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Abstract

In 1988, Haagerup and Størmer conjectured that every pointwise inner automorphism of a type III_1 factor is a composition of an inner and a modular automorphism. We study this conjecture and prove that every type III_1 factor with trivial bicentralizer indeed satisfies this condition. In particular, this shows that Haagerup and Størmer's conjecture holds in full generality if Connes' bicentralizer problem has an affirmative answer. Our proof is based on Popa's intertwining theory and Marrakchi's recent work on relative bicentralizers.

1. Introduction

Let M be a von Neumann algebra and $\mathrm{Aut}(M)$ the set of all automorphisms on M. We say that $\theta \in \mathrm{Aut}(M)$ is *pointwise inner* [HS87] if for each positive linear functional $\varphi \in M_*^+$, there is a unitary $u \in \mathcal{U}(M)$ such that $\theta(\varphi) = u\varphi u^*$, where we used the notation $\theta(\varphi) := \varphi \circ \theta^{-1}$ and $x\varphi y := \varphi(y \cdot x)$ for $x, y \in M$. In other words, θ is pointwise inner if the induced map $\theta\colon M_*^+ \to M_*^+$ is inner at a pointwise level. This notion naturally appeared in the study of unitary equivalence relations on state spaces on von Neumann algebras and was intensively studied by Haagerup and Størmer [HS87, HS88, HS91].

Any inner automorphism is pointwise inner. Indeed, we can choose the same unitary for all elements in M_*^+. A nontrivial and motivating example comes from *Tomita–Takesaki modular theory*, that is, every *modular automorphism* σ_t^φ is pointwise inner, where $\varphi \in M_*^+$ is a faithful state and $t \in \mathbb{R}$; see § 2. Hence, every composition $\mathrm{Ad}(u) \circ \sigma_t^\varphi$ of an inner and a modular automorphism is pointwise inner because the pointwise inner property is closed under compositions. Haagerup and Størmer conjectured that the opposite phenomenon also holds for every type III_1 factor [HS88].

Haagerup–Størmer conjecture. *Let M be a type III_1 factor with separable predual. Then every pointwise inner automorphism is a composition of an inner and a modular automorphism.*

We note that if M is a factor that is not of type III_1, similar results were obtained in [HS87, HS88], hence the only remaining problem for characterizing pointwise inner automorphisms is the case for type III_1 factors. We also note that the separability assumption is necessary; see [HS88, § 6] and [AH12, Remark 4.21].

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In the present paper, we give a partial but satisfactory answer to this conjecture. To explain it, we prepare some terminology. For an inclusion of von Neumann algebras $N \subset M$, we say it is *irreducible* if $N' \cap M \subset N$, and is *with expectation* if there exists a faithful normal conditional expectation from M onto N. For every faithful state, or more generally, every faithful semifinite normal weight, φ on M, we denote by M_φ the *centralizer of* φ, which is the fixed point algebra of σ^φ. When M is a type III_1 factor with separable predual, the existence of a faithful state $\varphi \in M_*$ with $M_\varphi' \cap M = \mathbb{C}$ is equivalent to having *trivial bicentralizer*; see § 2. In this case, $M_\varphi \subset M$ is an irreducible subfactor of type II_1 with expectation, and the existence of such a subfactor is very useful in many contexts. *Connes' bicentralizer problem*, which is one of the most important problems for type III factors, asks if every type III_1 factor with separable predual has trivial bicentralizer. To the best of our knowledge, all concrete examples of type III_1 factors have trivial bicentralizers. Further, if M is a type III_1 factor, then the tensor product $M \mathbin{\overline{\otimes}} R_\infty$ has trivial bicentralizer, where R_∞ is the Araki–Woods factor of type III_1 [Mar18].

We now introduce our main theorem. For objects in (3) below; see § 2.

Theorem A. *Let M be a type III_1 factor with separable predual and assume it has trivial bicentralizer. Fix a faithful state $\varphi \in M_*$ such that $M_\varphi' \cap M = \mathbb{C}$. Then for each $\theta \in \mathrm{Aut}(M)$, the following conditions are equivalent.*

(1) *The automorphism θ is pointwise inner.*
(2) *For each faithful state $\psi \in M_*$ with $M_\psi' \cap M = \mathbb{C}$, there exists $u \in \mathcal{U}(M)$ such that $\theta(x) = uxu^*$ for all $x \in M_\psi$.*
(3) *There exists $u \in \mathcal{U}(M)$ such that $\theta^\omega(x) = uxu^*$ for all $x \in (M^\omega)_{\varphi^\omega}$, where ω is a fixed free ultrafilter on $\mathbb{N}$.*
(4) *There exist $u \in \mathcal{U}(M)$ and $t \in \mathbb{R}$ such that $\theta = \mathrm{Ad}(u) \circ \sigma_t^\varphi$.*

By the implication (1)⇒(4), we immediately get the following corollary.

Corollary B. *The Haagerup–Størmer conjecture holds for every type III_1 factor with separable predual that has trivial bicentralizer. In particular, the conjecture holds in full generality if Connes' bicentralizer problem has an affirmative answer.*

We emphasize that Theorem A applies to all concrete examples of type III_1 factors because they have trivial bicentralizers. Although we do not know the complete answer to Connes' bicentralizer problem, if it is solved affirmatively, then our theorem solves the Haagerup–Størmer conjecture. Even if the problem has a negative answer, we can apply the theorem to every type III_1 factor up to a tensor product with R_∞. Thus, in any case, Theorem A covers a large class of type III_1 factors.

We explain item (2) in Theorem A. In all previous works for pointwise inner automorphisms, a specific state (or weight) φ on a factor M plays a crucial role. More precisely, the implication (1)⇒(2) in Theorem A for the specific ψ $(= \varphi)$ is the key step of the proof. For example, φ is a trace if M is of type II. If M is a type III_λ $(0 \leq \lambda < 1)$ factor, then φ is a lacunary weight. We note that, in these cases, the implication (2)⇒(4) also follows by the fact that φ is a specific one. Unfortunately, if M is a type III_1 factor, there are no such specific states or weights in general, and this fact is the main difficulty for the conjecture. We do have a dominant weight, but it is not enough for our purpose because $M_\varphi \subset M$ is not with expectation. In [HS91, HI24], φ is assumed to be an almost periodic state, but not all type III_1 factors admit such a state.

In Theorem A, we do not assume φ is such a specific state or weight. We nevertheless prove the implication (1)⇒(2). This is the main observation of the paper, so we briefly explain the idea of the proof. Recall that, if φ is almost periodic, each element in M has a Fourier decomposition

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along the eigenspaces of φ. In [HS91, HI24], the Fourier decomposition of a unitary element $u \in \mathcal{U}(M)$ satisfying $\theta(\varphi(a\,\cdot\,)) = u\varphi(a\,\cdot\,)u^*$ for some carefully chosen $a \in M_\varphi$ is important in the proof. In our setting, we cannot use such a decomposition since φ is not assumed to be almost periodic. Instead, we use the embedding

$$u\mathrm{W}^*\{a^{\mathrm{i}t} \otimes \lambda_t \mid t \in \mathbb{R}\}u^* \subset \theta(A) \mathbin{\overline{\otimes}} L\mathbb{R}, \quad A := \mathrm{W}^*\{a\},$$

where the inclusion holds at a continuous core of M. By considering the cases of a^p for $0 < p < 1$, we deduce the condition $A \preceq_M \theta(A)$ in the sense of *Popa's intertwining theory* [Pop01, Pop03]. Then, by the choice of A, we get a contradiction and the proof is done. Proving $A \preceq_M \theta(A)$ is the key technical step and we do it by the *measurable selection principle*.

Once we get (2), since it is a condition for *every* ψ with $M_\psi' \cap M = \mathbb{C}$, we can use recent results from Marrakchi's work on relative bicentralizers [Mar23] and get (4). Thus, our proof is new even for the Araki–Woods factor R_∞. This is why we can avoid the use of the classification theorem of $\mathrm{Aut}(R_\infty)$ [KST89], which was necessary in all previous works [HS91, HI24] for the type III_1 factor case.

By a result in [AHHM18], we can also prove the equivalence to item (3) of Theorem A, which is a condition on ultraproducts. This condition is interesting in the sense that φ^ω has more information than φ. Indeed, item (3) is a condition for a single φ, while item (2) is one for many ψ.

2. Preliminaries

For a von Neumann algebra M, the L^2-norm with respect to $\varphi \in M_*^+$ is denoted by $\|\cdot\|_\varphi$. We use the notation $M_p := pMp$ for each projection $p \in M \cup M'$.

2.1 Tomita–Takesaki theory and Connes cocycles

Let M be a von Neumann algebra and $\varphi \in M_*^+$ a faithful functional. Throughout the paper, the *modular action* for φ is denoted by $\sigma^\varphi\colon \mathbb{R} \curvearrowright M$. The crossed product $M \rtimes_{\sigma^\varphi} \mathbb{R}$ is called the *continuous core* (with respect to φ) and is denoted by $C_\varphi(M)$. We say that M is a *type* III_1 *factor* if $C_\varphi(M)$ is a factor. The *centralizer algebra* M_φ is the fixed point algebra of the modular action σ^φ. See [Tak03] for details of these objects.

Let $\alpha\colon \mathbb{R} \curvearrowright M$ be a continuous action and $p \in M$ a nonzero projection. We say that a σ-strongly continuous map $u\colon \mathbb{R} \to pM$ is a *generalized cocycle for* α *(with support projection* p*)* if it satisfies

$$u_{s+t} = u_s\alpha_s(u_t), \quad u_su_s^* = p, \quad u_s^*u_s = \alpha_s(p), \quad \text{for all } s, t \in \mathbb{R}.$$

In this case, by putting $\alpha_s^u(pxp) := u_s\alpha_s(pxp)u_s^*$ for all $x \in M$ and $s \in \mathbb{R}$, we have a continuous $\mathbb{R}$-action on pMp.

For $\varphi, \psi \in M_*^+$ with φ faithful and with $s(\psi)$ the support projection of ψ, consider the modular actions σ^φ on M and σ^ψ on $M_{s(\psi)}$. The *Connes cocycle* $(u_t)_{t\in\mathbb{R}}$ (for ψ with respect to φ) [Con72] is a generalized cocycle for σ^φ with support projection $s(\psi)$ such that $(\sigma^\varphi)^u\colon \mathbb{R} \curvearrowright M_{s(\psi)}$ coincides with σ^ψ. We denote it by $u_t = [D\psi : D\varphi]_t$ for $t \in \mathbb{R}$. See [Tak03, VIII.3.19–20] for this nonfaithful version of the Connes cocycle.

Let $\theta = \sigma_t^\varphi$ for some $t \in \mathbb{R}$ and we see that it is pointwise inner. Let $\psi \in M_*^+$ be a positive functional and take a faithful $\widetilde{\psi} \in M_*^+$ with $s(\psi)\widetilde{\psi} = \widetilde{\psi}s(\psi) = \psi$. Then since $s(\psi) \in M_{\widetilde{\psi}}$ and $\sigma_t^{\widetilde{\psi}}(\psi) = \psi$, $u_t := [D\varphi : D\widetilde{\psi}]_t$ satisfies

$$\theta(\psi) = \mathrm{Ad}(u_t) \circ \sigma_t^{\widetilde{\psi}}(s(\psi)\widetilde{\psi}) = u_t\psi u_t^*.$$

Hence θ is pointwise inner.

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By the uniqueness of Connes cocycles, it is straightforward to check that, if φ, ψ are faithful, $v \in M$ a partial isometry with $e := vv^* \in M_\psi$, $v^*v \in M_\varphi$, and $v\varphi v^* = e\psi e$, then for each $x \in M$ and $t \in \mathbb{R}$,

$$v\sigma_t^\varphi(v^*xv)v^* = \sigma_t^\psi(exe), \quad e[D\psi : D\varphi]_t = [De\psi e : D\varphi]_t = v\sigma_t^\varphi(v^*).$$

We prove some lemmas.

Lemma 2.1. *Let $N \subset M$ be an inclusion of σ-finite von Neumann algebras with expectation E_N. Let $\varphi_N \in N_*$ be a faithful tracial state and put $\varphi := \varphi_N \circ E_N$. Let $p, q \in N$ be projections such that $N_p' \cap M_p \subset N_p$. Assume that there is a $*$-isomorphism $\theta\colon M_p \to M_q$ such that $\theta(N_p) = N_q$. We write $\theta_N := \theta|_{N_p}$.*

(1) *We have $\theta^{-1} \circ E_N = E_N \circ \theta^{-1}$ on M_q. In particular,*

$$[Dq\theta(p\varphi p)q : D\varphi]_t = [Dq\theta_N(p\varphi_N p)q : D\varphi_N]_t \in N, \quad \text{for all } t \in \mathbb{R}.$$

(2) *There is a unique nonsingular, positive, self-adjoint element h, which is affiliated with N_q, such that $q\theta(p\varphi p)q = \varphi_h$.*

(3) *Assume that $\theta = \mathrm{Ad}(v) \circ \theta'$ for a $*$-isomorphism $\theta'\colon pMp \to rMr$, a projection $r \in M$, and a partial isometry $v \in qMr$. Then h in (2) satisfies*

$$h^{\mathrm{it}} = v[Dr\theta'(p\varphi p)r : D\varphi]_t\sigma_t^\varphi(v^*), \quad \text{for all } t \in \mathbb{R}.$$

Proof. Recall that a normal conditional expectation from M onto N is unique if $N \subset M$ is irreducible ([Con72, § 1.4], [Tak03, Proposition IX.4.3]). Observe that $N_p \subset M_p$ and $N_q = \theta(N_p) \subset \theta(M_p) = M_q$ are irreducible. In the proof below, for simplicity of notation, we sometimes omit p, for example $q\theta(\varphi)q = q\theta(p\varphi p)q$.

(1) Since $\theta \circ E_N \circ \theta^{-1}\colon M_q \to N_q$ is a normal conditional expectation, it coincides with $E_N|_{M_q}$ by uniqueness. We get $\theta_N^{-1} \circ E_N = E_N \circ \theta^{-1}$ on M_q, and hence

$$q\theta(\varphi)q = q(\varphi_N \circ E_N \circ \theta^{-1})q = q(\varphi_N \circ \theta_N^{-1} \circ E_N)q = q(\theta_N(\varphi_N) \circ E_N)q.$$

Take a faithful $\psi_N \in N_*^+$ with $q\psi_N q = q\theta_N(\varphi_N)q$. Then by [Con72, Lemma 1.4.4], for each $t \in \mathbb{R}$,

$$\begin{aligned}[Dq\theta(\varphi)q : D\varphi]_t &= q[D\psi_N \circ E_N : D\varphi_N \circ E_N]_t \\ &= q[D\psi_N : D\varphi_N]_t = [Dq\theta_N(\varphi_N)q : D\varphi_N]_t \in N.\end{aligned}$$

(2) Since φ_N is a faithful trace on N, there is a unique h, affiliated with N_q, such that $q\theta_N(\varphi_N)q = (\varphi_N)_h$ on N_q. Combined with (1), for each $t \in \mathbb{R}$, we have

$$[Dq\theta(\varphi)q : D\varphi]_t = [Dq\theta_N(\varphi_N)q : D\varphi_N]_t = h^{\mathrm{it}}.$$

Since h is affiliated with M_q, we also have $h^{\mathrm{it}} = [D\varphi_h : D\varphi]_t$, hence we get $q\theta(\varphi)q = \varphi_h$.

(3) Let $\psi \in M_*^+$ be a faithful element such that $r \in M_\psi$ and $r\psi r = r\theta'(\varphi)r$. Then for each $t \in \mathbb{R}$,

$$\begin{aligned}h^{\mathrm{it}} &= [Dq\theta(\varphi)q : D\varphi]_t = [Dv\theta'(\varphi)v^* : D\varphi]_t \\ &= [Dv\psi v^* : D\psi]_t[D\psi : D\varphi]_t \\ &= v\sigma_t^\psi(v^*)[D\psi : D\varphi]_t \\ &= v[D\psi : D\varphi]_t\sigma_t^\varphi(v^*) = v[Dr\psi r : D\varphi]_t\sigma_t^\varphi(v^*),\end{aligned}$$

and we are done. ∎

Lemma 2.2. *Let M be a factor with a faithful state $\varphi \in M_*$ such that $M_\varphi' \cap M = \mathbb{C}$. If $\theta \in \mathrm{Aut}(M)$ satisfies $\theta(M_\varphi) = M_\varphi$, then $\theta(\varphi) = \varphi$. In particular, $M_\varphi \subset M$ is singular in the sense that $uM_\varphi u^* = M_\varphi$ for $u \in \mathcal{U}(M)$ implies $u \in M_\varphi$.*

Proof. Put $N := M_\varphi$ and $\varphi_N := \varphi|_N$. We apply Lemma 2.1(2) for the case $p = q = 1$, and take h such that $\theta(\varphi) = \varphi_h$. Since h is affiliated with N, we have $\varphi_h|_N = (\varphi_N)_h$. Since N is a finite factor and since $\theta(\varphi)|_N$ is a trace, we have $\theta(\varphi)|_N = \varphi_N$ by the uniqueness of the trace. We get $(\varphi_N)_h = \varphi_N$ and $h = 1$. This means $\theta(\varphi) = \varphi$.

If $uM_\varphi u^* = M_\varphi$ for $u \in \mathcal{U}(M)$, we can apply the first part of the lemma to $\mathrm{Ad}(u) \in \mathrm{Aut}(M)$. We get $\mathrm{Ad}(u)(\varphi) = \varphi$ and hence u is contained in M_φ. □

Lemma 2.3. *Let M be a factor with a faithful state $\varphi \in M_*$ such that $M_\varphi' \cap M = \mathbb{C}$. Let $\theta \in \mathrm{Aut}(M)$ and define an embedding*

$$\iota_\theta \colon M \ni x \mapsto \begin{bmatrix} x & 0 \\ 0 & \theta(x) \end{bmatrix} \in M \otimes \mathbb{M}_2.$$

If $\theta(\varphi) = \varphi$, then the following conditions are equivalent:

(1) $\theta|_{M_\varphi}$ *is outer;*
(2) $\iota_\theta(M_\varphi)' \cap (M \otimes \mathbb{M}_2) \subset M \oplus M$.

Proof. If $\theta|_{M_\varphi} = \mathrm{Ad}(u)$ for some $u \in \mathcal{U}(M_\varphi)$, then $\left[\begin{smallmatrix} 0 & 0 \\ u & 0 \end{smallmatrix}\right]$ is contained in $\iota_\theta(M_\varphi)' \cap M \otimes \mathbb{M}_2$.

If (2) does not hold, then there is a nonzero $a \in M$ such that $\theta(x)a = ax$ for all $x \in M_\varphi$. By $M_\varphi' \cap M = \mathbb{C}$, a is a scalar multiple of a unitary, so there is a unitary $u \in \mathcal{U}(M)$ such that $\theta(x) = uxu^*$ for all $x \in M_\varphi$. Then $\mathrm{Ad}(u)$ globally preserves M_φ, hence u is in M_φ by Lemma 2.2. Thus (1) does not hold. □

2.2 Ultraproduct von Neumann algebras

Let M be a σ-finite von Neumann algebra and ω a free ultrafilter on $\mathbb{N}$. Put

$$\mathcal{I}_\omega = \{(x_n)_n \in \ell^\infty(\mathbb{N}, M) \mid x_n \to 0 \text{ $*$-strongly as } n \to \omega\},$$
$$\mathcal{M}^\omega = \{x \in \ell^\infty(\mathbb{N}, M) \mid x\mathcal{I}_\omega \subset \mathcal{I}_\omega \text{ and } \mathcal{I}_\omega x \subset \mathcal{I}_\omega\},$$

where $\ell^\infty(\mathbb{N}, M)$ is the set of all norm bounded sequences in M. The *ultraproduct von Neumann algebra* [Ocn85] is defined as the quotient C*-algebra $M^\omega := \mathcal{M}^\omega/\mathcal{I}_\omega$, which naturally admits a von Neumann algebra structure. The image of $(x_n)_n$ in M^ω is denoted by $(x_n)_\omega$. We have an embedding $M \subset M^\omega$ as constant sequences, and this inclusion is with expectation E_ω given by $E_\omega((x_n)_\omega) = \sigma\text{-weak}\lim_{n\to\omega} x_n$. For every faithful $\varphi \in M_*^+$, we can define a faithful positive functional $\varphi^\omega := \varphi \circ E_\omega$ on M^ω.

Every $\theta \in \mathrm{Aut}(M)$ induces $\theta^\omega \in \mathrm{Aut}(M^\omega)$ by the equation $\theta^\omega((x_n)_\omega) = (\theta(x_n))_\omega$. In particular, the modular action of φ^ω is given by $\sigma_t^{\varphi^\omega} = (\sigma_t^\varphi)^\omega$ for all $t \in \mathbb{R}$ [AH12]. For more on ultraproduct von Neumann algebras, we refer the reader to [Ocn85, AH12].

2.3 Relative bicentralizer algebras

Let $N \subset M$ be an inclusion of von Neumann algebras with separable predual and with expectation E_N. Let $\varphi \in N_*$ be a faithful state and extend it to M by E_N. We define the *asymptotic centralizer* and the *relative bicentralizer algebra* (with respect to φ) as

$$\mathrm{AC}_\varphi(N) := \{(x_n)_n \in \ell^\infty(\mathbb{N}, N) \mid \lim_{n\to\infty} \|x_n\varphi - \varphi x_n\| = 0\},$$

$$\mathrm{BC}_\varphi(N \subset M) := \{x \in M \mid \lim_{n\to\infty} \|x_n x - x x_n\|_\varphi \to 0 \text{ for every } (x_n)_n \in \mathrm{AC}_\varphi(N)\}.$$

This does not depend on the choice of φ up to a canonical isomorphism, if N is a type III_1 factor. In the case $N = M$, we write $\mathrm{BC}_\varphi(M) = \mathrm{BC}_\varphi(N \subset M)$ and we call it the *bicentralizer*. Then Connes' bicentralizer problem asks if every type III_1 factor M with separable predual has trivial bicentralizer, that is, $\mathrm{BC}_\varphi(M) = \mathbb{C}$. This condition is equivalent to having a faithful state $\varphi \in M_*$ such that $M_\varphi' \cap M = \mathbb{C}$ [Haa85]. Here is a convenient expression by ultraproducts ([HI15, Proposition 3.3], [AHHM18, Proposition 3.3]):

$$\mathrm{BC}_\varphi(N \subset M) = (N^\omega)'_{\varphi^\omega} \cap M,$$

where ω is a fixed free ultrafilter on $\mathbb{N}$.

The next theorem explains the importance of relative bicentralizers. This should be understood as a type III counterpart of Popa's work [Pop81].

Theorem 2.4 ([Haa85], [AHHM18, Theorem C]). *Assume that N is a type III_1 factor. If $\mathrm{BC}_\varphi(N \subset M) = N' \cap M$, then there exists an amenable subfactor $R \subset N$ of type II_1 with expectation such that $R' \cap M = N' \cap M$.*

Very recently, Marrakchi obtained a very general and useful criterion for computations of relative bicentralizers. We will need the following statement. Recall that the inclusion $N \subset M$ is *regular* if the normalizer $\mathcal{N}_M(N) := \{u \in \mathcal{U}(M) \mid uNu^* = N\}$ of N in M generates M as a von Neumann algebra.

Theorem 2.5 [Mar23, Theorem E(5)]. *Assume that N is a type III_1 factor. Assume that N has trivial bicentralizer and that $N \subset M$ is regular. If $N' \cap C_\varphi(M) \subset C_\varphi(N)$, then we have $\mathrm{BC}_\varphi(N \subset M) = \mathbb{C}$.*

Proof. By [Mar23, Theorem E(5)], we have

$$\mathrm{BC}_\varphi(N \subset C_\varphi(M)) = L\mathbb{R} \vee (N' \cap C_\varphi(M)) \subset C_\varphi(N),$$

where the relative bicentralizer algebra is defined for arbitrary inclusions. This implies

$$\mathrm{BC}_\varphi(N \subset M) = M \cap \mathrm{BC}_\varphi(N \subset C_\varphi(M)) \subset M \cap C_\varphi(N) = N.$$

We get $\mathrm{BC}_\varphi(N \subset M) \subset \mathrm{BC}_\varphi(N) = \mathbb{C}$. □

2.4 Popa's intertwining theory

We recall Popa's intertwining theory [Pop01, Pop03]. For this, we fix a σ-finite von Neumann algebra M and a finite von Neumann subalgebra $B \subset M$ with expectation E_B. Fix a trace τ_B on B. We can define a canonical trace Tr on the basic construction $\langle M, B\rangle$ satisfying $\mathrm{Tr}(xe_By) = \tau_B \circ E_B(yx)$ for $x, y \in M$ (see for example, [HI15, § 4]).

In this setting, we have the following equivalence. For the proof of this theorem, we refer the reader to [HI15, Theorem 4.3]. Since B is finite, the canonical operator-valued weight T_M from $\langle M, B\rangle$ to M, which appears in [HI15, Theorem 4.3 (6)], is unnecessary.

Theorem 2.6 [Pop01, Pop03]. *Retain the setting of this section. For a finite von Neumann subalgebra $A \subset M$ with expectation, the following conditions are equivalent.*

(1) *We have $A \preceq_M B$. This means that there exist projections $e \in A$, $f \in B$, a partial isometry $v \in eMf$, and a unital normal $*$-homomorphism $\theta\colon eAe \to fBf$ such that $v\theta(a) = av$ for all $a \in eAe$.*
(2) *There exists no net $(u_i)_i$ of unitaries in $\mathcal{U}(A)$ such that for every $a, b \in M$,*

$$\|E_B(b^* u_i a)\|_{\tau_B} \to 0, \quad \text{as } i \to \infty.$$

(3) *There exists a nonzero positive element $d \in A' \cap \langle M, B\rangle$ such that $\mathrm{Tr}(d) < \infty$.*

The next three lemmas are well known to experts. We include short proofs for the reader's convenience.

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Lemma 2.7. *Retain the setting as in Theorem 2.6. Let $A_1, A_2 \subset M$ be finite von Neumann subalgebras with expectation such that A_1 and A_2 are commuting with each other and that $A_1 \vee A_2 \subset M$ is a finite von Neumann subalgebra with expectation.*

Suppose that there are nonzero positive elements $d_i \in A_i' \cap \langle M, B\rangle$ satisfying the condition of item (3) in Theorem 2.6 for $i = 1, 2$ such that $d_1 d_2 \neq 0$. Then we have $A_1 \vee A_2 \preceq_M B$.

Proof. Consider Tr on $\langle M, B\rangle$ as in Theorem 2.6. Put

$$\mathcal{K} := \overline{\mathrm{conv}}^{\sigma\text{-weak}}\{u d_1 u^* \mid u \in \mathcal{U}(A_2)\} \subset A_1' \cap \langle M, B\rangle.$$

By the normality of Tr, every element in $\mathcal{K}$ has finite value in Tr. By regarding $\mathcal{K}$ as a closed subset of $L^2(\langle M, B\rangle, \mathrm{Tr})$ (see [HI15, Lemma 4.4]), take the unique element $d \in \mathcal{K}$ which has the minimum distance from 0. The uniqueness condition implies $udu^* = u$ for all $u \in \mathcal{U}(A_2)$, hence d is contained in $(A_1 \vee A_2)' \cap \langle M, B\rangle$. Then observe that $d_1, d_2 \in L^2(\langle M, B\rangle, \mathrm{Tr})$, and for each $u \in \mathcal{U}(A_2)$,

$$\langle u d_1 u^*, d_2\rangle_{\mathrm{Tr}} = \mathrm{Tr}(d_2 u d_1 u^*) = \mathrm{Tr}(u^* d_2 u d_1) = \mathrm{Tr}(d_2 d_1) = \mathrm{Tr}(d_1^{1/2} d_2 d_1^{1/2}) > 0.$$

This implies that $\mathcal{K}$ does not contain 0, hence $d \neq 0$. This implies $A_1 \vee A_2 \preceq_M B$. □

Lemma 2.8. *Let M be a von Neumann algebra, $\varphi \in M_*$ a faithful state, and $A, B \subset M_\varphi$ von Neumann subalgebras. If $A \not\preceq_M B$, then $A \mathbin{\overline{\otimes}} L\mathbb{R} \not\preceq_{C_\varphi(M)} B \mathbin{\overline{\otimes}} L\mathbb{R}$.*

Proof. Let $E_B \colon M \to B$ be the unique φ-preserving conditional expectation. Then since E_B commutes with σ^φ, we can extend E_B to one from $C_\varphi(M)$ onto $B \mathbin{\overline{\otimes}} L\mathbb{R}$. Take a net $(u_i)_i$ of $\mathcal{U}(A)$ as in item (2) in Theorem 2.6. Then it is easy to see that $(u_i \otimes 1_{L\mathbb{R}})_i$ works for $A \mathbin{\overline{\otimes}} L\mathbb{R} \not\preceq_{C_\varphi(M)} B \mathbin{\overline{\otimes}} L\mathbb{R}$. □

In the lemma below, we need the case that $A, B \subset M$ are *possibly nonunital* finite von Neumann subalgebras, where subalgebras $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ with units $1_A, 1_B$ are assumed to be with expectation. In this case, we can use the same definition for $A \preceq_M B$, and item (2) in Theorem 2.6 works by replacing $a, b \in M$ by $a, b \in M 1_B$. Item (3) is slightly more complicated, but we do not need it. See [HI15, Theorem 4.3] for more on the nonunital case.

Lemma 2.9. *Let $A, B_1, \ldots, B_n \subset M$ be (possibly nonunital) inclusions of σ-finite von Neumann algebras with expectation. Let $E_{B_k} \colon M_{1_{B_k}} \to B_k$ be faithful normal conditional expectations for all k and assume that $A, B_1, \ldots, B_n$ are finite with trace $\tau_k \in (B_k)_*$ for all k. If $A \not\preceq_M B_k$ for all k, then there exists a net $(u_i)_i$ in $\mathcal{U}(A)$ such that for all k,*

$$\|E_{B_k}(b^* u_i a)\|_{\tau_k} \to 0, \quad \text{for all } a, b \in M 1_{B_k}.$$

Proof. Consider $M_n := M \otimes \mathbb{C}^n$ and embeddings

$$\widetilde{A} := A \otimes \mathbb{C}1_{\mathbb{C}^n} \subset M_n, \quad \widetilde{B} := B_1 \oplus \cdots \oplus B_n \subset M_n.$$

Observe that $A \not\preceq_M B_k$ implies $\widetilde{A} \not\preceq_{M_n} B_k$ for all k (use item (2) of Theorem 2.6). Then by [HI15, Remark 4.2(2)], we have that $\widetilde{A} \not\preceq_{M_n} \widetilde{B}$. Then a net of unitaries in item (2) of Theorem 2.6 for $\widetilde{A} \not\preceq_{M_n} \widetilde{B}$ works. □

3. Proof of Theorem A: (1)⇒(2)

We need two lemmas. The first one uses the *measurable selection principle.*

Lemma 3.1. *Let M be a von Neumann algebra with separable predual, φ a faithful normal state, and $\theta \in \mathrm{Aut}(M)$ a pointwise inner automorphism. Then for each positive element $a \in M_\varphi$,*

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there exist a compact subset $K \subset (0,1)$ with positive Lebesgue measure, and a continuous map $K \ni p \mapsto u_p \in \mathcal{U}(M)$ such that

$$\theta(\varphi_{a^p}) = u_p \varphi_{a^p} u_p^*, \quad \text{for all } p \in K,$$

where $\mathcal{U}(M)$ is equipped with the $*$-strong topology.

Proof. Throughout the proof, we consider the $*$-strong topology for $\mathcal{U}(M)$ and we regard it as a Polish space. Set

$$\mathcal{S} := \{(p,u) \in (0,1) \times \mathcal{U}(M) \mid \theta(\varphi_{a^p}) = u\varphi_{a^p}u^*\}.$$

Then it is easy to see that $\mathcal{S}$ is a closed subset. Since θ is pointwise inner, we have $(0,1) = \pi_1(\mathcal{S})$, where π_1 is the projection onto the first coordinate. Hence, with the Lebesgue measure on $(0,1)$, we can apply the measurable selection principle (e.g. [KR97, Theorem 14.3.6]) and find a Lebesgue measurable map

$$\eta\colon (0,1) = \pi_1(\mathcal{S}) \to \mathcal{U}(M)$$

such that $(p, \eta(p)) \in \mathcal{S}$ for all $p \in (0,1)$. Since $\mathcal{U}(M)$ is a Polish space, we can apply Lusin's theorem to η and find a compact subset $K \subset (0,1)$ such that $\eta|_K$ is continuous and that $(0,1) \setminus K$ has arbitrarily small Lebesgue measure. The conclusion follows. □

The second lemma uses the first. This lemma is the key observation of the paper.

Lemma 3.2. *Let M be a von Neumann algebra with separable predual, $\varphi \in M_*$ a faithful state, and $\theta \in \mathrm{Aut}(M)$ a pointwise inner automorphism such that $\theta(\varphi) = \varphi$. Then, for each positive invertible element $a \in M_\varphi$, we have $A \preceq_M \theta(A)$, where $A := \mathrm{W}^*\{a\} \subset M_\varphi$.*

Proof. Observe that $\theta(M_\varphi) = M_\varphi$ and $\theta(A) \subset M_\varphi$. Take $u \in \mathcal{U}(M)$ such that $\varphi_{\theta(a)} = \theta(\varphi_a) = u\varphi_a u^*$. Then since

$$[D\varphi_{\theta(a)} : D\varphi_a]_t = [D\varphi_{\theta(a)} : D\varphi]_t [D\varphi : D\varphi_a]_t = \theta(a^{\mathrm{i}t})a^{-\mathrm{i}t}$$

and

$$[Du\varphi_a u^* : D\varphi_a]_t = u\sigma_t^{\varphi_a}(u^*) = ua^{\mathrm{i}t}\sigma_t^{\varphi}(u^*)a^{-\mathrm{i}t},$$

we get $\theta(a^{\mathrm{i}t}) = ua^{\mathrm{i}t}\sigma_t^{\varphi}(u^*)$ for all $t \in \mathbb{R}$. Define

$$\widetilde{A} := \mathrm{W}^*\{a^{\mathrm{i}t} \otimes \lambda_t \mid t \in \mathbb{R}\} \subset C_\varphi(M),$$

where λ_t denotes the left regular representation on $L^2(\mathbb{R})$, and observe that $\widetilde{A} \subset C_\varphi(M)$ is with expectation. Then it is easy to see that $\mathrm{Ad}(u)$ restricts to a $*$-homomorphism from $\widetilde{A}$ into $\theta(A) \mathbin{\overline{\otimes}} L\mathbb{R}$, so that we get $\widetilde{A} \preceq_{C_\varphi(M)} \theta(A) \mathbin{\overline{\otimes}} L\mathbb{R}$. More precisely, if we denote by e the Jones projection for a fixed faithful normal conditional expectation from $C_\varphi(M)$ onto $\theta(A) \mathbin{\overline{\otimes}} L\mathbb{R}$, then the projection u^*eu satisfies the condition of item (3) in Theorem 2.6.

Fix $0 < p < 1$, and we consider the positive invertible element a^p. Then, by the same reasoning as above, we have that

$$\widetilde{A}_p \preceq_{C_\varphi(M)} \theta(A) \mathbin{\overline{\otimes}} L\mathbb{R}, \quad \text{where } \widetilde{A}_p := \mathrm{W}^*\{a^{\mathrm{i}pt} \otimes \lambda_t \mid t \in \mathbb{R}\},$$

together with the projection $u_p^* e u_p$, where u_p is a unitary satisfying $\varphi_{\theta(a^p)} = u_p \varphi_{a^p} u_p^*$. We note that the algebra $\theta(A) \mathbin{\overline{\otimes}} L\mathbb{R}$ does not depend on p.

Now we claim that $A \mathbin{\overline{\otimes}} L\mathbb{R} \preceq_{C_\varphi(M)} \theta(A) \mathbin{\overline{\otimes}} L\mathbb{R}$. To see this, we first apply Lemma 3.1 and find a compact $K \subset (0,1)$ with positive measure and a continuous map $K \ni p \mapsto u_p$. Then we take a sequence $p_n \in K$ such that p_n converges to $p \in K$ and $p \notin \{p_n\}_n$. Then $u_{p_n} \to u_p$

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in the $*$-strong topology, hence

$$d_{p_n} := u_{p_n}^* e u_{p_n} \to u_p^* e u_p =: d_p$$

as well. In particular, we can find $p_n =: q$ such that $q \neq p$ and $d_q d_p \neq 0$. Observe that $\widetilde{A}_p \vee \widetilde{A}_q = A \mathbin{\overline{\otimes}} L\mathbb{R}$, which is a finite von Neumann subalgebra in $C_\varphi(M)$ with expectation. Then by Lemma 2.7, we have that

$$A \mathbin{\overline{\otimes}} L\mathbb{R} = \widetilde{A}_p \vee \widetilde{A}_q \preceq_{C_\varphi(M)} \theta(A) \mathbin{\overline{\otimes}} L\mathbb{R}.$$

Hence the claim is proven.

Finally, this condition implies $A \preceq_M \theta(A)$ by Lemma 2.8. □

Now we prove (1)⇒(2) of Theorem A. For the proof, we need Popa's $\mathcal{G}$-singular maximal abelian $*$-subalgebra (MASA) technique for type III_1 factors; see Theorem A.1.

Proof of Theorem A: (1)⇒(2). Let $\psi \in M_*^+$ be a faithful state such that $M_\psi' \cap M = \mathbb{C}$. Since θ is pointwise inner, there is $v \in \mathcal{U}(M)$ such that $\mathrm{Ad}(v) \circ \theta(\psi) = \psi$. So up to replacing $\mathrm{Ad}(v) \circ \theta$ by θ, we may assume $\theta(\psi) = \psi$. Then we show that $\theta|_{M_\psi}$ is inner.

Suppose by contradiction that θ is not inner on M_ψ. Then, putting $\mathcal{G} := \{\mathrm{id}_M,\ \theta^{-1}\}$, we apply Theorem A.1 and take a $\mathcal{G}$-singular MASA $A \subset M_\psi$. Take a generator a of A which is positive and invertible. We can apply Lemma 3.2 to a and get $A \preceq_M \theta(A)$. Since A and $\theta(A)$ are MASAs in M, by [HV12, Theorem 2.5], one can find a nonzero partial isometry $v \in M$ such that

$$v^* v \in A, \quad vv^* \in \theta(A), \quad vAv^* = \theta(A) vv^*.$$

Thus $\mathrm{Ad}(v)$ restricts to a $*$-isomorphism from Av^*v onto $\theta(A)vv^*$. Since A is diffuse, up to exchanging v^*v with a smaller projection in A (or vv^* with one in $\theta(A)$) if necessary, we may assume $v^*v \neq 1$ and $vv^* \neq 1$. Then since M is a type III factor, $1 - v^*v$ and $1 - vv^*$ are equivalent, so we can take a partial isometry $w \in M$ such that $w^*w = 1 - v^*v$ and $ww^* = 1 - vv^*$. Then $\widetilde{v} := v + w \in M$ is a unitary element satisfying $\widetilde{v}p = v$ where $p := v^*v$, so that

$$\widetilde{v} A p \widetilde{v}^* = vAv^* \subset \theta(A).$$

This means $\mathrm{Ad}(\theta^{-1}(\widetilde{v})) \circ \theta^{-1}(Ap) \subset A$. By $\mathcal{G}$-singularity, we have that $\theta^{-1} = \mathrm{id}_M$, a contradiction. □

4. Proof of Theorem A: (2)⇒(3) and (3)⇒(4)

Proof of Theorem A: (2)⇒(3). We may assume that θ is outer on M. Set $M_2 := M \otimes \mathbb{M}_2$ and consider an embedding $\iota_\theta \colon M \to M_2$ with expectation as in Lemma 2.3. Since θ is outer, we have $M' \cap M_2 = \mathbb{C} \oplus \mathbb{C}$. We consider the relative bicentralizer

$$\mathrm{BC}_\varphi(M \subset M_2) = (M^\omega)'_{\varphi^\omega} \cap M_2.$$

We claim $\mathrm{BC}_\varphi(M \subset M_2) \neq M' \cap M_2$.

To see this, suppose by contradiction that $\mathrm{BC}_\varphi(M \subset M_2) = M' \cap M_2$. Then by Theorem 2.4, there is an amenable II_1 factor $R \subset M$ with expectation E_R such that $R' \cap M_2 = M' \cap M_2 = \mathbb{C} \oplus \mathbb{C}$. Put $\psi := \tau_R \circ E_R$, where τ_R is the trace on R, and observe that $R \subset M_\psi \subset M$. Then since $M_\psi' \cap M \subset R' \cap M = \mathbb{C}$ (this follows by $R' \cap M_2 = \mathbb{C} \oplus \mathbb{C}$), by assumption, there is $u \in \mathcal{U}(M)$ such that $\mathrm{Ad}(u) \circ \theta|_{M_\psi} = \mathrm{id}_{M_\psi}$. Then $\left[\begin{smallmatrix} 0 & u \\ 0 & 0 \end{smallmatrix}\right]$ is contained in $M_\psi' \cap M_2 \subset R' \cap M_2 = \mathbb{C} \oplus \mathbb{C}$. This is a contradiction and the claim is proven.

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We take $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in (M^\omega)'_{\varphi^\omega} \cap M_2$, which is not contained in $\mathbb{C} \oplus \mathbb{C}$. Then since $(M^\omega)'_{\varphi^\omega} \cap M = \mathrm{BC}_\varphi(M) = \mathbb{C}$, we have $a, d \in \mathbb{C}$, hence $b \neq 0$ or $c \neq 0$. By taking the adjoint if necessary, we may assume $c \neq 0$. Then observe that $\theta^\omega(x)c = cx$ for all $x \in (M^\omega)_{\varphi^\omega}$. Since $(M^\omega)'_{\varphi^\omega} \cap M = \mathbb{C}$, by the polar decomposition, we can replace c by a unitary element. This means (3) holds. □

Proof Theorem A: (3)⇒(4). By assumption, take $u \in \mathcal{U}(M)$ such that $\mathrm{Ad}(u) \circ \theta^\omega = \mathrm{id}$ on $(M^\omega)_{\varphi^\omega}$. Up to replacing $\mathrm{Ad}(u) \circ \theta$ by θ, we may assume $u = 1$. Then by Lemma 2.2, we have that $\theta(\varphi) = \varphi$. In particular, θ commutes with the modular action of φ.

We set

$$H(\theta) := \{n \in \mathbb{Z} \mid \theta^n = \mathrm{Ad}(u) \circ \sigma_t^\varphi \text{ for some } u \in \mathcal{U}(M),\ t \in \mathbb{R}\}.$$

Observe that by $M'_\varphi \cap M = \mathbb{C}$ and $\theta|_{M_\varphi} = \mathrm{id}$, the equality $\theta^n = \mathrm{Ad}(u) \circ \sigma_t^\varphi$ implies $u \in \mathbb{C}$, so we can actually remove $u \in \mathcal{U}(M)$ in the definition of $H(\theta)$. Since $H(\theta)$ is a subgroup in $\mathbb{Z}$, we can write $H(\theta) = k\mathbb{Z}$ for some $k \geq 0$.

Assume that $k > 0$ and $\theta^k = \sigma_t^\varphi$. Then, putting $\widetilde{\theta} := \sigma_{-t/k}^\varphi \circ \theta$, observe that $H(\widetilde{\theta}) = H(\theta)$, $\widetilde{\theta}^k = \mathrm{id}$, and $\widetilde{\theta}^\omega = \mathrm{id}$ on $(M^\omega)_{\varphi^\omega}$. Hence up to replacing $\widetilde{\theta}$ by θ, we may assume $\theta^k = \mathrm{id}$. If $k = 0$, we do not need this replacement.

For each $k \geq 0$, set $G := \mathbb{Z}/k\mathbb{Z}$ and define an outer action $G \curvearrowright M$ given by $\{\theta^n\}_{n \in \mathbb{Z}}$. We denote this action again by θ. Put $\widetilde{M} := M \rtimes_\theta G$ with canonical expectation $E_M \colon \widetilde{M} \to M$ and note that $M' \cap \widetilde{M} = \mathbb{C}$ by the outerness of θ. We extend φ to $\widetilde{M}$ by E_M. Since θ commutes with the modular action of φ, the continuous core of $\widetilde{M}$ is such that

$$C_\varphi(\widetilde{M}) = M \rtimes_{\theta \times \sigma^\varphi} (G \times \mathbb{R}) = C_\varphi(M) \rtimes_\theta G,$$

where $\theta \colon G \curvearrowright C_\varphi(M)$ on the right-hand side is the canonical extension.

The next claim is important to us.

Claim. *We have $M' \cap C_\varphi(\widetilde{M}) = \mathbb{C}$.*

Proof of Claim. Since the canonical extension $\theta \colon G \curvearrowright C_\varphi(M)$ is still outer (e.g. [Tak03, Lemma XII.6.14]), the crossed product $C_\varphi(M) \rtimes_\theta G$ is a factor. Let $E := \varphi \otimes \mathrm{id} \colon \mathbb{B}(L^2(M)) \mathbin{\overline{\otimes}} \mathbb{B}(L^2(G \times \mathbb{R})) \to \mathbb{B}(L^2(G \times \mathbb{R}))$, where φ is extended to a normal state on $\mathbb{B}(L^2(M))$, and observe that it restricts to a conditional expectation $M \rtimes (G \times \mathbb{R}) \to L(G \times \mathbb{R})$ (because θ and σ^φ preserve φ).

Since the $G \times \mathbb{R}$ action is trivial on M_φ, we have that

$$M'_\varphi \cap C_\varphi(\widetilde{M}) \subset (M'_\varphi \cap M) \mathbin{\overline{\otimes}} \mathbb{B}(L^2(G \times \mathbb{R})) = \mathbb{C} \mathbin{\overline{\otimes}} \mathbb{B}(L^2(G \times \mathbb{R})).$$

We have $E(x) = x$ for all $x \in M'_\varphi \cap C_\varphi(\widetilde{M})$, so that $M'_\varphi \cap C_\varphi(\widetilde{M}) = L(G \times \mathbb{R})$. Then since $G \times \mathbb{R}$ is abelian,

$$M' \cap C_\varphi(\widetilde{M}) = M' \cap L(G \times \mathbb{R}) \subset \mathcal{Z}(C_\varphi(\widetilde{M})).$$

Since $C_\varphi(\widetilde{M})$ is a factor, the conclusion follows. □

To finish the proof, we have to show $k = 1$. For this, suppose by contradiction that $k \neq 1$ and $G = \mathbb{Z}/k\mathbb{Z}$ is a nontrivial group. Then observe that the inclusion $M \subset \widetilde{M}$ is with expectation, regular, $\mathrm{BC}_\varphi(M) = \mathbb{C}$, and $M' \cap C_\varphi(\widetilde{M}) \subset C_\varphi(M)$. Hence we can apply Theorem 2.5 and get that $\mathrm{BC}_\varphi(M \subset \widetilde{M}) = \mathbb{C}$. Since G is nontrivial, $1 \in G$ (the image of $1 \in \mathbb{Z}$ in G) is nontrivial. We denote by $\lambda_1^G \in LG$ the canonical unitary corresponding to $1 \in G$. Then since $\theta^\omega = \mathrm{id}$ on

$(M^\omega)_{\varphi^\omega}$ and since $\mathrm{Ad}(\lambda_1^G) \in \mathrm{Aut}(\widetilde{M})$ coincides with θ on M, we get

$$\lambda_1^G \in (M^\omega)'_{\varphi^\omega} \cap \widetilde{M} = B_\varphi(M \subset \widetilde{M}) = \mathbb{C}.$$

This is a contradiction because $1 \in G$ is nontrivial. Thus we get $k = 1$ and this finishes the proof. □

Appendix A. Popa's $\mathcal{G}$-singular MASAs in type $\mathrm{III_1}$ factors

The following theorem is an analogue of [Pop83, Theorem 4.2] in the type III setting. It covers the case that M is a type II_1 factor, a type III_λ factor $(0 < \lambda < 1)$, or a type III_1 factor with trivial bicentralizer. The proof is a rather straightforward adaptation of [Pop16, HP17], so we merely sketch it.

Theorem A.1. *Let M be a factor with separable predual and $\varphi \in M_*$ a faithful state such that $M'_\varphi \cap M = \mathbb{C}$. Let $\{\theta_n\}_{n\geq 1} \subset \mathrm{Aut}(M)$ be a countable subset such that*

$$\theta_n(\varphi) = \varphi \quad \text{and} \quad \theta_n|_{M_\varphi} \not\in \mathrm{Int}(M_\varphi)$$

for all $n \geq 1$. Put $\theta_0 := \mathrm{id}_M$ and $\mathcal{G} := \{\theta_n\}_{n\geq 0}$.

Then there is an abelian von Neumann subalgebra $A \subset M_\varphi$ such that A is a MASA in M and that A is $\mathcal{G}$-singular in the following sense: if $\theta \in \mathcal{G}$ satisfies

$$\mathrm{Ad}(u) \circ \theta(Ap) \subset A$$

for some nonzero projection $p \in A$ and $u \in \mathcal{U}(M)$, then $\theta = \mathrm{id}_M$ and $up = pu \in A$.

We note that, since id_M is contained in $\mathcal{G}$, the resulting MASA $A \subset M$ is *singular* in the sense that $uAu^* = A$ for $u \in \mathcal{U}(M)$ implies $u \in A$.

The next lemma is a variant of [Pop16, Lemma 1.2.2].

Lemma A.2. *Let M be a von Neumann algebra, $\varphi \in M_*$ a faithful normal state, and $N \subset M_\varphi$ a diffuse von Neumann subalgebra. Then for every finite-dimensional abelian von Neumann subalgebra $D \subset N$, finite subset $X \subset M$, and $\varepsilon > 0$, there exists a finite-dimensional abelian von Neumann subalgebra $A \subset N$ containing D such that*

$$\|E_{A'\cap M}(x) - E_{A\vee(N'\cap M)}(x)\|_\varphi < \varepsilon, \quad \text{for all } x \in X.$$

Proof. If M is a finite von Neumann algebra, this is an equivalent form of the last statement of [Pop16, Lemma 1.2.2]. For a general M, since N is contained in M_φ, we can use [Pop95, Theorem A.1.2] and then follow the proof of [Pop16, Lemma 1.2.2]. □

Proof of Theorem A.1. We fix the following setting.

- Let M be a von Neumann algebra with separable predual, $\varphi \in M_*$ a faithful state, and $N \subset M_\varphi$ a type II von Neumann subalgebra such that $N' \cap M \subset N$. Set

$$\mathrm{Aut}_\varphi(N \subset M) := \{\theta \in \mathrm{Aut}(M) \mid \theta(\varphi) = \varphi,\ \theta(N) = N\}.$$

- Let $\{\theta_n\}_{n\geq 1} \subset \mathrm{Aut}_\varphi(N \subset M)$ be such that

$$\iota_n(N)' \cap (M \otimes \mathbb{M}_2) \subset M \oplus M, \quad \text{for all } n \geq 1,$$

  where $\iota_n := \iota_{\theta_n}$ is as in Lemma 2.3.

We will find $A \subset N$ that satisfies the conclusion of Theorem A.1 for $\mathcal{G} = \{\theta_n\}_{n\geq 0}$ with $\theta_0 = \mathrm{id}$. Thanks to Lemma 2.3, this gives the proof of Theorem A.1.

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For a von Neumann subalgebra $B \subset M$ that is globally invariant by σ^φ, we denote by $E_B \colon M \to B$ the unique φ-preserving conditional expectation, and by e_N the corresponding Jones projection.

Lemma A.3. *Retain the setting of the above proof. We fix a finite-dimensional abelian von Neumann subalgebra $A \subset N$, a projection $f \in A' \cap N$, and finite subsets $X \subset M$ and $Y \subset \mathrm{Aut}_\varphi(N \subset M)$. Then, for each $\varepsilon > 0$, there exist a finite-dimensional abelian von Neumann subalgebra $D \subset N$, which contains A and f, and $v \in \mathcal{U}(Df)$ such that*

$$\|E_{D'\cap M}(y^*\theta(v)x)f^\perp\|_\varphi < \varepsilon \quad \textit{for all } x, y \in X \textit{ and } \theta \in Y.$$

Proof. Since $A \vee \mathbb{C}f \subset N$ is finite-dimensional, by [Pop81, Theorem 3.3] (see also [HP17, Lemma 2.2]), there is a commutative von Neumann subalgebra $B \subset N$ that is a MASA in M and that contains $A \vee \mathbb{C}f$. We can write $B = \bigvee_k B_k$, where $\{B_k\}_k$ is an increasing sequence of finite-dimensional von Neumann algebras such that $B_0 := A \vee \mathbb{C}f$.

Since $A_f' \cap N_f$ is of type II and B is of type I, we have

$$A_f' \cap N_f \not\preceq_M \theta^{-1}(B_{f^\perp}), \quad \text{for all } \theta \in Y.$$

By Lemma 2.9, take $v \in \mathcal{U}(A_f' \cap N_f)$ such that for all $\theta \in Y$,

$$\|E_B(f^\perp y^*\theta(v)xf^\perp)\|_\varphi = \|E_{\theta^{-1}(Bf^\perp)}(\theta^{-1}(f^\perp y^*)v\theta^{-1}(xf^\perp))\|_\varphi < \varepsilon, \quad \text{for all } x, y \in X.$$

By approximating v by finite sums of projections, we may assume that v is contained in a finite-dimensional abelian subalgebra in $A_f' \cap N_f$. Hence there exists a finite-dimensional commutative von Neumann algebra $D_1 \subset N_f$ containing v and A_f.

Since $e_{B_k'\cap M} \to e_{B'\cap M} = e_B$ strongly, there exists k such that

$$\|E_{B_k'\cap M}(y^*\theta(v)x)f^\perp\|_\varphi = \|E_{B_k'\cap M}(f^\perp y^*\theta(v)xf^\perp)\|_\varphi < \varepsilon, \quad \text{for all } x, y \in X, \quad \theta \in Y.$$

Here we can exchange $E_{B_k'\cap M}$ by $E_{D_2'\cap M_{f^\perp}}$, where $D_2 := (B_k)_{f^\perp}$. Then $D := D_1 \oplus D_2 \subset N_f \oplus N_{f^\perp} \subset N$ works. □

Take a $*$-strongly dense subset $\{x_n\}_{n\in\mathbb{N}} \subset (M)_1$. Take projections $\{e_n\}_{n\in\mathbb{N}}$ in N which are $*$-strongly dense in all projections in N. We may assume that $e_0 = x_0 = 1$ and that each e_k appears infinitely many times in $\{e_n\}_{n\in\mathbb{N}}$.

For each $n > 0$, we denote the inclusion $\iota_n(M) \subset M \otimes \mathbb{M}_2$ by $M \subset M_n$. For $n = 0$, we put $M_0 := M$.

Claim. *There exist an increasing sequence of finite-dimensional von Neumann abelian subalgebras $A_n \subset N$, projections $f_n \in A_n$, and unitaries $v_n \in \mathcal{U}(A_nf_n)$ such that*

- (P1) $\|f_n - e_n\|_\varphi \le 7\|e_n - E_{A_{n-1}'\cap N}(e_n)\|_\varphi$,
- (P2) $\|E_{A_n'\cap M}(x_i^*\theta_k(v_n)x_j)f_n^\perp\|_\varphi \le 1/n$ *for all* $0 \le i, j, k \le n$,
- (P3) $\|E_{A_n'\cap M}(x_j) - E_{A_n}(x_j)\|_\varphi \le 1/n$ *for all* $0 \le j \le n$,
- (P4) $\|E_{A_n'\cap M_k}(x_j) - E_{A_n\vee(N'\cap M_k)}(x_j)\|_\varphi \le 1/n$ *for all* $0 \le j \le n$ *and* $1 \le k \le n$.

Proof. Suppose that (A_k, f_k, v_k) are constructed for $0 \le k \le n$, and we will construct one for $k = n+1$. Following the proof of the first part of [HP17, Theorem 1], by using Lemma A.3, one can construct $(A_{n+1}, f_{n+1}, v_{n+1})$ that satisfies (P1), (P2), and (P3). To see (P4), we put $D_{n+1} := A_{n+1}$, and below we construct A_{n+1} that satisfies all the desired conditions.

We apply Lemma A.2 and find an abelian von Neumann subalgebra $B^0_{n+1} \subset N$ containing D_{n+1} such that

$$\|E_{(B^0_{n+1})' \cap M_0}(x_j) - E_{B^0_{n+1} \vee (N' \cap M_0)}(x_j)\|_\varphi \leq \frac{1}{n+1}, \quad \text{for all } 0 \leq j \leq n+1.$$

Next we again apply Lemma A.2 and find an abelian von Neumann subalgebra $B^1_{n+1} \subset N$ containing B^0_{n+1} such that the same inequality holds for the inclusion $B^1_{n+1} \subset N \subset M_1$. We repeat this procedure and construct $B^k_{n+1} \subset N$ containing B^{k-1}_{n+1} such that

$$\|E_{(B^k_{n+1})' \cap M_k}(x_j) - E_{B^k_{n+1} \vee (N' \cap M_k)}(x_j)\|_\varphi \leq \frac{1}{n+1}, \quad \text{for all } 0 \leq j, k \leq n+1.$$

Put $A_{n+1} := B^{n+1}_{n+1}$. Then since the corresponding Jones projections satisfy

$$e_{(B^k_{n+1})' \cap M_k} - e_{B^k_{n+1} \vee (N' \cap M_k)} \geq e_{A'_{n+1} \cap M_k} - e_{A_{n+1} \vee (N' \cap M_k)}$$

for all $0 \leq k \leq n+1$, A_{n+1} satisfies (P4).

We next check (P1), (P2), and (P3) for this new A_{n+1}. We do not use A_{n+1} in (P1), hence it automatically holds. Regarding (P2) and (P3), since $D_{n+1} \subset A_{n+1}$, one has

$$e_{A'_{n+1} \cap M} \leq e_{D'_{n+1} \cap M} \quad \text{and} \quad e_{A'_{n+1} \cap M} - e_{A_{n+1}} \leq e_{D'_{n+1} \cap M} - e_{D_{n+1}},$$

so that (P2) and (P3) still hold for A_{n+1}. □

Take A_n as in the claim, and define $A := \bigvee_n A_n \subset N$. Then (P3) and (P4) imply that A is a MASA in M and that

$$A' \cap M_n = A \vee (N' \cap M_n) \subset M \oplus M \quad \text{for all } n \geq 1.$$

We will show that A satisfies the conclusion of Theorem A.1.

Suppose now that there exist $\theta \in \mathcal{G}$, $u \in \mathcal{U}(M)$, and a nonzero projection $p \in A$ such that

$$\theta^u(Ap) \subset A, \quad \text{where } \theta^u := \mathrm{Ad}(u) \circ \theta.$$

Since $A, \theta^u(A)$ are MASAs in M, putting $q := \theta^u(p) \in A$, we have a $*$-isomorphism

$$\theta^u \colon pMp \to qMq \quad \text{such that } \theta^u(Ap) = Aq.$$

Assume first $\theta = \mathrm{id}$ and $\theta^u|_{Ap} = \mathrm{id}_{Ap}$. Then since A is a MASA in M, we get $up = pu \in A$, which is the conclusion. So from now on, we assume either that $\theta \neq \mathrm{id}$, or $\theta = \mathrm{id}$ and $\theta^u|_{Ap} \neq \mathrm{id}_{Ap}$. Then we will deduce a contradiction.

Claim. *There exists a nonzero projection $z \in Ap$ such that $\theta^u(z)z = 0$.*

Proof. If $\theta = \theta_n$ $(n \geq 1)$ and $\theta^u|_{Ap} = \mathrm{id}_{Ap}$, then $\left[\begin{smallmatrix} 0 & pu \\ 0 & 0 \end{smallmatrix}\right]$ is contained in $A' \cap M_n \subset M \oplus M$, hence we have a contradiction. So we have $\theta^u|_{Ap} \neq \mathrm{id}_{Ap}$ in this case. The same holds if $\theta = \theta_0 = \mathrm{id}_M$ by assumption.

Put $q := \theta^u(p)$. If $p = q$, then θ^u defines a nontrivial automorphism on Ap, so that $z \in Ap$ exists. If $p \neq q$, then, putting $z_0 := p - pq \in Ap$ and $w_0 := q - pq \in Aq$, we have $z_0 \neq 0$ or $w_0 \neq 0$. If $z_0 \neq 0$, then $z := z_0$ works. If $w_0 \neq 0$, then $z := (\theta^u)^{-1}(w_0)$ works. □

We apply Lemma 2.1(3) to $A \subset M$ and $(\theta^u)^{-1} \colon qMq \to pMp$ with $(\theta^u)^{-1}(Aq) = Ap$. By using $(\theta^u)^{-1} = \mathrm{Ad}(\theta^{-1}(u^*)) \circ \theta^{-1}$ and $\theta(\varphi) = \varphi$, there is a unique nonsingular positive self-adjoint element h, which is affiliated with Ap, such that

$$p(\varphi \circ \theta^u)p = \varphi_h, \quad \sigma^\varphi_t(u\theta(p)) = u\theta(h^{\mathrm{i}t}), \quad t \in \mathbb{R}.$$

There exists a spectral projection $p_0 \in Ap$ of h such that

$$p_0 z \neq 0, \quad \delta p_0 \leq h p_0 \leq \delta^{-1} p_0 \quad \text{for some } \delta > 0.$$

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To deduce a contradiction, up to replacing p, z, q by $p_0, zp_0, \theta^u(p_0)$, we may assume $p = p_0$ (while still $z \leq p$). Then $u\theta(z)$ is a partial isometry, for which there are $\kappa_1, \kappa_2 > 0$ such that

$$\|xu\theta(z)\|_\varphi \leq \kappa_1\|x\|_\varphi, \quad \|x(u\theta(z))^*\|_\varphi \leq \kappa_2\|x\|_\varphi, \quad \text{for all } x \in M.$$

CLAIM. *The following assertions hold.*

(a) $$\|zf_n\|_\varphi = \|\theta(v_n z)\|_\varphi \leq \kappa_1\|E_{A_n'\cap M}(u\theta(v_n z)u^*)z^\perp\|_\varphi, \quad \textit{for all } n \in \mathbb{N};$$

(b) $$\|E_{A_n'\cap M}(u\theta(v_n z)u^*)z^\perp\|_\varphi \leq \|E_{A_n'\cap M}(x\theta(v_n)y)z^\perp\|_\varphi + \kappa_2\|u - x\|_\varphi + \|x\|\|\theta(z)u^* - y\|_\varphi, \quad \textit{for all } x, y \in M \textit{ and } n \in \mathbb{N};$$

(c) $$\|e_n - f_n\|_\varphi \leq 7\|e_n - z\|_\varphi, \quad \textit{for all } n \in \mathbb{N}.$$

Proof. We have only to follow arguments in [HP17, Theorem 1], but we give proofs for the reader's convenience. We explain only (a) and (c), since (b) is straightforward.

For (a), since $zf_n = f_n z$, $v_n^* v_n = f_n$ and $\theta(\varphi) = \varphi$,

$$\|f_n z\|_\varphi = \|\theta(v_n z)\|_\varphi = \|u^* u\theta(v_n z)u^* u\theta(z)\|_\varphi \leq \kappa_1\|u\theta(v_n z)u^*\|_\varphi.$$

Then since $u\theta(v_n z)u^* = \theta^u(v_n z)\theta^u(z)$, $z\theta^u(z) = 0$, and $\theta^u(v_n z) \in Aq \subset A' \cap M$, one has

$$\|u\theta(v_n z)u^*\|_\varphi = \|u\theta(v_n z)u^* z^\perp\|_\varphi = \|E_{A_n'\cap M}(u\theta(v_n z)u^*)z^\perp\|_\varphi.$$

We thus have (a).

For (c), since $z \in A \subset A_{n-1}' \cap N$, combined with (P1), one has

$$\|e_n - f_n\|_\varphi \leq 7\|e_n - E_{A_{n-1}'\cap N}(e_n)\|_\varphi = 7\|(e_n - z) - E_{A_{n-1}'\cap N}(e_n - z)\|_\varphi \leq 7\|e_n - z\|_\varphi.$$

We thus have (c). □

Take a subsequence $\{n_k\}_k$ such that $e_{n_k} \to z$ strongly. Then (c) implies that $e_{n_k} - f_{n_k}$ converges to 0 strongly, so that $zf_{n_k} \to z$ as well. We show that zf_{n_k} also converges to 0, a contradiction.

Fix $\varepsilon > 0$ and take x_i, x_j such that $\kappa_2\|u - x_i^*\|_\varphi + \|\theta(z)u^* - x_j\|_\varphi < \varepsilon$. Then by (a) and (b),

$$\begin{aligned}
\kappa_1^{-1}\|zf_n\|_\varphi &\leq \|E_{A_n'\cap M}(u\theta(v_n z)u^*)z^\perp\|_\varphi \\
&\leq \|E_{A_n'\cap M}(x_i^*\theta(v_n)x_j)z^\perp\|_\varphi + \varepsilon \\
&\leq \|E_{A_n'\cap M}(x_i^*\theta(v_n)x_j)f_n^\perp\|_\varphi + \|z^\perp - f_n^\perp\|_\varphi + \varepsilon.
\end{aligned}$$

Then (P2) implies

$$\limsup_k \|zf_{n_k}\|_\varphi \leq \kappa_1 \limsup_k(\|E_{A_{n_k}'\cap M}(x_i^*\theta(v_{n_k})x_j)f_{n_k}^\perp\|_\varphi + \|z^\perp - f_{n_k}^\perp\|_\varphi) + \kappa_1\varepsilon = \kappa_1\varepsilon.$$

Letting $\varepsilon \to 0$, we get $zf_{n_k} \to 0$ strongly, as desired. □

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CONFLICTS OF INTEREST
None

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