $p(G)$ paths using the previous decomposition of $G - H_1$. The resulting numbers t, s, and f are $t = t' + 1$, $s = s'$, $f = f'$, or $t = t'$, $s = s' + 1$, $f = f'$, according respectively to whether H_1 is considered to be a two-way infinite path or a one-way infinite path. Thus t and s are not uniquely determined, and the lemma is proved.

LEMMA 6. *If G is not an Euler graph* (i.e. connected, finite, and having no odd vertices), *and if no even wing of G contains any infinite vertices, then if $P_1, \ldots, P_k$ is a minimal decomposition of G, all of the path ends of the P_i are distinct odd vertices of G.*

First of all, none of the P_i are closed paths. For if, say, P_1 were closed, then either $k > 1$, in which case, by connectedness of G, P_1 would have a common vertex with some other P_i, say P_2, or $k = 1$, in which case G is an Euler graph. The latter case is ruled out by hypothesis. In the former case, we can consider $P_1 + P_2$ to be a single path, thus violating the minimality of the decomposition. Thus if a vertex is a path end twice among the P_i, it must be the end of two different paths, say P_1 and P_2. But in this case $P_1 + P_2$ can be considered as a single path, again violating the minimality of the decomposition. So there is at most one path end at any vertex of G. In particular, this excludes even

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vertices from being path ends. To complete the proof we show that there are no path ends at infinite vertices.

Suppose that P_1 has an end at an infinite vertex u. Then $G(u)$ is decomposable into k paths also, since $P_1 + L(u, u')$ is either a one-way or a two-way infinite path. Let $H_1, \ldots, H_l$ be a restricted l-splitting of G. Then by (2) we have an $(l + 1)$-splitting $H_1, \ldots, H_l, L(u, u') - E_u$ of the $(l + 1)$-limited graph $G(u)$, where E_u is the subgraph of $L(u, u')$ consisting of u, u', and the single edge between them. Since u is infinite, by hypothesis of this lemma, u must be in an odd wing of G. Let H_l be the member of this wing in the splitting. Then we have $\rho_G(H_l) + 1 = \rho_{G(u)}(H_l)$, since the edge of E_u is counted in the right-hand side of this equation. Thus the wing of $G(u)$ to which H_l belongs is even. Then the number of even wings of $G(u)$ is one greater than for G, while the number of odd wings is the same, since the wing of $G(u)$ which contains $L(u, u')$ is odd, compensating for the change of the wing containing H_l from odd to even. Thus $p(G(u)) = p(G) + 1$. But we already have a decomposition of $G(u)$ into $k = p(G)$ paths. This contradicts Theorem 1. Hence the assumption that P_1 has an end at an infinite vertex must be false, and the lemma is proved.

For a graph satisfying the hypotheses of Lemma 6, the number of path ends in any minimal decomposition is at most the number of odd vertices of the graph. On the other hand, every odd vertex must be a path end for some path in the decomposition Thus the number of path ends is at least the number of odd vertices, and

$$m = 2f + s \tag{16}$$

for any minimal decomposition of a graph satisfying the hypotheses of Lemma 6.

Let P be a path and v a vertex in it. Let the associated path sequence from which P is derived be $\ldots, v_{n-1}, e_{n-1}, v_n, e_n, v_{n+1}, \ldots$, where $v = v_n$. Then we say that v *divides* the path P into the two paths Q and R, derived respectively from the sequences $\ldots, v_{n-1}, e_{n-1}, v_n$ and $v_n, e_n, v_{n+1}, \ldots$.

LEMMA 7. *If there is a minimal decomposition of G such that two of the paths of the decomposition are one-way infinite, then there is a minimal decomposition of G with two one-way infinite paths which have a common vertex.*

Let $P_1, \ldots, P_k$ be a minimal decomposition of G, and assume that P_1 and P_2 are one-way infinite. Since G is connected, there are finite paths in G with one end at a vertex of P_1 and the other at a vertex of P_2. Let P be such a path with smallest length d. If $d = 0$, the two paths P_1 and P_2 have a common vertex. So assume that $d > 0$. Let v_1 and v_2 be the ends of P in P_1 and P_2 respectively. By the minimality of d, there is only one edge of P which has an end-point in P_1; call it E. E thus has one end-point at v_1 and the other at a vertex v_3 which is not in P_1, since v_3 is connected to vertices of P_2 by a path of length $d - 1$. The edge E must be in one of the paths of the decomposition, and this path must thus contain v_1 and v_3. Hence it is distinct from P_1 and from P_2. Let P_3 be the path containing E. Now v_1 divides P_1 into a finite path Q_1 and an infinite path R_1. Similarly, v_1 divides P_3 into two paths, Q_3 and R_3, one of which, say R_3, contains E. Suppose R_3 is infinite. Then we can replace P_1 and P_3 in the minimal decomposition by the one-way infinite path $Q_1 + R_3$ and the path $Q_3 + R_1$, thus obtaining a new minimal decomposition. If R_3 is finite, then we can replace P_1 and P_3 by the one-way infinite path $R_1 + R_3$ and the path $Q_1 + Q_3$, again obtaining a new minimal decomposition. In either case the new decomposition still includes P_2, and it includes a one-way infinite path, different from P_2, which contains v_3. Thus there is a minimal decomposition of G with two one-way infinite paths such that a shortest path between them is at most of length $d - 1$. Now we can repeat this construction until we reduce the shortest path length to 0, when the paths have a common vertex.

LEMMA 8. *If there is a minimal decomposition of G which includes either a finite and a two-way infinite path which have a common vertex, or two one-way infinite paths which contain a common vertex, then t, s, and f are not uniquely determined.*

Let $P_1, P_2, \ldots, P_k$ be a minimal decomposition of G. Suppose the paths P_1 and P_2 are one-way infinite and have a common vertex u. Then u divides each of them into a finite part, Q_1 and Q_2 respectively, and an infinite part, R_1 and R_2 respectively. Then $Q_1 + Q_2$, $R_1 + R_2$, $P_3, \ldots, P_k$ is a minimal decomposition with two fewer one-way infinite paths than the original minimal decomposition. In a similar way, if we start with P_1 and P_2 finite and two-way infinite respectively, we can construct a new minimal decomposition with two more one-way infinite paths than the original one. In either case we see that t, s, and f are not uniquely determined.

LEMMA 9. *If s is uniquely determined, and $s = 1$, then if $m > 1$, and $\alpha + \frac{1}{2}(\beta - 1) > 0$, there is an edge E of G such that $G - E$ contains a finite component which contains all the odd vertices of G.*

Since s is uniquely determined, Lemma 5 implies that no even wing of G contains an infinite vertex. Since G is not finite, G satisfies the hypotheses of Lemma 6. Hence we can use (16) along with (15) and the equation $s = 1$ to deduce that $t = \alpha + \frac{1}{2}(\beta - 1) > 0$ and $f = \frac{1}{2}(m - 1) > 0$ are uniquely determined. By Lemma 8 none of the finite paths in a minimal decomposition of G can have any vertices in common with any of the two-way infinite paths of the decomposition. Let P be the single one-way infinite path for some minimal decomposition. Then since G is connected, P must have vertices in common with at least one of the finite paths of the decomposition and with at least one of the two-way infinite paths. Let $v_1, e_1, v_2, \ldots$ be a path sequence from which P is derived, and let v_a be the first vertex in the sequence which is in one of the two-way infinite paths. Let v_b be a vertex of P which is in one of the finite paths, say F. Then v_b divides F into two finite paths, A and B, and it divides P into a finite path Q and an infinite path R. We can then replace P and F in the minimal decomposition by $Q + A$ and $R + B$, this latter path being the one-way infinite path in the resulting minimal decomposition. If $b \geqslant a$, then the finite path $Q + A$ of this minimal decomposition has the vertex v_a in common with a two-way infinite path of the decomposition. By Lemma 8 this violates the uniqueness of t, s, and f. Thus $b < a$. That is, all vertices of P which are in any of the finite paths of the decomposition precede v_a in the sequence for P. Let v_r be the last vertex of the sequence which is in one of the finite paths. The value $a - r > 0$ depends on the choice of the sequence for P. Define $d(P)$ to be the minimum value of $a - r$ for all choices of the sequence for P.

Now select a minimal decomposition of G, with finite paths $F_1, \ldots, F_f$, two-way infinite paths $T_1, \ldots, T_t$, and a one-way infinite path P, such that $d(P)$ is a minimum for all such selections. Let $v_1, e_1, v_2, \ldots, v_r, \ldots, v_a, \ldots$ be a path sequence for P, v_a, and v_r defined as before, such that $d(P) = a - r$. Then P is divided into three paths by v_a and v_r as follows: P_1, derived from the sequence $v_1, e_1, \ldots, v_r$; P_2 derived from the sequence $v_r, e_r, \ldots, v_a$; P_3 derived from the sequence $v_a, e_a, v_{a+1}, \ldots$.

Suppose for $b \leqslant r < a \leqslant c$ that we have $v_b = v_c$. Then the path R derived from the sequence $v_1, e_1, \ldots, e_{b-1}, v_b, e_c, v_{c+1}, \ldots$ is a one-way infinite path, and the path S derived from the sequence $v_b, e_b, \ldots, v_r, \ldots, v_a, \ldots, e_{c-1}, v_c$ is closed. Let F_1 be a finite path in the decomposition which contains v_r, and T_1 a two-way infinite path which contains v_a. Replacing F_1 and P in the minimal decomposition by the finite path $F_1 + S$ and the one-way infinite path R yields a new minimal decomposition. But $F_1 + S$ and T_1 have v_a in common, violating the uniqueness of t, s, f, by Lemma 8. Hence $v_b \neq v_c$; that is, P_1 and P_3 are disjoint. Now suppose that for some vertex v_d of P_1 other than v_r we have $v_d = v_e$ for $d < r < e \leqslant a$. Let A be the part of P_1 derived from the sequence $v_1, e_1, \ldots, v_d$, and let B be the part derived from $v_d, e_d, \ldots, v_r$. Let C be the part of P_2 derived from $v_r, e_r, \ldots, v_e$, and let D be the part derived from $v_e, e_e, \ldots, v_a$. Then $A + D + P_3$ is one-way infinite, and $B + C$ is closed. Thus we can form a new minimal decomposition of G by replacing P and F_1 by the one-way infinite path $A + D + P_3$ and the finite path $F_1 + B + C$. For this decomposition we have

$$d(A + D + P_3) \leqslant a - e < a - r = d(P),$$

since $A + D + P_3$ contains v_e and v_a. But this violates the choice of P. Thus the assumption that there is a vertex of P_1 other than v_r which is also in P_2 is false. A similar argument shows that the only vertex of P_3 which is also in P_2 is v_a.

If H is the sum of P_1 and the finite paths $F_1, \ldots, F_f$, and K is the sum of P_3 and the two-way infinite paths $T_1, \ldots, T_t$, then H and K are disjoint, since by the choice of v_a and v_r we know that P_1 and P_3 are disjoint respectively from the T's and F's, and, as we have just seen, P_1 and P_3 are disjoint. By the choice of v_a and v_r we see that P_2 has only v_r in common with the F's and only v_a in common with the T's, and thus, by what was shown above, P_2 has precisely v_r in common with H and v_a in common with K.

Suppose there were two edge-disjoint paths Q_1 and Q_2 in P_2 with end-points at v_r and v_a. The only odd vertices of P_2 are v_r and v_a, and since $Q_1 + Q_2$ is a closed path, these are the only odd vertices of $P_2 - (Q_1 + Q_2)$. Then they must be in the same component P_2' of $P_2 - (Q_1 + Q_2)$ (**3**, Theorem 1.2.1). Let $P_2'' = P_2 - P_2'$. Then P_2'' is connected since P_2 is connected. P_2'' has no odd vertices since it is just the sum of $Q_1 + Q_2$ and those components of $P_2 - (Q_1 + Q_2)$ which have no odd vertices. Thus P_2'' is a closed path. (**3**, Theorem 3.1.1). P_2' is a finite path with ends at v_r and v_a (**3**, Theorem 3.1.2). Hence $P_1 + P_2' + P_3$ is a one-way infinite path, and $F_1 + P_2''$ is a finite path. Replacing F_1 and P in the minimal decomposition with $F_1 + P_2''$ and $P_1 + P_2' + P_3$ gives a new minimal decomposition. But the finite path $P_2'' + F_1$ contains v_a, since P_2'' contains Q_1. Since v_a is in T_1, this contradicts the uniqueness of t, s, and f by Lemma 8. Thus there must be no pair of edge-disjoint paths in P_2 with ends at v_a and v_r. Then there must be an edge E of P_2 such that v_a and v_r are in different components of $P_2 - E$ (**3**, Theorem 12.3.1). Let C_r and C_a be respectively the components of $P_2 - E$ which contain v_r and v_a. The components of $G - E$ are just $H + C_r$ and $K + C_a$. $H + C_r$ is finite, and since it contains all the finite paths of the decomposition along with P_1, it contains all of the path ends for the decomposition. Thus it contains all of the odd vertices of G. Hence E is the desired edge, and the lemma is proved.

LEMMA 10. *If s is uniquely determined, and $s = 0$, then either $m = 0$, or $\alpha = \beta = 0$.*

Since $s = 0$, and G is connected, either $t = 0$ or $f = 0$ by Lemma 8. If $t = 0$, then G is finite and $\alpha = \beta = 0$. If $t > 0$, then $f = 0$, and there are no path ends, so $m = 0$.

LEMMA 11. *If G has no even wings with infinite vertices, and if either $m \leqslant 1$ or $\alpha + \frac{1}{2}(\beta - 1) \leqslant 0$, then t, s, f are uniquely determined.*

If $m = 0$, then either G is finite, in which case it is an Euler graph and $f = 1$, $s = t = 0$ (**3**, Theorem 3.1.1), or G is infinite, in which case Lemma 6 implies that there are no path ends, and thus that $s = f = 0$, and $t = p(G)$. If $m = 1$, then Lemma 6 implies that there is just one path end, and thus that $s = 1$, $f = 0$, $t = \alpha + \frac{1}{2}(\beta - 1)$. Finally, if $\alpha + \frac{1}{2}(\beta - 1) \leqslant 0$, then either $\alpha = \beta = 0$, in which case G is finite and $s = t = 0, f = p(G)$, or $\alpha = 0$, $\beta = 1$, in which case we may employ (15) and (16) to obtain $s = 1$, $t = 0$, $f = \frac{1}{2}(m - 1)$.

LEMMA 12. *If G has no even wings with any infinite vertices, and there is an edge*

E of G such that $G - E$ has a finite component which contains all the odd vertices of G, then t, s, f are uniquely determined.

We may assume that G is infinite and that $m > 0$; otherwise Lemma 11 applies. For any minimal decomposition of G precisely one of the paths, say P_1, must contain E. Then P_1 is derived from some sequence $\ldots, v_n, E, v_{n+1}, \ldots$. Let P_1' be the path derived from the sequence of terms preceding E, and P_1'' be the path derived from the terms following E. Then one of these two paths is in each of the components of $G - E$. Suppose that P_1' is in the finite component. Then P_1' must be finite, and thus P_1 has an end in the finite component of $G - E$. On the other hand, P_1'' must be infinite, since otherwise P_1 would have an end in the infinite component of $G - E$, which would violate Lemma 6 since there are no odd vertices there. Thus P_1 is one-way infinite. There is no other one-way infinite path in the decomposition. If there were, then it could not contain E, and thus would be entirely in either the finite component of $G - E$ or the infinite component. The former case is clearly impossible. In the latter case, since there are no odd vertices of G in the infinite component, the path could have no end by Lemma 6, and thus could not be one-way infinite. Hence $s = 1$ is uniquely determined, and with (15) and (16) this yields $f = \frac{1}{2}(m - 1)$, $t = \alpha + \frac{1}{2}(\beta - 1)$.

THEOREM 2. *Necessary and sufficient conditions that t, s, and f be uniquely determined for a connected, limited (m, q)-graph G are:*

(A) *No even wing of G contains any infinite vertex.*

(B) *At least one of the following conditions holds:*

(i) $m \leqslant 1$,

(ii) $\alpha + \frac{1}{2}(\beta - 1) \leqslant 0$,

(iii) *there is an edge E of G such that $G - E$ has a finite component containing all the odd vertices of G.*

The necessity of condition A is established by Lemma 5. Lemmas 7 and 8 show the necessity of $s \leqslant 1$, which, together with Lemmas 9 and 10, implies the necessity of condition B. The sufficiency of A and B is established in Lemmas 11 and 12.

COROLLARY. *Necessary and sufficient conditions that t, s, and f be uniquely determined for a limited (m, q)-graph which is not necessarily connected are that* A *and* B *hold for each component of the graph.*

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