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SUPER-ŁUKASIEWICZ PROPOSITIONAL LOGICS

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§ 0. Introduction

In [8] (1920), Łukasiewicz introduced a 3-valued propositional calculus with one designated truth-value and later in [9], Łukasiewicz and Tarski generalized it to an m-valued propositional calculus (where m is a natural number or $\aleph_0$) with one designated truth-value. For the original 3-valued propositional calculus, an axiomatization was given by Wajsberg [16] (1931). In a case of $m \neq \aleph_0$, Rosser and Turquette gave an axiomatization of the m-valued propositional calculus with an arbitrary number of designated truth-values in [13] (1945). In [9], Łukasiewicz conjectured that the $\aleph_0$-valued propositional calculus is axiomatizable by a system with modus ponens and substitution as inference rules and the following five axioms: $p \supset q \supset p$, $(p \supset q) \supset (q \supset r) \supset p \supset r$, $p \vee q \supset q \vee p$, $(p \supset q) \vee (q \supset p)$, $(\sim p \supset \sim q) \supset q \supset p$. Here we use $P \vee Q$ as the abbreviation of $(P \supset Q) \supset Q$. We associate to the right and use the convention that $\supset$ binds less strongly than $\vee$. In [15] p. 51, it is stated as follows: "This conjecture has proved to be correct; see Wajsberg [17] (1935) p. 240. As far as we know, however, Wajsberg's proof has not appeared in print." Rose and Rosser gave the first proof of it in print in [12] (1958). Their proof was essentially due to McNaughton's theorem [10], so it was metamathematical in nature. An algebraic proof was given by Chang [1] [2] (1959).

On the other hand, Rose [11] (1953) showed that the cardinality of the set of all super-Łukasiewicz propositional logics is $\aleph_0$. Surprisingly it was before Rose and Rosser's completeness theorem [12]. The proof in Rose [11] was also due to McNaughton's theorem. Some of our theorems in this paper have already been obtained by Rose [11]. But our proofs are completely algebraic.

In our former paper [5], we gave a complete description of super-

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Łukasiewicz implicational logics (SLIL). In this paper, we will give a complete description of super-Łukasiewicz propositional logics (SLL). We need the completeness of a theory on some ordered abelian groups in [6] to give the complete description of SLL. In the first three sections, we will develope a theory without need of the result in [6]. So some of the results in § 1–§ 3 are included in more generalized forms in the later sections.

In § 1, we will give a complete description of these SLLs which are obtained by adding only C formulas to the smallest SLL ***Lu***. In § 2, we will discuss the inclusion relations between SLLs. And we will have the theorem stated in [15] p. 48 without proof. In § 3, we will give a characterization of SLLs without finite model property. § 4 is the main section of this paper. A complete description of SLLs will be given in it. In § 5, we will give some applications of the complete description of SLLs. In § 6, we will discuss the lattice structure of all SLLs and illustrate a finite sub-structure of it.

We suppose familiarity with [4] and [5]. Only in § 4, we suppose familiarity with [6]. *A CN formula* (or simply, *formula*) is an expression constructed from propositional variables and logical connectives $\supset$ and $\sim$ in the usual way. By *a super-Łukasiewicz propositional logic* (SLL), we mean a set of formulas which is closed with respect to substitution and modus ponens, and contains the following five formulas:

A1. $p \supset q \supset p$,

A2. $(p \supset q) \supset (q \supset r) \supset p \supset r$,

A3. $p \vee q \supset q \vee p$,

A4. $(p \supset q) \vee (q \supset p)$,

A5. $(\sim p \supset \sim q) \supset q \supset p$.

A C algebra is an algebra $\langle A; 1, \rightarrow \rangle$ which satisfies the following axioms, where A is a non empty set and 1 and $\rightarrow$ are 0-ary and 2-ary functions on A respectively.

B1. $1 \rightarrow x = x$.

B2. $x \rightarrow y \rightarrow x = 1$.

B3. $(x \rightarrow y) \rightarrow (y \rightarrow z) \rightarrow x \rightarrow z = 1$.

B4. $x \cup y = y \cup x$.

B5. $(x \rightarrow y) \cup (y \rightarrow x) = 1$.

We abbreviate $(x \to y) \to y$ by $x \cup y$. We use the same convention as before. *A CN algebra* is an algebra $\langle A; 1, \to, \neg \rangle$ which satisfies the following axiom, where $\langle A; 1, \to \rangle$ is a C algebra and $\neg$ is an 1-ary function on A.

$$\text{C1.} \quad \neg x \to \neg y \leq y \to x .$$

Here we denote $x \to y = 1$ by $x \leq y$. We say simply that A is a *CN* algebra, when $\langle A; 1, \to, \neg \rangle$ is a *CN* algebra. If a formula contains no connective other than $\supset$, it is called *a C formula*. In [5], we denote the set of C formulas valid in a C algebra A by $L(A)$. In this paper, we denote the set of formulas valid in a *CN* algebra A by $L(A)$. The set of C formulas valid in a *CN* algebra A is denoted by $L_I(A)$. $\boldsymbol{Lu}$ denotes the set of formulas derivable from A1–A5, that is, $\boldsymbol{Lu}$ is the smallest SLL. For any SLL $\boldsymbol{L}$, $\boldsymbol{L}_I$ denotes the set of C formulas contained in $\boldsymbol{L}$. Let $\boldsymbol{H}$ be any set of formulas and $\boldsymbol{L}$ be any SLL. Then we denote the smallest SLL which includes $\boldsymbol{L} \cup \boldsymbol{H}$ by $\boldsymbol{L} + \boldsymbol{H}$. Sometimes, $\boldsymbol{L} + \{P_1, \cdots, P_n\}$ is denoted by $\boldsymbol{L} + P_1 + \cdots + P_n$. A SLL $\boldsymbol{L}$ is called to be finitely axiomatizable if there exists a finite set $\boldsymbol{H}$ such that $\boldsymbol{L} = \boldsymbol{Lu} + \boldsymbol{H}$.

We denote the set $\{0, 1/m, 2/m, \cdots, (m-1)/m, 1\}$ and the set of all rationals in the interval $[0, 1]$ by S_m $(m \geq 1)$ and S_ω, respectively. We define the functions $\to$ and $\neg$ on S_m $(1 \leq m \leq \omega)$ by $x \to y = \min(1, 1 - x + y)$ and $\neg x = 1 - x$, respectively. Then we can regard S_m as a *CN* algebra. S_m is the well-known Łukasiewicz $(m+1)$-valued (or $\aleph_0$-valued if $m = \omega$) model. We denote also the *CN* algebra with only one element by S_0.

§1. SLLs obtained by adding only C formulas

Let A be a *CN* algebra. A non-empty subset J of A is a *filter* of A if it satisfies the following two conditions:

1) $1 \in J$,
2) $x \in J$ and $x \to y \in J \Rightarrow y \in J$.

Let A be a *CN* algebra, x be an element of A other than 1. A is *irreducible with respect to x* if x is contained within any filter of A which contains at least an element other than 1. A is *irreducible*, if there exists an element such that A is irreducible with respect to the element or A has only one element. By Theorem 2.10 in [4], we have

THEOREM 1.1. *Any irreducible CN algebra is linearly ordered.*

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We can, similarly to Theorems 3.8 and 3.9 in [5], show the following theorems.

THEOREM 1.2. *If a CN algebra B is a subalgebra of a CN algebra A, or $B = A/J$ for some filter J of A, then $L(B) \supseteq L(A)$.*

THEOREM 1.3. *For any SLL L, there exists a set $\{A_\lambda\}_{\lambda \in \Lambda}$ of irreducible CN algebras such that $L = \bigcap_{\lambda \in \Lambda} L(A_\lambda)$.*

Next theorem gives a complete description of SLLs obtained by adding only C formulas.

THEOREM 1.4. *Let $\{A_i \mid i \in I\}$ be a set of C formulas. If $\boldsymbol{L} = \boldsymbol{Lu} + \{A_i \mid i \in I\}$, then $\boldsymbol{L} = \bigcap_{k \leq n} L(S_k)$ for some $n \leq \omega$.*

Proof. By Theorem 4.1 in [5], if $A_i \notin \boldsymbol{Lu}$, then A_i is interdeducible in $\boldsymbol{Lu}$ with $(p\supset)^m q \vee p$ for some m. Here we define $(P\supset)^n(Q)$ as $(P\supset)^0(Q) = Q$ and $(P\supset)^{n+1}(Q) = P \supset (P\supset)^n(Q)$, and we denote $(P\supset)^n(Q)$ by $(P\supset)^n Q$ when no confusion occurs. Because $Lu + (p\supset)^m q \vee p \ni (p\supset)^l q \vee p$ for $l \geq m$, there exists n such that $\boldsymbol{L} = \boldsymbol{Lu} + (p\supset)^n q \vee p$. As $(p\supset)^n q \vee p$ is valid in S_k for any $k \leq n$, $\boldsymbol{L} \subseteq \bigcap_{k \leq n} L(S_k)$. We can easily shown that if $(p\supset)^n q \vee p \in L(A)$, then $\mathrm{ord}(A) \leq n$. Here we give same definition of order of a CN algebra as a C algebra, that is, $\mathrm{ord}(A) = \sup\{\mathrm{ord}(x) \mid x \in A\}$ and $\mathrm{ord}(x)$ is the least integer n such that $x \cup (x\rightarrow)^n y = 1$ for any element y of A ($\mathrm{ord}(x) = \omega$, if no such integer n exists). Therefore, we have that if $(p\supset)^n q \vee p \in L(A)$ and A is irreducible, then A is isomorphic to S_k for some $k \leq n$. Then, we have $\boldsymbol{L} = \bigcap_{k \leq n} L(S_k)$. Clearly, if $A_i \in \boldsymbol{Lu}$ for any $i \in I$, then $\boldsymbol{L} = \boldsymbol{Lu} = \bigcap_{k < \omega} L(S_k) = \bigcap_{k \leq \omega} L(S_k)$. Q.E.D.

If $\boldsymbol{L}_I \not\subseteq \boldsymbol{Lu}$, that is, $\boldsymbol{L}_I \neq \boldsymbol{Lu}_I$, there exists a non-negative integer n such that $(p\supset)^n q \vee p \in \boldsymbol{L}$. Let I be the set of non-negative integers $\{i \mid \boldsymbol{L} \subseteq L(S_i)$ and $i \leq n\}$. Then, we can show that $\boldsymbol{L} = \bigcap_{i \in I} L(S_i)$. Let J be the set of non-negative integers $\{i \mid \boldsymbol{L} \not\subseteq L(S_i)$ and $i \leq n\}$. For each $i \in J$, there exists a formula P_i such that $P_i \in \boldsymbol{L}$ and $P_i \notin L(S_i)$. Let $\boldsymbol{H}$ be the set of formulas $\{P_i \mid i \in J\}$. Then, without being depend on the representative P_i chosen, we have that $\boldsymbol{L} = \boldsymbol{Lu} + (p\supset)^n q \vee p + \boldsymbol{H}$. Therefore, we have the following theorems.

THEOREM 1.5. *If $\boldsymbol{L}_I \neq \boldsymbol{Lu}_I$, then there exists a finite set I of non-negative integers such that $\boldsymbol{L} = \bigcap_{i \in I} L(S_i)$.*

THEOREM 1.6. *If $\boldsymbol{L}_I \neq \boldsymbol{Lu}_I$, then is finitely axiomatizable.*

COROLLARY 1.7. *The cardinality of the set* $\{L | L$ *is a SLL such that* $L_I \neq Lu_I\}$ *is countable.*

§ 2. Inclusion relations between SLLs

Though $L_I(S_n) \subseteq L_I(S_m)$ for $n \geq m$ in SLILs, we can easily know that $L(S_3) \not\subseteq L(S_2)$. In [9], it is stated that Lindenbaum proved that $L(S_n) \subseteq L(S_m)$ if and only if m is a divisor of n. We will generalize Lindenbaum's theorem. We define *the CN algebras* S_n^ω $(n = 1, 2, 3, \cdots)$ as follows.

$$S_n^\omega = \{(x, y) \mid x \in \{1/n, 2/n, \cdots, (n-1)/n\}, y \in Z\} \cup \{(0, y) \mid y \in N\} \cup \{(1, -y) \mid y \in N\} ,$$

where Z and N are the set of all integers and the set of all non-negative integers, respectively.

$$(x, y) \to (z, u) = \begin{cases} (1, 0) & \text{if } z > x \,, \\ (1, \min(0, u - y)) & \text{if } z = x \,, \\ (1 - x + z, u - y) & \text{otherwise} \,. \end{cases}$$

$$\neg(x, y) = (1 - x, -y) \,.$$

When $n = 1$, the first term in S_1^ω is regarded as an empty set. S_1^ω is essentially equivalent to the MV-algebra C defined in Chang [1]. We can check easily that $\langle S_n^\omega; (1, 0), \to, \neg \rangle$ is a CN algebra.

THEOREM 2.1. *Let I and J be finite sets of positive integers.*

$$\bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j^\omega) \subseteq L(S_m)$$

if and only if there exists $n \in I \cup J$ such that m is a divisor of n.

Proof. If there exists $n \in I \cup J$ such that m is a divisor of n, S_m is isomorphic to a subalgebra of S_n (or S_n^ω). Therefore, we have $\bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j^\omega) \subseteq L(S_m)$. Conversely, suppose that $\bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j^\omega) \subseteq L(S_m)$. Let r be $\max I \cup J$ and P be the formula

$$[[(p\supset)^{m-2} \sim p \supset p]\supset]^{r+1}[(p\supset)^{m-1} \sim p\supset]^{r+1}p \,.$$

If f assigns the element $(m-1)/m$ of S_m for p, then $f(P)$ is also $(m-1)/m$. Hence, we have $P \notin L(S_m)$. Therefore, we have $P \notin \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j^\omega)$. Hence, there exists $i \in I$ such that $P \notin L(S_i)$ or there exists $j \in J$ such that $P \notin L(S_j^\omega)$. Suppose that $P \notin L(S_j^\omega)$. Let g be an assignment of S_j^ω such that $g(P) \neq (1, 0)$. We can show that for any $x, y \in S_j^\omega$ and any $l > j$, if

$(x\to)^{\ell} y \neq (1, 0)$ then x is of the form $(1, *)$. Here by $c = (b, *)$ we mean that the first component of c is b. Hence, $(a\to)^{m-2} \neg a \to a = (1, *)$ and $(a\to)^{m-1} \neg a = (1, *)$, where a denotes $g(p)$. Let $a = (1 - k/j, *)$. Then we have $(m-1)k/j \leq (j-k)/j$ and $mk/j \geq 1$. Hence, we have that $j = mk$. When $P \notin L(S_i)$, the proof is similar. Q.E.D.

COROLLARY 2.2 (Lindenbaum). *$L(S_n) \subseteq L(S_m)$ if and only if m is a divisor of n $(1 \leq m < \omega,\ 1 \leq n < \omega)$.*

THEOREM 2.3. *Let I and J be finite sets of positive integers.*

$$\bigcap_{i\in I} L(S_i) \cap \bigcap_{j\in J} L(S_j^{\omega}) \subseteq L(S_m^{\omega})$$

if and only if there exists $n \in J$ such that m is a divisor of n.

Proof. If there exists $n \in J$ such that m is a divisor of n, $\bigcap_{i\in I} L(S_i) \cap \bigcap_{j\in J} L(S_j^{\omega}) \subseteq L(S_m^{\omega})$ because S_m^{ω} is isomorphic to a subalgebra of S_n^{ω}. Conversely, suppose that $\bigcap_{i\in I} L(S_i) \cap \bigcap_{j\in J} L(S_j^{\omega}) \subseteq L(S_m^{\omega})$. Let r be $\max I \cup J$ and P be the formula

$$[[(p\supset)^{m-2} \sim p \supset p]\supset]^{r+1}[(p\supset)^{m-1} \sim p\supset]^{r+1}[(q\supset)^{r} s \vee q]\,.$$

Let f be an assignment of S_m^{ω} such that $f(p) = ((m-1)/m, 0)$, $f(q) = (1, -1)$ and $f(s) = (0, 0)$. Then $f(P) = (1, -1)$. Hence, we have $P \notin L(S_m^{\omega})$. Therefore, we have $P \notin \bigcap_{i\in I} L(S_i) \cap \bigcap_{j\in J} L(S_j^{\omega})$. Because $P \in \bigcap_{i\in I} L(S_i)$, there exists $j \in J$ such that $P \notin L(S_j^{\omega})$. Similarly to the proof of Theorem 2.1, we have this theorem. Q.E.D.

COROLLARY 2.4. *$L(S_n^{\omega}) \subseteq L(S_m^{\omega})$ if and only if m is a divisor of n $(1 \leq m < \omega,\ 1 \leq n < \omega)$.*

§ 3. SLLs without fmp

By the result of [5], we know that any SLIL has the finite model property (fmp). We will show that there exist SLLs without fmp.

DEFINITION 3.1. A SLL L has *fmp* if there exists a set of finite *CN* algebras $\{A_i \mid i \in I\}$ such that $L = \bigcap_{i\in I} L(A_i)$.

A finite irreducible *CN* algebra is isomorphic to S_n for some n. Therefore, by Theorem 1.3, we have

THEOREM 3.2. *A SLL L has fmp if and only if there exists a set I of non-negative integers such that $L = \bigcap_{k\in I} L(S_k)$.*

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THEOREM 3.3. *If $L \neq \boldsymbol{Lu}$, then $L_I \neq \boldsymbol{Lu}_I$ if and only if L has fmp.*

Proof. By Theorem 1.5, L has fmp if $L_I \neq \boldsymbol{Lu}_I$. Conversely, L has fmp. Then there exists a set I of non-negative integers such that $L = \bigcap_{k \in I} L(S_k)$. Because $L \neq \boldsymbol{Lu}$, I is a finite set. So $(p\supset)^n q \vee p \in L$ where $n = \max I$. Hence $L_I \neq \boldsymbol{Lu}_I$. Q.E.D.

For any positive integers m, n, S_n^ω has a subalgebra isomorphic to S_m if we regard S_m and S_n^ω as C algebras. Then we have

LEMMA 3.4. *$L_I(S_k^\omega) = \boldsymbol{Lu}_I$ for any positive integer k.*

THEOREM 3.5. *If both I and J are finite sets of positive integers, $J \neq \phi$ and $L = \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j^\omega)$, then L has not fmp.*

Proof. $L \neq \boldsymbol{Lu}$ because $I \cup J$ is a finite set. By $J \neq \phi$ and Lemma 3.4, $L_I = \boldsymbol{Lu}_I$. Therefore, L has not fmp by Theorem 3.3. Q.E.D.

COROLLARY 3.6. *$L(S_n^\omega)$ has not fmp for any positive integer n.*

§ 4. A complete description of SLLs

This section is the main part of this paper.

DEFINITION 4.1. Let A be a linearly ordered CN algebra, and a be the maximum element of A. An element x of A is called *almost maximum* if $(x\rightarrow)^n \neg a \neq a$ for any positive integer n. An element of x is called *infinitesimal* if $\neg x$ is almost maximum. If A has an element other than the maximum element, the set M_A of all almost maximum elements of A is a filter of A. The CN algebra A/M_A is denoted by $\tilde{A}$. rank (A) is defined by rank $(A) = \text{ord}\,(\tilde{A})$.

Clearly, only one almost maximum element of $\tilde{A}$ is the maximum element, that is, $\tilde{A}$ is locally finite (This is Chang's terminology [1].).

THEOREM 4.2. *Let A be a linearly ordered CN algebra. If* rank $(A) = \omega$, *then $L(A) = \boldsymbol{Lu}$.*

Proof. By Theorem 1.2, $L(A) \subseteq L(\tilde{A})$. Because $\tilde{A}$ is locally finite, $\tilde{A}$ is isomorphic to a subalgebra of the CN algebra of all real numbers between 0 and 1 (cf. [2] p. 78). By ord $(\tilde{A}) = \omega$, A has an infinite number of members. Therefore, $L(\tilde{A}) = \boldsymbol{Lu}$ (cf. [12] p. 5). Hence, we have $L(A) = \boldsymbol{Lu}$. Q.E.D.

For a given model G of $\boldsymbol{SS}$ (cf. [6]), let the segment $G[c]$ determined by a positive element c of G be the set of all elements $x \in G$ such that $0 \leq x \leq c$. We define the functions $\rightarrow$ and $\neg$ on $G[c]$ as follows:

$$x \rightarrow y = \min(c, c - x + y)\,,$$
$$\neg x = c - x\,.$$

Then we can easily prove the following lemma.

LEMMA 4.3. *The algebra* $\langle G[c]; c, \rightarrow, \neg\rangle$ *defined above is a linearly ordered CN algebra. If* m *satisfies* $-1 < 2(m - c) < 1$, *then* $\operatorname{rank}(G[c]) = m$.

We now wish to establish the converse to Lemma 4.3. Let A be a linearly ordered CN algebra and 0 be the minimum element of A. We let A^* be the set $\{(s, x) \mid s \in \{+, -\}, x$ is an infinitesimal element of $A\}$. We identify $(+, 0)$ with $(-, 0)$ and denote $(\pm, x)$ by $\pm x$, respectively. On the set A^* we define the functions $+$ and $-$ and the relation $0<$ as follows:

$$(+, x) + (+, y) = (+, \neg x \rightarrow y)\,,$$
$$(-, x) + (-, y) = (-, \neg x \rightarrow y)\,,$$
$$(+, x) + (-, y) = (-, y) + (+, x) = \begin{cases} (+, \neg(x \rightarrow y)) & \text{if } y \leq x\,, \\ (-, \neg(y \rightarrow x)) & \text{if } x < y\,, \end{cases}$$
$$-(+, x) = (-, x)\,,$$
$$-(-, x) = (+, x)\,,$$
$$0 < (s, x) \Leftrightarrow s = + \quad \text{and} \quad x \neq 0\,.$$

Then the algebra $\langle A^*; +, -, 0<\rangle$ is a totally ordered abelian group. Generally, the group $ZG = Z \times G$ is a model of $\boldsymbol{SS}$ is G is a totally ordered abelian group, where $Z \times G$ is ordered as $0 < (x, y)$ if and only if either $0 < x$ or $x = 0$ and $0 < y$. Hence ZA^* is a model of $\boldsymbol{SS}$.

LEMMA 4.4. *Let A be a linearly ordered CN algebra,* $\operatorname{ord}(A) = \omega$ *and* $\operatorname{rank}(A) = n$. *Then there exists an infinitesimal element b of A such that* $b \neq 0$ *and* $A \cong ZA^*[(n, +b)]$.

Proof. By $\operatorname{rank}(A) = n$, $\tilde{A} \cong S_n$. Let φ be an isomorphism from $\tilde{A}$ to S_n and α be an element of $\tilde{A}$ (and hence an equivalence class of A) such that $\varphi(\alpha) = (n - 1)/n$. Since $\operatorname{ord}(A) = \omega$, we can take a sufficiently large element x of α such that $(x\rightarrow)^n\, 0 < a$ (a is the maximum element of A). We can show that for any $y \neq a$ there is an unique infinitesimal element z of A such that $y = (x\rightarrow)^m z$ or $y = (x\rightarrow)^{m-1} \neg (\neg x \rightarrow z)$ if $\varphi([y]) = m/n$. Let b detote $\neg(x\rightarrow)^n 0$. Let f be a function from A to $ZA^*[(n, +b)]$ such that $f((x\rightarrow)^m z) = (m, +z)$, $f((x\rightarrow)^{m-1} \neg (\neg x \rightarrow z)) = (m, -z)$ and $f(a) = (n, +b)$. Then f is an isomorphism from A onto $ZA^*[(n, +b)]$.

Q.E.D.

The first order language $\mathscr{L}'$ is the same as in [6], which consists of $0, 1, -, +, 0<, n|$ (for each integer $n > 0$) and $=$. Let $\mathscr{L}''$ be the language obtained from $\mathscr{L}'$, by adding a binary function symbol *min*. The language of the theory $\boldsymbol{SS}'$ is $\mathscr{L}''$ and the set of axioms of $\boldsymbol{SS}'$ is obtained from $\boldsymbol{SS}$ by adding the following axiom:

$$(j) \quad z = \min(x, y) \leftrightarrow (x < y \rightarrow z = x) \wedge (y \leq x \rightarrow z = y) .$$

It is clear that each model of $\boldsymbol{SS}$ can be regarded also as a model of $\boldsymbol{SS}'$. In $\boldsymbol{SS}'$, for any formula $A(x)$, the following is derivable:

$$A(\min(s, t)) \leftrightarrow (s \leq t \rightarrow A(s)) \wedge (t < s \rightarrow A(t)) .$$

Therefore, for any formula F of $\mathscr{L}''$ we can construct the formula F^* of $\mathscr{L}'$ such that $F \leftrightarrow F^*$ is derivable in $\boldsymbol{SS}'$ and each variable of which some occurrence is bound in F^* is also bound in F. Especially, F^* is open if F is open. Hence, by Corollary 2.3 in [6], we have

LEMMA 4.5. *For any open formula F of $\mathscr{L}''$ and any model A of $\boldsymbol{SS}' \cup (i)$, F is valid in ZQ if and only if F is valid in A.*

We now define the term P^* of $\mathscr{L}''$ corresponding to a formula P of SLL in the following manner:

$$p^* = h(p) ,$$
$$(P \supset Q)^* = \min(c - P^* + Q^*, c) ,$$
$$(\sim P)^* = c - P^* .$$

Here h is an injective mapping from the set of propositional variables of SLL to the set of variables of $\mathscr{L}''$ such that $h(p) \neq c$ for any p. We assume that $x_1, x_2, \cdots, x_n$ are the only variables occurring in P^*. Next, we define the formula P^0 as $P^0 = (0 \leq x_1 \leq c \wedge \cdots \wedge 0 \leq x_n \leq c \rightarrow P^* = c)$.

LEMMA 4.6. *For any formula P of SLL and any linearly ordered CN algebra A such that* $\operatorname{ord}(A) = \omega$ *and* $\operatorname{rank}(A) = n$, *$P$ is valid in A if $-1 < 2(n - c) < 1 \rightarrow P^0$ is valid in ZQ.*

Proof. Suppose that P is not valid in A. There exists an assignment

f of A such that $f(P) < a$ where a is the maximum element of A. By Lemma 4.4, there exists an isomorphism φ from A to $ZA^*[(n, +b)]$. Let g be an assignment of ZA^* such that $g(x) = \varphi(f(h^{-1}(x)))$ and $g(c) = (n, +b)$. Then $-1 < 2(n - c) < 1 \to P^0$ is not true under g. Since ZA^* is a model of $SS' \cup$ (i), $-1 < 2(n - c) < 1 \to P^0$ is not valid in ZQ by Lemma 4.5.

Q.E.D.

LEMMA 4.7. *For any linearly ordered CN algebra A such that* ord $(A) = \omega$ *and* rank $(A) = n$, $L(A) \subseteq L(ZZ[(n, 1)])$.

Proof. By Lemma 4.4, $A \cong ZA^*[(n, +b)]$. A subalgebra of $ZA^*[(n, +b)]$ generated by $(1, 0)$ is isomorphic to $ZZ[(n, 1)]$. Q.E.D.

LEMMA 4.8. *For any integer k,*

$$L(ZZ[(n, 0)]) \subseteq L(ZZ[(n, k)]) \subseteq L(ZZ[(n, 1)]) .$$

Proof. By Lemma 4.7, $L(ZZ[(n, k)]) \subseteq L(ZZ[(n, 1)])$. Suppose that P is not valid in $ZZ[(n, k)]$. Let f be an assignment of $ZZ[(n, k)]$ such that $f(P) = (u, v) \neq (n, k)$. Let g be an assignment of $ZZ[(n, nk)]$ such that $g(p) = (m, nl)$ if $f(p) = (m, l)$ for any propositional variable p. Then $g(P) = (u, nv) \neq (n, nk)$. $ZZ[(n, nk)]$ is isomorphic to $ZZ[(n, 0)]$ (isomorphism φ is given by $\varphi((m, l)) = (m, l - mk)$). Hence, P is not valid in $ZZ[(n, 0)]$.

Q.E.D.

LEMMA 4.9. *For any integer k,*

$$L(ZZ[(n, 0)]) = L(ZZ[(n, k)]) = L(ZZ[(n, 1)]) .$$

Proof. By Lemma 4.8, it suffices to show that $L(ZZ[(n, 0)]) \subseteq L(ZZ[(n, 1)])$. Let P be a formula which is not valid in $ZZ[(n, 0)]$ and f be an assignment of $ZZ[(n, 0)]$ such that $f(P) \leq (n, -1)$. Let $g_m: ZZ[(n, 0)] \to ZZ[(n, 0)]$ be a homomorphism such that $(i, j) \mapsto (i, mj)$. Let f' be an assignment of $ZZ[(n, 1)]$ such that $f'(p) = g_m f(p)$ for any propositional variable p. For any formula F with the degree d (that is, the number of occurrences of logical connectives in the formula F is d), we shall show by induction on d that

$$g_m f(F) - (0, d) \leq f'(F) \leq g_m f(F) + (0, d) .$$

Suppose F is $G \supset H$ and the degrees of G and H are e and e', respectively. By the inductive hypothesis,

$$g_m f(G) - (0, e) \leq f'(G) \leq g_m f(G) + (0, e)\,,$$
$$g_m f(H) - (0, e') \leq f'(H) \leq g_m f(H) + (0, e')\,.$$

Since

$$f'(G \supset H) = \min\left((n, 1) - f'(G) + f'(H),\ (n, 1)\right),$$
$$g_m f(G \supset H) = \min\left((n, 0) - g_m f(G) + g_m f(H),\ (n, 0)\right)$$

and $d = e + e' + 1$, we have

$$g_m f(G \supset H) - (0, d) \leq f'(G \supset H) \leq g_m f(G \supset H) + (0, d)\,.$$

The case that F is $\sim G$ is similar. Therefore, we have that $f'(P) \leq (n, d - m)$. If $m \geq d$, P is not true in $ZZ[(n, 1)]$ under the assignment f'. Q.E.D.

We are now in a position to prove the following key theorem.

THEOREM 4.10. *For any linearly ordered CN algebra A such that* $\operatorname{ord}(A) = \omega$ *and* $\operatorname{rank}(A) = n$, $L(A) = L(ZZ[(n, 0)])$.

Proof. By Lemma 4.7 and Lemma 4.9, we have $L(A) \subseteq L(ZZ[(n, 0)])$. We shall show that $L(A) \supseteq L(ZZ[(n, 0)])$. Let P be a formula valid in $ZZ[(n, 0)]$. By Lemma 4.9, P is valid in $ZZ[(n, k)]$ for any integer k. Hence $-1 < 2(n - c) < 1 \to P^0$ is valid in ZZ. By Lemma 4.5, $-1 < 2(n - c) < 1 \to P^0$ is valid in ZQ. By Lemma 4.6, P is valid in A. Q.E.D.

$ZZ[(n, 0)]$ is isomorphic to S_n^ω defined in § 2. Now, we can prove the main theorem.

THEOREM 4.11. *For any* SLL, *there exist sets of non-negative integers I, J such that $\boldsymbol{L} = \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j^\omega)$. If $\boldsymbol{L} \neq \boldsymbol{Lu}$, then both sets I and J are finite.*

Proof. By Theorem 1.3, there exists a set $\{A_\lambda\}_{\lambda \in \Lambda}$ of irreducible CN algebras such that $\boldsymbol{L} = \bigcap_{\lambda \in \Lambda} L(A_\lambda)$. By Theorem 3.13 in [5], $L(A_\lambda) = L(S_n)$ if $\operatorname{ord}(A_\lambda) = n$. By Theorem 4.10, $L(A_\lambda) = L(S_n^\omega)$ if $\operatorname{ord}(A_\lambda) = \omega$ and $\operatorname{rank}(A_\lambda) = n$. By Theorem 4.2, $L(A_\lambda) = \boldsymbol{Lu} = \bigcap_{k < \omega} L(S_k)$ if $\operatorname{rank}(A_\lambda) = \omega$. Therefore, $\boldsymbol{L} = \bigcap_{i \in I} L(S_i) \cap \bigcap_{j \in J} L(S_j^\omega)$ for some I and J. If $I \cup J$ is infinite, then $\boldsymbol{L} \subseteq \bigcap_{i \in I \cup J} L(S_i)$ because $L(S_n^\omega) \subseteq L(S_n)$. By Theorem 20 in [15] p. 49, $\bigcap_{i \in I \cup J} L(S_i) = \boldsymbol{Lu}$. Hence, we have $\boldsymbol{L} = \boldsymbol{Lu}$. Q.E.D.

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§ 5. Applications of the main theorem

By Theorem 4.11, Theorem 3.5 gives a complete characterization of SLLs without fmp. For example, we can show as follows that $\boldsymbol{Lu}+P$ has not fmp, where P is the formula $(p \supset \sim p) \supset (\sim p \supset p) \supset p \vee \sim p$. Because $P \in L(S_3) \cap L(S_1^\omega)$ and $P \notin L(S_n)$ for $n=2$ or $n \geq 4$ and $P \notin L(S_n^\omega)$ for $n \geq 2$, we have $\boldsymbol{Lu}+P = L(S_3) \cap L(S_1^\omega)$ by Theorem 4.11. Hence, $\boldsymbol{Lu}+P$ has not fmp by Theorem 3.5.

The following theorem, that was proved in Rose [10], is easily obtained from Theorem 4.11.

THEOREM 5.1. *The cardinality of the set of all SLLs is countable.*

Rose [11] also showed that any SLL is finitely axiomatizable. We will show it as follows.

LEMMA 5.2. $\boldsymbol{Lu}+A_n = \bigcap_{k \leq n} L(S_k^\omega)$, *where*

$$A_n = [(p\supset)^{2n} \sim p] \supset [(p\supset)^{n-1} \sim p \supset p] \supset (p\supset)^{n-1} \sim p \vee p\,.$$

Proof. By Theorem 4.11 and $L(S_k^\omega) \subseteq L(S_k)$ for any k, it suffices to show that (1) $A_n \in L(S_k^\omega)$ for $k \leq n$ and that (2) $A_n \notin L(S_k)$ for $k > n$.

Proof of (1). Let f be an assignment of S_k^ω. If $f(p) \leq ((k-1)/k, 0)$ or $f(p) = (1, 0)$, then $f((p\supset)^{n-1} \sim p \vee p) = (1, 0)$. Therefore, $f(A_n) = (1, 0)$. If $f(p) = ((k-1)/k, *)$, then $f((p\supset)^{n-1} \sim p \supset p) \leq f((p\supset)^{n-1} \sim p)$. Therefore, $f(A_n) = (1, 0)$. If $f(p) = (1, *)$, then $f((p\supset)^{2n} \sim p) \leq f(p)$. Hence, $f(A_n) = (1, 0)$.

Proof of (2). Let f be an assignment of S_k such that $f(p) = 1 - [k/n + 1]\cdot 1/k$, where $[x]$ is the integral part of x. Then $f((p\supset)^{2n} \sim p) = 1$, $f((p\supset)^{n-1} \sim p \supset p) = 1$ and $f((p\supset)^{n-1} \sim p \vee p) \neq 1$. Therefore, $f(A_n) \neq 1$. Q.E.D.

THEOREM 5.3. *Any SLL is finitely axiomatizable.*

Proof. Let $\boldsymbol{L}$ be a SLL. If $\boldsymbol{L} = \boldsymbol{Lu}$, then $\boldsymbol{L}$ is finitely axiomatizable. Suppose that $\boldsymbol{L} \neq \boldsymbol{Lu}$. Then there exists a positive integer n such that $\bigcap_{j \leq n} L(S_j^\omega) \subseteq \boldsymbol{L}$. Hence $A_n \in \boldsymbol{L}$. Because $A_n \notin L(S_k)$ and $A_n \notin L(S_k^\omega)$ for any $k > n$, there exist two sets of positive integers I' and J' such that $\boldsymbol{L} = \bigcap_{i \in I'} L(S_i) \cap \bigcap_{j \in J'} L(S_j^\omega)$ and $I', J' \subseteq \{i \mid i \leq n\}$. Let I and J be the sets of positive integers $\{i \mid \boldsymbol{L} \not\subseteq L(S_i)$ and $i \leq n\}$ and $\{j \mid \boldsymbol{L} \not\subseteq L(S_j^\omega)$ and $j \leq n\}$, respectively. For each $i \in I$ ($j \in J$), there exists a formula $P_i(Q_j)$ such that

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$P_i \in L$ $(Q_j \in L)$ and $P_i \notin L(S_i)$ $(Q_j \notin L(S_j^\omega))$. Let G and H be the set of formulas $\{P_i \mid i \in I\}$ and $\{Q_j \mid j \in J\}$, respectively. Then, we have that $L = Lu + G + H + A_n$. Q.E.D.

We denote the set of all formulas by W. By Theorem 4.11, $W - L$ is recursive enumerable for any SLL L. By Theorem 5.3, L is recursive enumerable for any SLL L. Hence we have

THEOREM 5.4. *Any SLL is decidable.*

Krzystek and Zachorowski [7] proved that $L(S_n)$ $(2 \leq n \leq \omega)$ has not Interpolation Property. Quite similarly, we can prove the following theorem.

THEOREM 5.5. *Any SLL except W and $L(S_1)$ has not Interporation Property.*

Proof. Let L be a SLL except W and $L(S_1)$. Let P and Q be the formulas $((r \supset r \supset p) \supset r \supset p) \supset p$ and $(s \supset s \supset p) \supset s \supset p$, respectively. The formula $P \supset Q$ is valid in S_ω. Hence we have $P \supset Q \in Lu$. Let A be a CN algebra such that A is S_n $(n \geq 2)$ or S_n^ω $(n \geq 1)$. Let f be an assignment of A such that $f(r)$, $f(s) \notin \{0, 1\}$ and $f(p) = 0$. It is easy to observe that $f(P)$, $f(Q) \notin \{0, 1\}$ but for every formula R, built up from the variable p only, $f(R) \in \{0, 1\}$. Hence, for every such R, $P \supset R \notin L(A)$ or $R \supset Q \notin L(A)$. By Theorem 2.1 and Theorem 4.11, $L \subseteq L(S_n)$ for some $n \geq 2$ or $L \subseteq L(S_1^\omega)$. Therefore, $P \supset Q \in L$ but for every R, built up from the variable p only, $P \supset R \notin L$ or $R \supset Q \notin L$. Q.E.D.

§6. Lattice structures of SLLs

Hosoi [3] showed that the set $\mathscr{L}$ of all intermediate propositional logics is a pseudo-Boolean algebra (PBA). We can similarly prove that the set $\mathscr{SL}$ of all SLLs is a PBA. Let $\{L_\lambda\}_{\lambda \in \Lambda}$ be a set of SLLs. Then $\bigcap_{\lambda \in \Lambda} L_\lambda$ is naturally a SLL but $\bigcup_{\lambda \in \Lambda} L$ is not always a SLL. But there exists the minimum SLL including $\bigcup_{\lambda \in \Lambda} L_\lambda$. So, by $\bigcup_{\lambda \in \Lambda} L_\lambda$, we mean the minimum SLL including $\bigcup_{\lambda \in \Lambda} L_\lambda$. By the definition, we have

THEOREM 6.1. *$\mathscr{SL}$ forms a complete lattice with $\subseteq$ as the order relation.*

Further, we have

THEOREM 6.2. $\bigcup_{\lambda \in \Lambda} L_\lambda \cap L = \bigcup_{\lambda \in \Lambda} (L_\lambda \cap L)$.

Proof. It suffices to prove that $\bigcup_{\lambda \in \Lambda} L_\lambda \cap L \subseteq \bigcup_{\lambda \in \Lambda} (L_\lambda \cap L)$. Suppose that $P \in \bigcup_{\lambda \in \Lambda} \boldsymbol{L}_\lambda \cap \boldsymbol{L}$. Then there exist formulas $Q_1, Q_2, \cdots, Q_n \in \bigcup_{\lambda \in \Lambda} \boldsymbol{L}_\lambda$ such that $Q_1 \supset Q_2 \supset \cdots \supset Q_n \supset P \in \boldsymbol{Lu}$. Hence, $Q_1 \vee P \supset Q_2 \vee P \supset \cdots \supset Q_n \vee P \supset P \in \boldsymbol{Lu}$ because $(Q_1 \supset Q_2 \supset \cdots \supset Q_n \supset P) \supset Q_1 \vee P \supset Q_2 \vee P \supset \cdots \supset Q_n \vee P \supset P \in \boldsymbol{Lu}$. On the other hand, as each Q_i belongs to some $\boldsymbol{L}_\lambda$, each $Q_i \vee P$ belongs to some $\boldsymbol{L}_\lambda \cap \boldsymbol{L}$. So P belongs to $\bigcup_{\lambda \in \Lambda} (\boldsymbol{L}_\lambda \cap \boldsymbol{L})$. Q.E.D.

Remark. $\bigcap_{\lambda \in \Lambda} \boldsymbol{L}_\lambda \cup \boldsymbol{L} = \bigcap_{\lambda \in \Lambda} (\boldsymbol{L}_\lambda \cup \boldsymbol{L})$ does not always hold. For example, $\bigcap_{i \in N} L(S_i) \cup L(S_1^\omega) = L(S_1^\omega) \neq L(S_1) = \bigcap_{i \in N} (L(S_i) \cup L(S_1^\omega))$.

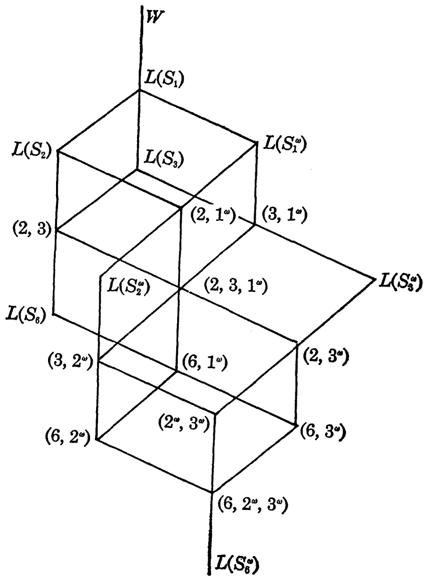
Theorem 6.2 is a necessary and sufficient condition for a complete lattice to be a PBA.

THEOREM 6.3. *$\mathscr{SL}$ is a PBA with $\boldsymbol{W}$ and $\boldsymbol{Lu}$ as the maximum element and the minimum element, respectively.*

We denote by $\mathscr{SL}(\boldsymbol{L})$ the set of all SLLs including $\boldsymbol{L}$. By Theorem 4.11, $\mathscr{SL}(\boldsymbol{L})$ is a finite set if $\boldsymbol{L} \neq \boldsymbol{Lu}$. Hence we have

THEOREM 6.4. *If $\boldsymbol{L} \neq \boldsymbol{Lu}$, then $\mathscr{SL}(\boldsymbol{L})$ is a finite PBA.*

We illustrate the lattice structure of $\mathscr{SL}(L(S_6^\omega))$ in the following Figure using Theorems 2.1, 2.3 and 4.11. Here we use the abbreviation such as $(2, 3, 1^\omega) = L(S_2) \cap L(S_3) \cap L(S_1^\omega)$.



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