INTROENUMERABILITY, AUTOREDUCIBILITY, AND RANDOMNESS

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Abstract. We define Ψ-autoreducible sets given an autoreduction procedure Ψ. Then, we show that for any Ψ, a measurable class of Ψ-autoreducible sets has measure zero. Using this, we show that classes of cototal, uniformly introenumerable, introenumerable, and hyper-cototal enumeration degrees all have measure zero.

By analyzing the arithmetical complexity of the classes of cototal sets and cototal enumeration degrees, we show that weakly 2-random sets cannot be cototal and weakly 3-random sets cannot be of cototal enumeration degree. Then, we see that this result is optimal by showing that there exists a 1-random cototal set and a 2-random set of cototal enumeration degree. For uniformly introenumerable degrees and introenumerable degrees, we utilize Ψ-autoreducibility again to show the optimal result that no weakly 3-random sets can have introenumerable enumeration degree. We also show that no 1-random set can be introenumerable.

§1. Introduction. In 1959, Friedberg and Rogers [4] introduced enumeration reducibility. A set $A \subseteq \omega$ is *enumeration reducible* to another set $B \subseteq \omega$ if there is a c.e. set W such that $A = \{x : (\exists y)\langle x, y\rangle \in W \text{ and } D_y \subseteq B\}$, where $\{D_y\}_{y\in\omega}$ gives a computable listing of all finite sets. We call the c.e. set W that witnesses this reduction an enumeration operator and write $A = W(B)$. The degree structure induced by enumeration reduction $\leq_e$ consists of the enumeration degrees. We can identify subsets of ω with infinite strings in the Cantor space 2^ω. Therefore, we can consider the measure of different classes of enumeration degrees (often abbreviated by e-degrees), including cototal e-degrees, uniformly introenumerable e-degrees, introenumerable e-degrees, and hyper-cototal e-degrees.

Given a set A of natural numbers and any number n, we may ask whether the membership of n in A can be determined using the oracle A without asking "is n in A". If so, A has a kind of self-reducibility. The notion of autoreducibility introduced by Trakhtenbrot [12] in 1970 is a formalization of this idea. A set A is said to be *autoreducible* if there is a Turing functional Φ such that for any n, $A(n) = \Phi^{A-\{n\}}(n)$. We will generalize the autoreduction notion by defining Ψ-autoreducibility for any autoreduction procedure Ψ, which is a function from $\omega \times 2^\omega$ to $\{0, 1\}$. The classes of enumeration degrees mentioned above all have natural autoreducibility by replacing

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the Turing functional with different autoreduction procedures. Next, we will show that any measurable class of Ψ-autoreducible sets has measure zero for any Ψ. Then, we use this property of classes of Ψ-autoreducible sets to show that the classes of above e-degrees all have measure zero.

Intuitively, given a set $A \subseteq \omega$, or equivalently an infinite string in 2^ω, it is random if it is hard to compress or one cannot predict the next bit or it has no rare properties. In 1966, Martin-Löf introduced a randomness notion using the latter idea that a random set is in no effective measure zero set in [8]. An infinite string $A \in 2^\omega$ is *Martin-Löf random* (*n-random*), if A is not in $\bigcap_m G_m$, where $\{G_m\}_{m\in\omega}$ is any uniformly Σ^0_1 (Σ^0_n, respectively) sequence of open sets such that the measure of each G_m is smaller than 2^{-m}. A set A is *weakly n-random* if A avoids all Π^0_n classes. Generally, a set is random if it avoids a particular kind of null classes. Such null classes can be arithmetical as above or even go beyond arithmetical.

Since our classes of e-degrees have measure zero, sufficiently random sets must avoid such measure zero classes. Therefore, we can ask questions about what level of randomness the above sets or e-degrees can reach, and what level of randomness the above sets or e-degrees must avoid. We answer such questions for cototal sets, cototal e-degrees, uniformly introenumerable sets, uniformly introenumerable e-degrees, introenumerable sets, and introenumerable e-degrees. For references for randomness notions, see [2] or [10].

We start by giving the definitions of the sets and e-degrees we mentioned. First, a set A is *total* if $\overline{A} \leq_e A$. It is named total because the degree of a total set is the degree of the graph of a total function. In [1], the notion of cototality is given by reversing the relationship between A and $\overline{A}$.

Definition 1.1. A set A is *cototal* when $A \leq_e \overline{A}$.

Definition 1.2. An infinite set X is *uniformly introenumerable* if there is an enumeration operator Γ such that for every infinite subset Y of X, $\Gamma(Y) = X$.

In [7], Jockush introduced the notion of uniform introenumerability. The definition of uniform introenumerability we give here is slightly different by using an enumeration operator instead of a c.e. operator, though the two definitions were shown to be equivalent in [6] by Greenberg et al. Recently, Goh et al. [5] also showed that Jockush's notion of (non-uniform) introenumerability is equivalent to the following notion:

Definition 1.3. An infinite set X is *introenumerable* if, for every infinite subset Y of X, there is an enumeration operator Γ such that $\Gamma(Y) = X$.

In [11], Sanchis introduced a reduction that is related to hyperarithmetical reduction and only uses positive information about membership in the set:

Definition 1.4. Let A and B be sets such that, for some c.e. set W, the following relation holds: $x \in B$ if and only if

$$(\forall f \in \omega^{<\omega})(\exists n, y)[\langle f \upharpoonright n, x, y\rangle \in W \wedge D_y \subseteq A].$$

Then we say that B is *hyper-enumeration* reducible to A and write this relation: $B \leq_{he} A$.

Definition 1.5. A is called *hyper-cototal* if $A \leq_{he} \overline{A}$.

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THEOREM 1.6. *The relationship of enumeration degrees of the above notions is the following*:

$$\text{Cototal} \rightarrow \text{Uniformly Introenumerable} \dashrightarrow \text{Introenumerable} \rightarrow \text{Hyper-cototal}.$$

REMARK 1.7. The solid arrows are strict. For proof of the first arrow, see [9]. The third arrow and the strictness of the first arrow are proved in [5] by Goh et al. It is still unknown whether there is a set of introenumerable e-degree that does not have uniformly introenumerable e-degree.

§2. Measure of classes with autoreduction. In this section, we define Ψ-autoreducible sets given an autoreduction procedure Ψ and show that any measurable class of Ψ-autoreduction sets has measure zero. Next, we apply the autoreducibility of hyper-cototal e-degrees to show that the measure of the class of such e-degrees is zero.

DEFINITION 2.1. Given a function $\Psi : \omega \times 2^\omega \rightarrow \{0, 1\}$, A set A is Ψ-*autoreducible* if and only if

$$(\forall n)[A(n) = \Psi(n, A - \{n\})].$$

Here, we say that the function Ψ is an autoreduction procedure.

Next, to show that the measure of a class of Ψ-autoreducible sets is zero, we use the Lebesgue density theorem.

THEOREM 2.2. *Fix an autoreduction procedure* Ψ, *a measurable class* S *of* Ψ-*autoreducible sets has measure zero.*

PROOF. Suppose a class S of Ψ-autoreducible sets has positive measure. By the Lebesgue density theorem, for any $\varepsilon > 0$, there is a string $\sigma \in 2^{<\omega}$ such that $\frac{\mu(S\cap[\sigma])}{\mu(S)} \geq 1 - \varepsilon$. Fix $\varepsilon = \frac{1}{4}$ along with the corresponding string σ. Consider an $n \in \omega$ larger than $|\sigma|$. Define subsets $P_i (i = 0, 1)$ of S as follows:

$$P_i = \{X \in S : \Psi(n, X - \{n\}) = i\}.$$

Since P_0 and P_1 partition S, one of them must have the following relative measure: $\frac{\mu(P_i\cap[\sigma])}{\mu(S)} \geq \frac{1-\varepsilon}{2} = \frac{3}{8}$. Without loss of generality, assume that such subset is P_0. Now, consider the set

$$P_2 = \{\hat{X} : X \in P_0, \hat{X}(n) = 1, (\forall i \neq n)[X(i) = \hat{X}(i)]\}.$$

Notice that if $x \in P_0$, $X(n) = 0$. So, P_2 also has relative measure $\frac{\mu(P_2\cap[\sigma])}{\mu(S)} \geq \frac{3}{8} > \frac{1}{4}$. Therefore, $\frac{\mu(P_2\cap S\cap[\sigma])}{\mu(S)} > 0$. So, $P_2 \cap S$ is not empty. For any $Y \in P_2 \cap S$, $\Psi(n, Y - \{n\}) = 0 \neq 1 = Y(n)$. This is a contradiction. Therefore, S has measure zero. ⊣

REMARK 2.3. In this theorem, the assumption that the class S is measurable is necessary. Consider the finite difference equivalence classes: two sets A and B are in the same equivalence class if and only if $(A - B) \cup (B - A)$ is finite. Now, we can define a class S_0 that contains exactly one element from each of the equivalence classes. It is not difficult to see that S_0 is not measurable. We can define a function Ψ_0 such that if $A \in S_0$ and $n \in \omega$, then $\Psi_0(n, A - \{n\}) = A(n)$. It is well-defined because, for any $B \in S_0$ and $B - \{n\} = A - \{n\}$, $\Psi_0(n, B - \{n\})$ has to equal $A(n)$ by the definition of S_0. Therefore, S_0 is a class consisting of Ψ_0-autoreducible sets that does not have measure zero since it is not measurable.

Now we use the above theorem to show that the measure of the class of hyper-cototal e-degrees is zero. First, we discuss the autoreducibility of hyper-cototal sets.

LEMMA 2.4. *For every hyper-cototal set A, there is a Ψ such that A is Ψ-autoreducible.*

PROOF. Suppose A is hyper-cototal and there is some hyper-enumeration operator Δ such that $A = \Delta(\overline{A})$. When $n \in A$, $\overline{A} \subseteq \overline{A - \{n\}}$. Therefore, $n \in \Delta^{\overline{A}} \subseteq \Delta^{\overline{A-\{n\}}}$. When $n \notin A$, $n \notin \Delta^{\overline{A}} = \Delta^{\overline{A-\{n\}}}$. So, $A(n) = \Delta^{\overline{A-\{n\}}}(n)$. Then, we can define $\Psi(n, X) := \Delta^{\overline{X}}(n)$. ⊣

In fact, each set of hyper-cototal degree is Ψ-autoreducible for some autoreduction procedure Ψ as well.

LEMMA 2.5. *Any set in the class of hyper-cototal e-degrees is a hyper-cototal set.*

PROOF. In [11], Sanchis proved that If $A \leq_e B$, then $A \leq_{he} B$ and $\overline{A} \leq_{he} \overline{B}$. Suppose A has hyper-cototal e-degree and $A \equiv_e B$, where B is a hyper-cototal set. Then, $A \equiv_{he} B \leq_{he} \overline{B} \equiv_{he} \overline{A}$. ⊣

Next, in order to apply Theorem 2.2 to show that the measure of the classes of hyper-cototal e-degrees is 0, we first need to show that the class of hyper-cototal e-degrees is measurable by analyzing the arithmetical complexity of

$$\{A : A \leq_{he} \overline{A}\} = \bigcup_{\Gamma} \{A : (\forall n)[n \in A \rightarrow n \in \Gamma^{\overline{A}} \wedge n \notin A \rightarrow n \notin \Gamma^{\overline{A}}]\}.$$

Notice that $n \in \Gamma^{\overline{A}}$ and $n \notin \Gamma^{\overline{A}}$ are Π^1_1 and Σ^1_1, respectively, for a hyper-enumeration operator Γ by Definition 1.4. So, the class of hyper-cototal e-degrees is the difference of two Π^1_1 classes. Recall that Π^1_1 sets are measurable. Therefore, the class of hyper-cototal e-degrees is measurable. Now, we use the results from above to see that the class of hyper-cototal e-degrees has measure zero.

LEMMA 2.6. *The classes of hyper-cototal, introenumerable, uniformly introenumerable, and cototal e-degrees all have measure zero.*

PROOF. Suppose the class of hyper-cototal e-degrees has positive measure. Because there are only countably many hyper-enumeration operators, there exists a Γ such that the class of hyper-cototal e-degrees witnessed by this operator has positive measure. However, any set in this class would be Γ-autoreducible by Lemma 2.4. Now, applying Theorem 2.2 gives us a contradiction. By the relationship between the e-degrees mentioned above in Theorem 1.6, we see that the measure of these classes are all zero. ⊣

§3. Bounds of randomness. Notice that, for any class of measure zero, sufficiently random sets avoid it. So, we now discuss what level of randomness these e-degrees could and could not have. In this section, all necessary background knowledge of

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randomness is from Nies' book [10]. We first discuss the class of cototal sets and the class of cototal e-degrees.

THEOREM 3.1. *Weakly* 2*-random sets are not cototal.*

PROOF. The class of cototal sets $\{A : A \leq_e \overline{A}\}$ is defined by

$$\bigcup_e \{A : A = \Gamma_e^{\overline{A}}\} = \bigcup_e \{A : \forall n[n \in A \to (\exists D_y \subseteq \overline{A})[\langle n, y\rangle \in \Gamma_e\rangle] \\ \wedge\, n \notin A \to (\forall y)[\langle n, y\rangle \in \Gamma_e \to D_y \cap A \neq \emptyset]]\},$$

where Γ_e's are enumeration operators. Therefore, the class of cototal sets is a union of Π^0_2 classes. By Lemma 2.6, all such classes have measure zero. Because any weakly 2-random set avoids all null Π^0_2 classes, weakly 2-random sets are not cototal. ⊣

To see that weak 2-randomness is optimal, we show that the 1-random Chaitin's Ω is a cototal set.

THEOREM 3.2. *There exists a* 1*-random cototal set*.

PROOF. Because Ω is left-c.e., there is a non-descending computable sequence $\{q_n\}$ of rationals such that $\Omega = \lim_{n\to\infty} q_n$. For any enumeration of $\overline{\Omega}$, we can enumerate Ω using this computable sequence. First, to determine whether 0 is in Ω or not, either for some n, we see the dyadic expansion of q_n starts with 1 or we see 1 enter $\overline{\Omega}$. Only for the first case, we enumerate 0 in Ω. Then, we can iteratively do this process for each nature number in order. Eventually, we obtain an enumeration of Ω. Therefore, $\Omega \leq_e \overline{\Omega}$. ⊣

For the class of cototal e-degrees, we first discuss what level of randomness is enough to avoid them.

THEOREM 3.3. *Weakly* 3*-random sets do not have cototal e-degree.*

PROOF. Notice that the class of cototal e-degrees defined by an enumeration operator Γ_e is

$$\{A : A = \Gamma_e^{\overline{K_A}}\} = \{A : (\forall n)[n \in A \to (\exists y)[\langle n, y\rangle \in \Gamma_e \to D_y \cap K_A = \emptyset] \\ \wedge n \notin A \to (\forall y)[\langle n, y\rangle \in \Gamma_e \to D_y \cap K_A \neq \emptyset]]\}.$$

Since $D \cap K_A = \emptyset$ and $D \cap K_A \neq \emptyset$ are Π^0_1 and Σ^0_1 respectively, the class of cototal e-degrees defined by Γ_e is Π^0_3. Since each of these classes is null, weakly 3-random sets avoid them all. So, we conclude that weakly 3-random sets do not have cototal e-degree. ⊣

Next, we see that weak 3-randomness is optimal by showing that there is a 2-random set of cototal e-degree even though any cototal set cannot be weakly 2-random.

THEOREM 3.4. *There exists a* 2*-random set of cototal e-degree.*

PROOF. Consider Chaitin's Ω relativized to $\emptyset'$, i.e., $\Omega^{\emptyset'}$, which is 2-random. Let L be $\{q \in \mathbb{Q}_2 : q < \Omega^{\emptyset'}\}$. Then, $L \leq_e \Omega^{\emptyset'} \leq_e L \oplus \overline{L}$. Notice that L is Σ^0_2. In [1], it was shown that every Σ^0_2 set has cototal e-degree. So, there exists M

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such that $M \equiv_e L$ and $\overline{M} \geq_e M$. Then, $\overline{\Omega^{\emptyset'} \oplus L \oplus M} \geq_e \overline{L} \oplus \overline{M} \geq_e \overline{L} \oplus M \equiv_e \overline{L} \oplus L \geq_e \Omega^{\emptyset'} \equiv_e \Omega^{\emptyset'} \oplus L \oplus L \equiv_e \Omega^{\emptyset'} \oplus L \oplus M$. Hence, we have a cototal set that is enumeration equivalent to $\Omega^{\emptyset'}$. ⊣

In the proofs above, we did not use autoreducibility since it is enough to analyze the arithmetical complexities of the class of cototal sets and the class of cototal e-degrees to show the optimal level of randomness the sets in these classes must avoid. However, a similar analysis would not work for the classes of (uniform) introenumerable sets or e-degrees. We can verify the complexity of the collection of uniformly introenumerable e-degrees:

$$\bigcup_e \{A : \exists i, m \forall B [\forall a [a \in A \leftrightarrow \exists b [\langle a, b\rangle \in \Gamma_m \wedge D_b \subseteq \Gamma_i(A)]] \wedge$$
$$[B \subseteq \Gamma_i(A) \wedge [\forall p \in B \exists q > p] \rightarrow$$
$$\forall t [t \in \Gamma_i(A) \leftrightarrow \exists s [\langle t, s\rangle \in \Gamma_e \wedge D_s \subseteq B]]]]\}.$$

This is Π^1_1. We suspect that the class of uniformly introenumerable e-degrees is Π^1_1-complete. This was shown to be true for the class of uniformly introreducible sets in [6]. Assuming that there is no simpler definition, the analysis we used for cototal e-degrees would not work. Instead, for each set A of uniform introenumerable e-degree, we show Ψ-autoreducibility for some autoreduction procedure Ψ so that we can apply Theorem 2.2 again.

THEOREM 3.5. *Weakly* 3-*random sets do not have uniformly introenumerable e-degree.*

PROOF. We will show that uniformly introenumerable e-degrees are contained in a countable union of measure zero Π^0_3 classes. To do this, we show that each set A of uniformly introenumerable e-degree is Ψ-autoreducible for some Ψ. Since A has uniformly introenumerable e-degree, there is a set B, enumeration operators Φ, Γ, and Δ such that $A = \Delta(B)$, $B = \Phi(A)$, and for any infinite subset C of B, $\Gamma(C) = B$. Let

$$\Psi(n, Z) = \begin{cases} 1, & n \in \Delta(\Gamma(\Phi(Z))) \\ & \text{ or } \Phi(Z) \text{ is finite,} \\ 0, & \text{otherwise.} \end{cases}$$

Note that n has to be in A when $\Phi(A - \{n\})$ is finite. So, A is Ψ-autoreducible. Now we consider the class of Ψ-autoreducible sets:

$$\{D : \forall n[[n \in D \rightarrow n \in \Delta(\Gamma(\Phi(D - \{n\})))$$
$$\vee (\exists p \forall t > p)[t \notin \Phi(D - \{n\})]]$$
$$\wedge [n \notin D \rightarrow (\forall q \exists s > q)[s \in \Phi(D - \{n\})]$$
$$\wedge\, n \notin \Delta(\Gamma(\Phi(D - \{n\})))]]\}.$$

This is a Π^0_3 class. By Theorem 2.2, this is a null class. Because weakly 3-random sets cannot be in any Π^0_3 null class, weakly 3-random sets do not have uniformly introenumerable e-degree. ⊣

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Meanwhile, there also exists 2-random uniformly introenumerable e-degrees because of Theorem 3.4 and the fact that every set of cototal e-degree has uniform introenumerable e-degree.

With more work, the previous result can be improved to show that weakly 3-random sets do not have introenumerable e-degree either.

THEOREM 3.6. *No weakly* 3*-random set has introenumerable e-degree.*

PROOF. Suppose a weakly 3-random set A has introenumerable e-degree. Let B be an introenumerable set such that there are enumeration operators Φ and Δ with $A = \Delta(B)$ and $B = \Phi(A)$. For a contradiction, we define $C = \bigcup_i c_i$ as an infinite subset of B such that $\Gamma_i(C) \neq B$ for any enumeration operator Γ_i (here we identified strings c_i with corresponding sets). When we are constructing C, we also define a set D_i at each stage i. Let $c_0 = \emptyset$ and $D_0 = \emptyset$. Suppose c_i and D_i have been defined. By inductive assumption, $\Phi(A - D_i)$ is infinite. First, we consider whether there is an extension e of c_i such that $e \preccurlyeq c_i\Phi(A - D_i) \restriction [|c_i|, \infty)$, and $\Gamma_i(e) - B \neq \emptyset$. If so, we define c_{i+1} to be the least such e that contains at least one more element than c_i and $D_{i+1} = D_i$. If not, we consider whether there is an extension e of c_i such that for some n, $\Phi(A - D_i \cup \{n\})$ is infinite, $e \preccurlyeq c_i\Phi(A - D_i) \restriction [|c_i|, \infty)$, and $\Gamma_i(e\Phi(A - D_i \cup \{n\}) \restriction [|e|, \infty)) \subsetneq B$. If so, we define c_{i+1} to be the least such e that contains at least one more element than c_i, and $D_{i+1} = D_i \cup \{n\}$. If not, we can define

$$\Psi(n, Z) = \begin{cases} 1, & n \in \Delta(\Gamma_i(c_i\Phi(Z - D_i) \restriction [|c_i|, \infty))) \\ & \text{ or } \Phi(Z - D_i) \text{ is finite,} \\ 0, & \text{otherwise,} \end{cases}$$

similar to the proof in Theorem 3.5. Notice that A is Ψ-autoreducible and the class of Ψ-autoreducible sets is Π^0_3. This is impossible because A is weakly 3-random. This is a contradiction. Therefore, at least one of the two cases we considered has to be true. In this way, we obtain an infinite $C = \bigcup_i c_i \subseteq B$. Now we show that $\Gamma_i(C) \neq B$ for any i. For any i, if the first case we considered is true, then $\Gamma_i(C)$ contains an element not in B. If the second case is true, $\Gamma_i(C) \subseteq \Gamma_i(c_{i+1}\Phi(A - D_{i+1}) \restriction [|c_{i+1}|, \infty)) \subsetneq B$. ⊣

Again, by Theorems 1.6 and 3.4, we conclude that there exists 2-random introenumerable e-degree while there is no weakly 3-random introenumerable e-degree. Next, we consider the class of uniformly introenumerable sets. We use the proof ideas of Proposition 8 given by Figueira, Miller, and Nies in [3] that showed no random is autoreducible.

THEOREM 3.7. *No* 1*-random set is uniformly introenumerable.*

PROOF. We will apply Schnorr's theorem. To do so, we will show that the initial segment of any uniformly introenumerable set A can be compressed beyond any fixed constant.

Let Γ be the enumeration operator such that $\Gamma(B) = A$ for any infinite subset B of A. For each m, there is a least n_m such that $n_m > n_p$ for any $p < m$ and $\Gamma_{n_m}(0^m A \restriction [m, n_m)) \restriction m = A \restriction m$ since $A - \{0, 1, \dots, m-1\}$ is an infinite subset of A. Let c_m be the number of 1's in the string $A \restriction m$.

Now we define a prefix-free machine M that outputs $A \restriction n_m$ with input $\gamma = 0^{|\sigma|}1\sigma 0^{|\tau|}1\tau A \restriction [m, n_m)$, where σ, τ are binary strings corresponding to m, c_m.

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M first obtains the length of σ by reading until the first 1 and then obtains the number m by reading $|\sigma|$ many bits after the first 1. Next, M can find out c_m in the same way by reading the input until τ. Now, M's read head keeps on moving forward to read $A \restriction [m, n_m)$ bit by bit to do the enumeration of $\Gamma(0^m A \restriction [m, n_m)) \restriction m$ step by step to enumerate $A(x)$ for x between 0 and $m-1$ until c_m many of such $A(x)$ is determined to be 1, which means the other bits on $A \restriction m$ are zeros. M can output $A \restriction n_m$ by concatenation. Therefore, $K(A \restriction n_m) \leq^+ n_m - m + 4\log(m)$. By Schnorr's theorem, A is not 1-random. ⊣

For introenumerable sets, we combine the methods used in Theorems 3.6 and 3.7.

THEOREM 3.8. *No* 1-*random set is introenumerable.*

PROOF. Suppose there is a 1-random introenumerable set A. We prove the theorem by constructing an infinite subset $B = \bigcup_i b_i$ of A such that $\Gamma_i(B) \neq A$ for any enumeration operator Γ_i (here we identified the strings b_i with its corresponding set).

Let $b_0 = \emptyset$. Suppose we have already defined b_i. There are two possible cases. One of the two cases must hold for it to be 1-random.

First, We consider whether there is an n such that $\Gamma_i(b_i A \restriction [|b_i|, n))$ contains an element that is not in A. If so, we let $b_{i+1} = b_i A \restriction [|b_i|, n)$. In this case, we have a finite extension b_{i+1} of b_i such that b_{i+1} is a subset of A, and for any infinite extension B of b_{i+1}, $\Gamma_i(B)$ has an element not in A.

Second, if there is no such n in the first case, we consider whether there is an m such that $\Gamma_i(b_i 0^m A \restriction [|b_i| + m, \infty)) \subsetneq A$. If so, we let $b_{i+1} = b_i 0^m$. In this case, we have a finite extension b_{i+1} of b_i such that applying Γ_i to A's subset $b_{i+1} A \restriction [|b_{i+1}|, \infty)$ does not output A.

If one of the cases holds for every i, we can show that for any i, $\Gamma_i(B) \neq A$, contradicting introenumerability. If the first case holds for i, then for any extension B_0 of b_{i+1}, $\Gamma_i(B_0) \neq A$. If the first case does not hold, notice that B is a subset of $B_1 = b_i 0^m A \restriction [|b_i| + m, \infty)$. Then, $\Gamma_i(B) \subseteq \Gamma_i(B_1) \subsetneq A$.

If neither cases hold for some i, we show that A is not 1-random using a method similar to the one used in the proof of the above theorem. For each m, there is a least n_m such that $n_m > n_p$ for any $p < m$ and

$$\Gamma_{i,n_m}(b_i 0^m A \restriction [|b_i| + m, n_m)) \restriction |b_i| + m = A \restriction |b_i| + m$$

because the failure of the second case guarantees that eventually numbers in $A \restriction |b_i| + m$ will be enumerated and no other numbers would be enumerated by the failure of the first case. Let c_m be the number of 1s in the string $A \restriction [|b_i|, |b_i| + m)$. Now we define a prefix-free machine M that outputs $A \restriction n_m$ with input $\gamma = 0^{|\sigma|} 1 \sigma 0^{|\tau|} 1 \tau A \restriction [|b_i| + m, n_m)$, where σ, τ are binary strings corresponding to m, c_m. M obtains m, c_m in the same way as the proof above by reading until τ. Then, M obtains the first $|b_i|$ bits of A using Γ_i. Next, its read head keeps on moving forward to read $A \restriction [|b_i| + m, n_m)$ bit by bit to do the enumeration of $\Gamma_i(b_i 0^m A \restriction [|b_i| + m, n_m))$ step by step to enumerate $A(x)$ for x between $|b_i|$ and $|b_i| + m - 1$ until c_m many of such $A(x)$ is determined to be 1 and output $A \restriction n_m$ by concatenation. Therefore, $K(A \restriction n_m) \leq^+ n_m - m + 4\log(m)$. By Schnorr's theorem, A is not 1-random. ⊣

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