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A RECOLLEMENT APPROACH TO GEIGLE–LENZING WEIGHTED PROJECTIVE VARIETIES

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Abstract. We introduce a new method for expanding an abelian category and study it using recollements. In particular, we give a criterion for the existence of cotilting objects. We show, using techniques from noncommutative algebraic geometry, that our construction encompasses the category of coherent sheaves on Geigle–Lenzing weighted projective lines. We apply our construction to some concrete examples and obtain new weighted projective varieties, and analyze the endomorphism algebras of their tilting bundles.

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§1. Introduction

In their famous paper [GL], Geigle and Lenzing introduced an important class of abelian categories with a tilting object (see Definition 2.4), which have subsequently been called *coherent sheaves on Geigle–Lenzing (GL) weighted projective lines*. This category has played an important role in many fields, in particular representation theory of finite-dimensional algebras. It was recently generalized in [HIMO] to include higher-dimensional

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projective spaces. A different interpretation of these categories was discovered in [CI, RVdB] for the dimension-1 case and more generally in [IL], where these categories are shown to be equivalent to module categories $\mathsf{mod}\, A$ of a certain order A on $\mathbb{P}^d$, which we call a *GL order* (see below). Both interpretations build on Artin and Zhang's theory of noncommutative projective schemes [AZ].

Viewing GL weighted projective spaces as module categories allows for further, very fruitful, generalizations which is what we explore in this paper. The idea is rather simple: in [IL], all GL orders that were considered were always sheaves on $\mathbb{P}^d$, now we allow the center to be other varieties.

DEFINITION 1.1. Fix a scheme X over a field k, and for $i = 1, \ldots, n$ fix prime divisors L_i on X and integer weights $p_i \geqslant 2$. The *Geigle–Lenzing order* A (GL order, for short) with *center* X associated to this data is the sheaf of noncommutative algebras

$$A = \bigotimes_{i=1}^{n} A_i, \quad \text{where } A_i := \begin{bmatrix} \mathcal{O} & \mathcal{O}(-L_i) & \ldots & \mathcal{O}(-L_i) & \mathcal{O}(-L_i) \\ \mathcal{O} & \mathcal{O} & \ldots & \mathcal{O}(-L_i) & \mathcal{O}(-L_i) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \ldots & \mathcal{O} & \mathcal{O}(-L_i) \\ \mathcal{O} & \mathcal{O} & \ldots & \mathcal{O} & \mathcal{O} \end{bmatrix} \subset \mathcal{O}^{p_i \times p_i}$$

and $\mathcal{O} = \mathcal{O}_X$.

The aim of this paper is to study the category $\mathsf{mod}\, A$ of GL orders A. In particular, we give a criterion on the existence of tilting sheaves. First, we give a description of $\mathsf{mod}\, A$ in terms of *grid categories* $\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}]$, which are constructed from an abelian category $\mathcal{A}$ with endofunctors $F_i : \mathcal{A} \to \mathcal{A}$ and natural transformations $\eta_i : F_i \to \mathsf{id}_{\mathcal{A}}$ for $i = 1, \ldots, n$. Moreover, we give a sufficient condition for $\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}]$ to have a tilting object. Then, we apply these results to GL orders and obtain the following result, where, for each subset $I \subseteq \{1, \ldots, n\}$, we denote

$$\mathcal{O}_I = \bigotimes_{i \in I} \mathcal{O}_{L_i} = \mathcal{O}_{\cap_{i \in I} L_i}.$$

THEOREM 1.2. (Theorem 6.7) *Let A be a GL order on a smooth projective variety over an algebraically closed field k, and suppose that $\sum L_i$ is a simple normal crossing divisor. Assume that there is a collection of tilting objects $T_I \in \mathsf{mod}\, \mathcal{O}_I$ for all $I \subseteq \{1, \ldots, n\}$, such that*

- $T_I \otimes \mathcal{O}_J \otimes \mathcal{O}(-L_j) \to T_I \otimes \mathcal{O}_J$ *is injective, whenever* I, J *and* $\{j\}$ *are pairwise disjoint;*
- $\operatorname{Ext}^i_{\mathcal{O}_J}(T_I \otimes \mathcal{O}_J, T_{I \cup J}) = 0$ *for all* $i > 0$, *whenever* $I \cap J = \emptyset$.

Then,

$$\bigoplus_{I \subseteq \{1,\dots,n\}} \left(\bigotimes_{i \notin I} A_i f_i \otimes \bigotimes_{i \in I} \frac{A_i}{\langle e_i \rangle} \otimes T_I \right)$$

is tilting in $\mathsf{mod}\, A$, *where* e_i *and* f_i *are matrices of size* $p_i \times p_i$ *with* 1 *in the bottom right (respectively top left) position, and* 0 *elsewhere.*

We apply this result to several concrete projective varieties. For instance, let $X = \mathbb{P}^1 \times \mathbb{P}^1$, and let L_1 and L_2 be $(1,1)$-divisors. Suppose that $L_1 \cap L_2 = p + q$. Consider

$$A = A_1 \otimes A_2 = \begin{bmatrix} \mathcal{O} & \mathcal{O}(-L_1) \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \otimes \begin{bmatrix} \mathcal{O} & \mathcal{O}(-L_2) \\ \mathcal{O} & \mathcal{O} \end{bmatrix}.$$

Then, $T_\emptyset = \mathcal{O} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,1)$, $T_i = \mathcal{O}_{L_i}(1) \oplus \mathcal{O}_{L_i}(2)$ for $i = 1, 2$, and $T_{1,2} = \mathcal{O}_p \oplus \mathcal{O}_q$ satisfies the assumptions of the theorem. Hence, a tilting object in $\mathsf{mod}\, A$ is

$$(A_1 f_1 \otimes A_2 f_2 \otimes T_\emptyset) \oplus \left(\frac{A_1}{\langle e_1 \rangle} \otimes A_2 f_2 \otimes T_1 \right) \oplus \left(A_1 f_1 \otimes \frac{A_2}{\langle e_2 \rangle} \otimes T_2 \right) \\ \oplus \left(\frac{A_1}{\langle e_1 \rangle} \otimes \frac{A_2}{\langle e_2 \rangle} \otimes T_{1,2} \right).$$

Our approach is rather general and categorical. In Section 2, we begin with an abelian category $\mathcal{A}$ and an integer $n \geqslant 1$, we fix endofunctors F_i, natural transformations η_i and integer weights $p_i \geqslant 2$, and construct a new category $\mathcal{A}[\sqrt[p_1]{\eta_1}, \dots, \sqrt[p_n]{\eta_n}]$. In Section 3, we analyze the case $n = 1$ and, using recollements, give a criterion for this category to have a cotilting object. Our emphasis on cotilting, as opposed to tilting, is because the cotilting criterion is easier to check in $\mathsf{mod}\, A$. (Abstractly, a dual statement for tilting holds, but it does not seem to apply to the concrete setup as readily.) Due to the existence of Serre duality in $\mathsf{mod}\, A$, cotilting and tilting are actually equivalent, and so this subtlety causes no issues in practice. In Section 4, we analyze the situation for an arbitrary n. In Section 5, the global dimension of these categories is computed, showing that it often coincides with the global dimension of the original category. In Section 6, we translate the categorical results to orders, to obtain the main result as stated above. Finally, in Section 7, we show how our results may be applied to concrete situations: to Hirzebruch surfaces and to projective spaces. In the $\mathbb{P}^d$ case, we show that the tilting bundle we obtain is in fact a generalization of the *squid* algebra.

§2. Setup and notation

Throughout, k denotes an algebraically closed field. Let $\mathcal{A}$ be a k-linear, Hom-finite, abelian category. Throughout, we compose morphisms left to right. Fix, for $i = 1, \ldots, n$, commuting exact functors $F_i : \mathcal{A} \to \mathcal{A}$, natural transformations $\eta_i : F_i \to \mathsf{id}_{\mathcal{A}}$ and integer weights $p_i \geqslant 2$. For any $M \in \mathcal{A}$, we denote

$$\eta_i(M) : F_i M \to M,$$

instead of the more conventional notation $\eta_{i,M}$. Using these data, we now define a new category

$$\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}]$$

of n-dimensional grids of size $(p_1 + 1) \times \cdots \times (p_n + 1)$ of commuting morphisms. To make this precise, we need to introduce some notation. Let

$$S = \{1, \ldots, p_1\} \times \cdots \times \{1, \ldots, p_n\} \subseteq \mathbb{Z}^n,$$

and denote by $\mathbf{e}_i$ the ith basis vector in $\mathbb{Z}^n$.

Throughout, for compact notation, whenever objects or morphisms are indexed by S we also allow nonpositive indices and interpret them via $M_{\mathbf{a}} := F_i M_{\mathbf{a}+p_i\mathbf{e}_i}$, and similar for morphisms. Note that the assumption that the F_i commute makes this well defined even if several indices are nonpositive.

With this notation, we define objects of $\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}]$ to be tuples

$$((M_{\mathbf{a}})_{\mathbf{a}\in S}, (f^i_{\mathbf{a}} : M_{\mathbf{a}-\mathbf{e}_i} \to M_{\mathbf{a}})_{\substack{1\leqslant i\leqslant n\\ \mathbf{a}\in S}})$$

of objects and morphisms in $\mathcal{A}$, subject to the following conditions.

- Commutativity condition: for any $i, j \in \{1, \dots, n\}$ and $\mathbf{a} \in S$, we have $f^j_{\mathbf{a}-\mathbf{e}_i} f^i_{\mathbf{a}} = f^i_{\mathbf{a}-\mathbf{e}_j} f^j_{\mathbf{a}}$. That is, the following diagram commutes:

$$\begin{tikzcd} M_{\mathbf{a}-\mathbf{e}_i-\mathbf{e}_j} \arrow[r, "f^j_{\mathbf{a}-\mathbf{e}_i}"] \arrow[d, "f^i_{\mathbf{a}-\mathbf{e}_j}"'] & M_{\mathbf{a}-\mathbf{e}_i} \arrow[d, "f^i_{\mathbf{a}}"] \\ M_{\mathbf{a}-\mathbf{e}_j} \arrow[r, "f^j_{\mathbf{a}}"] & M_{\mathbf{a}} \end{tikzcd} .$$

- Cycle condition: for any $i \in \{1, \dots, n\}$ and $\mathbf{a} \in S$, we have $f^i_{\mathbf{a}-(p_i-1)\mathbf{e}_i} \cdots f^i_{\mathbf{a}-\mathbf{e}_i} f^i_{\mathbf{a}} = \eta_i(M_{\mathbf{a}})$.

A morphism $\varphi : (M_{\mathbf{a}}, f^i_{\mathbf{a}}) \to (N_{\mathbf{a}}, g^i_{\mathbf{a}}) \in \mathcal{A}[\sqrt[p_1]{\eta_1}, \dots, \sqrt[p_n]{\eta_n}]$ is a set of morphisms $\varphi_{\mathbf{a}} : M_{\mathbf{a}} \to N_{\mathbf{a}}$ in $\mathcal{A}$ with $\mathbf{a} \in S$, such that the following diagram commutes:

$$\begin{tikzcd} M_{\mathbf{a}-\mathbf{e}_i} \arrow[r, "f^i_{\mathbf{a}}"] \arrow[d, "\varphi_{\mathbf{a}-\mathbf{e}_i}"'] & M_{\mathbf{a}} \arrow[d, "\varphi_{\mathbf{a}}"] \\ N_{\mathbf{a}-\mathbf{e}_i} \arrow[r, "g^i_{\mathbf{a}}"] & N_{\mathbf{a}} \end{tikzcd}$$

Remark 2.1. A version of the grid category with one weight (see Example 2.2 below) has been studied by Lenzing [L] under the name *category of p-cycles*. His construction, in turn, was inspired by Seshadri's discussion [S] of quasiparabolic structures of filtration length p.

Example 2.2. If $n = 1$, then objects in $\mathcal{A}[\sqrt[p]{\eta}]$ are sequences

$$M_0 = FM_p \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_p} M_p$$

such that the composition

$$FM_{p-d-1} \xrightarrow{Ff_{p-d}} \dots \xrightarrow{Ff_p} FM_p \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_{p-d}} M_{p-d}$$

is equal to $\eta(M_{p-d}) : FM_{p-d} \to M_{p-d}$ for all $0 \leqslant d \leqslant p-1$.

In particular, it is worth noting that $\mathcal{A}[\sqrt[1]{\eta}]$ is isomorphic to $\mathcal{A}$: objects of $\mathcal{A}[\sqrt[1]{\eta}]$ are diagrams $FM_1 \xrightarrow{\eta(M_1)} M_1$, and thus are uniquely given by an object M_1 of $\mathcal{A}$. One easily sees that the morphism spaces also match.

EXAMPLE 2.3. Suppose that $n=2$, $p_1=2$ and $p_2=3$. Then, objects in $\mathcal{A}[\sqrt[p_1]{\eta_1},\ \sqrt[p_2]{\eta_2}]$ are

$$\begin{tikzcd}
F_1F_2M_{2,3} \arrow[r, "F_1f^2_{(2,1)}"] \arrow[d, "F_2f^1_{(1,3)}"'] & F_1M_{2,1} \arrow[r, "F_1f^2_{(2,2)}"] \arrow[d, "f^1_{(1,1)}"'] & F_1M_{2,2} \arrow[r, "F_1f^2_{(2,3)}"] \arrow[d, "f^1_{(1,2)}"'] & F_1M_{2,3} \arrow[d, "f^1_{(1,3)}"'] \\
F_2M_{1,3} \arrow[r, "f^2_{(1,1)}"] \arrow[d, "F_2f^1_{(2,3)}"'] & M_{1,1} \arrow[r, "f^2_{(1,2)}"] \arrow[d, "f^1_{(2,1)}"'] & M_{1,2} \arrow[r, "f^2_{(1,3)}"] \arrow[d, "f^1_{(2,2)}"'] & M_{1,3} \arrow[d, "f^1_{(2,3)}"'] \\
F_2M_{2,3} \arrow[r, "f^2_{(2,1)}"] & M_{2,1} \arrow[r, "f^2_{(2,2)}"] & M_{2,2} \arrow[r, "f^2_{(2,3)}"] & M_{2,3}
\end{tikzcd}$$

where all of the squares commute and the rows and columns satisfy the cycle conditions.

In this paper, we are primarily concerned with the existence of cotilting objects in $\mathcal{A}[\sqrt[p_1]{\eta_1},\ldots,\sqrt[p_n]{\eta_n}]$.

DEFINITION 2.4. Let $\mathcal{A}$ be an abelian category. We say that an object T in $\mathcal{A}$ is *tilting* (resp. *cotilting*), if satisfies the following two conditions.

- Rigidity: $\mathrm{Ext}^i_{\mathcal{A}}(T,T)=0$ for all $i>0$.
- Generation (resp. cogeneration): $\mathrm{Ext}^i_{\mathcal{A}}(T,M)=0$ (resp. $\mathrm{Ext}^i_{\mathcal{A}}(M,T)=0$) for all $i\geqslant 0$ implies $M=0$.

In the next two sections, we focus on proving results regarding cotilting objects, rather than tilting. Analogous results can be derived for the latter; however, the corresponding results are of little practical use in the applications to orders that we have in mind in Sections 6 and 7. However, due to the existence of Serre duality in the order setting, tilting and cotilting objects coincide.

§3. Cotilting for the case with only one weight

In this section, we analyze the situation where $n=1$, that is, the category $\mathcal{A}[\sqrt[p]{\eta}]$. Recall that this category was already introduced in Example 2.2. The results we obtain will be useful when we study the general case.

We denote by $\mathcal{A}_\eta$ the full subcategory of $\mathcal{A}$ with objects given by

$$\mathcal{A}_\eta := \{M\in\mathcal{A} \mid \eta(M)=0\}.$$

We begin by considering the following special case, which will prove to be an important ingredient of the general discussion later. Let $F=0$, and η be

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the zero transformation $F \to \mathsf{id}_{\mathcal{A}}$. To ease notation, we employ the following convention.

Convention 3.1. Whenever η is the zero transformation, we consider it as a natural transformation from the zero functor to the identity functor.

(A priori it would make sense to consider zero transformations between any two functors. However, in this paper, we will soon assume (see 3.8) that η is injective on sufficiently many objects. Thus, for us it makes sense to only consider zero transformations starting in zero functors.)

In this case,

$$\mathcal{A}[\sqrt[p]{0}] = \mathrm{rep}_{\mathcal{A}}\vec{A}_p := \mathrm{Fun}(\vec{A}_p, \mathcal{A}),$$

where $\vec{A}_p$ is the linearly oriented quiver of Dynkin type A and p vertices and viewed as a (finite) category in the obvious way. We have an exact functor $\delta : \mathcal{A} \to \mathcal{A}[\sqrt[p]{0}]$, with

$$\delta(M) := (0 \to M \to \cdots \to M) \oplus \cdots \oplus (0 \to M \to 0 \to \cdots \to 0),$$

which has an exact left adjoint

$$\delta_\lambda(0 \to M_1 \to \cdots \to M_p) = M_1 \oplus \cdots \oplus M_p.$$

In the following, we prove that exact adjoint functors are also adjoint with respect to Ext. Since the most usual way to see this is to use projective or injective resolutions, which we do not assume to exist here, we give a small argument using Yoneda extension groups.

LEMMA 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be exact categories, and let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be a pair of exact adjoint functors. Then,*

$$\mathrm{Ext}^n_{\mathcal{A}}(A, RB) \simeq \mathrm{Ext}^n_{\mathcal{B}}(LA, B)$$

functorial in $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proof. We view the Ext groups as Yoneda Ext groups and use the following notation. Given $\mathbb{E} \in \mathrm{Ext}^i(A, B)$ and maps $\alpha : B \to B'$ and $\beta : A' \to A$, we denote by $\alpha_*\mathbb{E} \in \mathrm{Ext}^i(A, B')$ the extension obtained by taking the pushout along α and by $\beta^*\mathbb{E} \in \mathrm{Ext}^i(A', B)$ the extension obtained by taking the pullback along β. Note that by [ML, Lemma 3.1.6] we have

$$\alpha_*\beta^*\mathbb{E} \simeq \beta^*\alpha_*\mathbb{E}.$$

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To prove the lemma, we give two maps, and show that they are mutually inverse to each other. From left to right, let $\mathbb{E} \in \operatorname{Ext}^n_{\mathcal{A}}(A, RB)$. Since L is exact, we may apply it to $\mathbb{E}$, obtaining $L(\mathbb{E}) \in \operatorname{Ext}^n_{\mathcal{B}}(LA, LRB)$. Now consider the counit of the adjunction $\varepsilon_B : LRB \to B$. Taking the pushout along this map, we obtain $\varepsilon_{B*}L(\mathbb{E}) \in \operatorname{Ext}^n_{\mathcal{B}}(LA, B)$.

Conversely, from right to left, we send $\mathbb{E} \in \operatorname{Ext}^n_{\mathcal{B}}(LA, B)$ to $\omega_A^* R(\mathbb{E}) \in \operatorname{Ext}^n_{\mathcal{A}}(A, RB)$, where $\omega_A : A \to RLA$ denotes the unit of the adjunction.

Both constructions are well defined on the Yoneda extension groups, and are functorial. It remains to see that they are mutually inverse. Here, we check that going from left to right and then back again one obtains the extension one started with. Checking that this also works the other way around is very similar.

Therefore, let $\mathbb{E} \in \operatorname{Ext}^n_{\mathcal{A}}(A, RB)$. Applying L to $\mathbb{E}$ and sending it to $\operatorname{Ext}^n_{\mathcal{B}}(LA, B)$ via a pushout along ε_B, and then applying R and sending it back to $\operatorname{Ext}^n_{\mathcal{A}}(A, RB)$ via the pullback along ω_A, we obtain $\omega_A^* R(\varepsilon_{B*}L(\mathbb{E})) \in \operatorname{Ext}^n_{\mathcal{A}}(A, RB)$. Since R is exact, it commutes with pushouts, so this is the same as applying RL to $\mathbb{E}$ and then taking the pushout along $R(\varepsilon_B)$ followed by a pullback along ω_A. Thus, we have

$$\omega_A^* R(\varepsilon_{B*}L(\mathbb{E})) \simeq \omega_A^* R(\varepsilon_B)_* RL(\mathbb{E}) \simeq R(\varepsilon_B)_* \omega_A^* RL(\mathbb{E}).$$

Moreover, ω is a natural transformation $\mathsf{id} \to RL$, so $\omega_A^* RL(\mathbb{E}) \simeq \omega_{RB*}\mathbb{E}$. Thus,

$$\omega_A^* R(\varepsilon_{B*}L(\mathbb{E})) \simeq R(\varepsilon_B)_* \omega_{RB*}\mathbb{E}.$$

Now, the proof is completed using the general fact for adjoint pairs that $R(\varepsilon_B) \circ \omega_{RB} = \mathsf{id}_{RB}$. ∎

Applying Lemma 3.2 to the special case of the exact adjoint functors δ_λ and δ from above, we obtain the following.

Lemma 3.3. *Let $M, N \in \mathcal{A}$ and $i \geqslant 1$. Then, $\operatorname{Ext}^i_{\mathcal{A}}(M, N) = 0$ if and only if $\operatorname{Ext}^i_{\mathcal{A}[\sqrt[p]{0}]}(\delta M, \delta N) = 0$.*

Proof. Since δ and δ_λ are exact,

$$\operatorname{Ext}^i_{\mathcal{A}[\sqrt[p]{0}]}(\delta M, \delta N) = \operatorname{Ext}^i_{\mathcal{A}}(\delta_\lambda \delta M, N) = \operatorname{Ext}^i_{\mathcal{A}}(M^n, N),$$

with $n = \frac{1}{2}p(p+1)$. ∎

Proposition 3.4. *If T is a cotilting object in $\mathcal{A}$, then $\delta(T)$ is a cotilting object in $\mathcal{A}[\sqrt[p]{0}]$.*

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Proof. By Lemma 3.3 and the fact that T is cotilting in $\mathcal{A}$, we see that $\delta(T)$ is rigid. We now prove that $\delta(T)$ cogenerates $\mathcal{A}[\sqrt[p]{0}]$. Let $M \in \mathcal{A}[\sqrt[p]{0}]$, and suppose that $\mathrm{Ext}^i_{\mathcal{A}[\sqrt[p]{0}]}(M, \delta(T)) = 0$. Then,

$$0 = \mathrm{Ext}^i_{\mathcal{A}[\sqrt[p]{0}]}(M, \delta T) = \mathrm{Ext}^i_{\mathcal{A}}(\delta_\lambda M, T) = \mathrm{Ext}^i_{\mathcal{A}}(M_1 \oplus \cdots \oplus M_p, T),$$

and so, since T cogenerates $\mathcal{A}$, $M_1 = \cdots = M_p = 0$, that is, $M = 0$. ∎

We now define recollements. These are used to study categories $\mathcal{A}[\sqrt[p]{\eta}]$ here, and in the next section are also applied to the study of $\mathcal{A}[\sqrt[p_1]{\eta_1}, \dots, \sqrt[p_n]{\eta_n}]$ (for an arbitrary n).

Definition 3.5. [BBD] Let $\mathcal{A}', \mathcal{A}, \mathcal{A}''$ be abelian categories. A *recollement* is the following diagram of additive functors:

$$\begin{tikzcd}[column sep=huge]
\mathcal{A}' \arrow[r, "\iota"] & \mathcal{A} \arrow[l, bend right, "\iota_\lambda"'] \arrow[l, bend left, "\iota_\rho"] \arrow[r, "\pi"] & \mathcal{A}'' \arrow[l, bend right, "\pi_\lambda"'] \arrow[l, bend left, "\pi_\rho"]
\end{tikzcd}$$

such that

(1) $(\iota_\lambda, \iota, \iota_\rho)$ and $(\pi_\lambda, \pi, \pi_\rho)$ are adjoint triples;
(2) ι, π_λ and π_ρ are fully faithful;
(3) $\mathrm{im}\, \iota = \ker \pi$.

Example 3.6. Let A be a ring, and let e be an idempotent. Denote by $\mathsf{mod}\, A$ the category of left A-modules. Then, we have the following recollement:

$$\begin{tikzcd}[column sep=huge]
\mathsf{mod}\,\dfrac{A}{\langle e\rangle} \arrow[r, "\iota"] & \mathsf{mod}\, A \arrow[l, bend right, "\iota_\lambda"'] \arrow[l, bend left, "\iota_\rho"] \arrow[r, "\pi"] & \mathsf{mod}\, eAe \arrow[l, bend right, "\pi_\lambda"'] \arrow[l, bend left, "\pi_\rho"]
\end{tikzcd}$$

where

$$\iota = \text{inclusion}, \qquad \pi = e(-), \qquad \pi_\lambda = Ae \otimes_{eAe} -, \qquad \pi_\rho = \mathrm{Hom}_{eAe}(eA, -),$$
$$\iota_\lambda = \frac{A}{\langle e\rangle} \otimes_A -, \qquad \iota_\rho = \mathrm{Hom}_A\left(\frac{A}{\langle e\rangle}, -\right).$$

If

$$A = \begin{bmatrix} R & I & I \\ R & R & I \\ R & R & R \end{bmatrix} \ni e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where R is a ring and I is an ideal, then

$$\frac{A}{\langle e\rangle} = \begin{bmatrix} R/I & 0 & 0 \\ R/I & R/I & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so the recollement becomes

$$\begin{tikzcd} \mathsf{mod} \begin{bmatrix} R/I & 0 \\ R/I & R/I \end{bmatrix} \arrow[r, "\iota"] & \mathsf{mod} \begin{bmatrix} R & I & I \\ R & R & I \\ R & R & R \end{bmatrix} \arrow[l, bend right, "\iota_\lambda"'] \arrow[l, bend left, "\iota_\rho"] \arrow[r, "\pi"] & \mathsf{mod}\, R \arrow[l, bend right, "\pi_\lambda"'] \arrow[l, bend left, "\pi_\rho"] \end{tikzcd}$$

with

$$\pi_\lambda = \begin{bmatrix} I \\ I \\ R \end{bmatrix} \otimes_R - \qquad \pi_\rho = \begin{bmatrix} R \\ R \\ R \end{bmatrix} \otimes_R -.$$

We now return to the category $\mathcal{A}[\sqrt[p]{\eta}]$. Recall that we denote by $\mathcal{A}_\eta$ the subcategory of $\mathcal{A}$ consisting of all objects on which η vanishes.

PROPOSITION 3.7. *The following is a recollement. (Recall that, by Convention 3.1, on the left side we consider the functor* 0*, and not the functor* F*, which is considered in the middle term.)*

$$\begin{tikzcd} \mathcal{A}_\eta[\sqrt[p-1]{0}] \arrow[r, "\iota"] & \mathcal{A}[\sqrt[p]{\eta}] \arrow[l, bend right, "\iota_\lambda"'] \arrow[l, bend left, "\iota_\rho"] \arrow[r, "\pi"] & \mathcal{A} \arrow[l, bend right, "\pi_\lambda"'] \arrow[l, bend left, "\pi_\rho"] \end{tikzcd}$$

where the functors are defined by the following:

$$\iota(0 \to M_1 \to \cdots \to M_{p-1}) = (0 \to M_1 \to \cdots \to M_{p-1} \to 0),$$
$$\pi(FM_p \to M_1 \to \cdots \to M_p) = M_p,$$
$$\pi_\lambda M = (FM \xrightarrow{\mathsf{id}} FM \to \cdots \to FM \xrightarrow{\eta(M)} M),$$
$$\pi_\rho M = (FM \xrightarrow{\eta(M)} M \xrightarrow{\mathsf{id}} M \to \cdots \to M),$$

$$\begin{aligned} &\iota_\lambda(FM_p \xrightarrow{f_1} M_1 \xrightarrow{f_2} \ldots \xrightarrow{f_p} M_p) \\ &\quad = (0 \to \operatorname{cok} f_1 \to \operatorname{cok} f_1 f_2 \to \cdots \to \operatorname{cok}(f_1 \cdots f_{p-1})), \end{aligned}$$

$$\begin{aligned}&\iota_\rho(FM_p \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_p} M_p)\\&\quad = (0 \to \ker(f_2 \cdots f_p) \to \ker(f_3 \cdots f_p) \to \cdots \to \ker f_p).\end{aligned}$$

Proof. Straightforward. ∎

As we shall see in Proposition 6.1, the recollement from Example 3.6 is a special case of the above more general recollement.

We observe that in the recollement of Proposition 3.7, the functors ι, π, π_λ and π_ρ are all exact. In particular, Lemma 3.2 implies that

$$\mathrm{Ext}^i(M, \pi N) \simeq \mathrm{Ext}^i(\pi_\lambda M, N) \qquad \text{and} \qquad \mathrm{Ext}^i(\pi M, N) \simeq \mathrm{Ext}^i(M, \pi_\rho N)$$

for all $i \geqslant 0$.

The situation is slightly more involved for ι, since none of the functors ι_λ or ι_ρ is exact. To be able to still control its effect on Ext spaces we will need the following assumption.

Assumption 3.8. For the remainder of this section, we assume that $\mathcal{A}$ has enough objects M such that $\eta(M)$ is a monomorphism. That is, for all objects $X \in \mathcal{A}$, there exists an object $M \in \mathcal{A}$ with $\eta(M)$ a monomorphism, and a surjection $M \twoheadrightarrow X$.

Lemma 3.9. *With the above assumption, the subcategory of $\mathcal{A}$ given by*

$$\mathcal{E} := \{FM_p \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_p} M_p \in \mathcal{A}[\sqrt[p]{\eta}] \mid f_1 \dots f_{p-1} \text{ is a monomorphism}\}$$

is a resolving subcategory.

The reason for choosing this particular subcategory is because $\operatorname{im} \iota \subseteq \mathcal{E}$, a fact that we need later.

Proof. It is clear from the definition that subobjects of objects in $\mathcal{E}$ are in $\mathcal{E}$ again. In particular, $\mathcal{E}$ is closed under kernels of epimorphisms. Thus, it remains to check that for any object $M \in \mathcal{A}[\sqrt[p]{\eta}]$ there is an epimorphism $E \to M$ for some $E \in \mathcal{E}$.

Let $M^\bullet = FM_p \xrightarrow{f_1} M_1 \xrightarrow{f_2} \dots \xrightarrow{f_p} M_p \in \mathcal{A}[\sqrt[p]{\eta}]$. For each $i = 1, \dots, p$, let

$$M_i^\bullet = FM_i \to FM_i \to \dots FM_i \xrightarrow{\eta(M_i)} M_i \to \cdots \to M_i,$$

where the $\eta(M_i)$ is the ith arrow from the right. Note that we have a surjective map $\oplus M_i^\bullet \twoheadrightarrow M^\bullet$. Furthermore, by the assumption on $\mathcal{A}$, for all

i, there exists an $X_i \in \mathcal{A}$ such that $X_i \twoheadrightarrow M_i$ and $\eta(X_i)$ is a monomorphism. Since F is exact, $F(X_i) \twoheadrightarrow F(M_i)$, and so

$$X_i^\bullet := FX_i \to FX_i \to \cdots \to FX_i \hookrightarrow X_i \to \cdots \to X_i \twoheadrightarrow M_i^\bullet,$$

with $X_i^\bullet \in \mathcal{E}$. Thus, we have $\oplus X_i^\bullet \twoheadrightarrow M^\bullet$, and we are done. ∎

LEMMA 3.10. *ι_λ is exact on $\mathcal{E}$.*

Proof. Let $0 \to (X, f) \to (Y, g) \to (Z, h) \to 0$ be an exact sequence in $\mathcal{E}$. For each $i = 1, \ldots, p-1$, we end up with the following commutative diagram, where all rows and columns are exact:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & FZ_p \arrow[r] \arrow[u] & Z_i \arrow[r] \arrow[u] & \operatorname{cok}(h_1 \ldots h_i) \arrow[r] & 0 \\
0 \arrow[r] & FY_p \arrow[r] \arrow[u] & Y_i \arrow[r] \arrow[u] & \operatorname{cok}(g_1 \ldots g_i) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & FX_p \arrow[r] \arrow[u] & X_i \arrow[r] \arrow[u] & \operatorname{cok}(f_1 \ldots f_i) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

From the snake lemma we see that

$$0 \to \operatorname{cok}(f_1 \ldots f_i) \to \operatorname{cok}(g_1 \ldots g_i) \to \operatorname{cok}(h_1 \ldots h_i) \to 0$$

is exact, and so we are done. ∎

These two lemmas, together with Lemma 3.2, give us the following.

PROPOSITION 3.11. *Let $M \in \mathcal{E}$, and let $N \in \mathcal{A}_\eta[\sqrt[p-1]{0}]$. Then, for any n we have*

$$\operatorname{Ext}^n_{\mathcal{A}_\eta[\sqrt[p-1]{0}]}(\iota_\lambda M, N) = \operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(M, \iota N).$$

In particular, for $M, N \in \mathcal{A}_\eta[\sqrt[p-1]{0}]$ one obtains

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota M, \iota N) = \operatorname{Ext}^n_{\mathcal{A}_\eta[\sqrt[p-1]{0}]}(M, N).$$

Proof. By the two lemmas above, we know that ι_λ and ι form a pair of exact adjoint functors between the exact categories $\mathcal{E}$ and $\mathcal{A}_\eta[\sqrt[p-1]{0}]$. Thus, it follows from Lemma 3.2 that

$$\operatorname{Ext}^n_{\mathcal{A}_\eta[\sqrt[p-1]{0}]}(\iota_\lambda M, N) = \operatorname{Ext}^n_{\mathcal{E}}(M, \iota N).$$

Now, since $\mathcal{E}$ is resolving in $\mathcal{A}[\sqrt[p]{\eta}]$, we have

$$\operatorname{Ext}^n_{\mathcal{E}}(X, Y) = \operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(X, Y)$$

for $X, Y \in \mathcal{E}$. In particular,

$$\operatorname{Ext}^n_{\mathcal{A}_\eta[\sqrt[p-1]{0}]}(\iota_\lambda M, N) = \operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(M, \iota N).$$

The "in particular" part now follows, since

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota M, \iota N) = \operatorname{Ext}^n_{\mathcal{A}_\eta[\sqrt[p-1]{0}]}(\underbrace{\iota_\lambda \iota M}_{=M}, N). \qquad \square$$

PROPOSITION 3.12. *Suppose that T is a cotilting object in $\mathcal{A}_\eta[\sqrt[p-1]{0}]$ and U is cotilting in $\mathcal{A}$. Then, $E = \iota T \oplus \pi_\rho U$ is cotilting in $\mathcal{A}[\sqrt[p]{\eta}]$ if and only if $\operatorname{Ext}^i(\pi_\rho U, \iota T) = 0$ for all $i > 0$.*

Proof. Cogeneration: suppose that $\operatorname{Ext}^i(M, E) = 0$ for all $i \geqslant 0$. Then, $0 = \operatorname{Ext}^i(M, \pi_\rho U) = \operatorname{Ext}^i(\pi M, U)$, and since U is cogenerating this implies $\pi M = 0$. In this case, $M \simeq \iota N$ for some N. However, then $0 = \operatorname{Ext}^i(\iota N, \iota T) = \operatorname{Ext}^i(\iota_\lambda \iota N, T) = \operatorname{Ext}^i(N, T)$ implies, using that T is cogenerating, that $N = 0$, and so $M = 0$.

Rigidity: for $i > 0$, we have $\operatorname{Ext}^i(\iota T, \iota T) = \operatorname{Ext}^i(\iota_\lambda \iota T, T) = \operatorname{Ext}^i(T, T) = 0$, and similarly $\operatorname{Ext}^i(\pi_\rho U, \pi_\rho U) = \operatorname{Ext}^i(\pi \pi_\rho U, U) = \operatorname{Ext}^i(U, U) = 0$. Moreover, $\operatorname{Ext}^i(\iota T, \pi_\rho U) = \operatorname{Ext}^i(\pi \iota T, U) = \operatorname{Ext}^i(0, U) = 0$. Finally, by assumption, $\operatorname{Ext}^i(\pi_\rho U, \iota T) = 0$, and so we are done. $\square$

We now analyze this condition further. We define an exact functor

$$\begin{aligned} \Delta : \mathcal{A}_\eta &\longrightarrow \mathcal{A}_\eta[\sqrt[p-1]{0}] \\ M &\longmapsto (0 \to M \to M \to \cdots \to M), \end{aligned}$$

which has left and right adjoints

$$\Delta_\lambda(0 \to M_1 \to \cdots \to M_{p-1}) = M_{p-1},$$
$$\Delta_\rho(0 \to M_1 \to \cdots \to M_{p-1}) = M_1.$$

PROPOSITION 3.13. *Let* $N \in \mathcal{A}_\eta[\sqrt[p-1]{0}]$, *and let* $M \in \mathcal{A}$. *Then,* $\operatorname{Ext}^i(\pi_\rho M, \iota N) = 0$ *for all* $i > 0$ *if*

- $\operatorname{Ext}^i(\Delta \ker \eta(M), N) = 0$ *for all* $i \geqslant 0$ *and*
- $\operatorname{Ext}^i(\Delta \operatorname{cok} \eta(M), N) = 0$ *for all* $i > 0$.

Proof. We have the following exact sequence:

$$0 \to \iota\Delta \ker \eta(M) \to \pi_\lambda M \to \pi_\rho M \to \iota\Delta \operatorname{cok} \eta(M) \to 0,$$

which we break up as follows:

$$0 \to \iota\Delta \ker \eta(M) \to \pi_\lambda M \to C \to 0,$$
$$0 \to C \to \pi_\rho M \to \iota\Delta \operatorname{cok} \eta(M) \to 0.$$

Since for all $i \geqslant 0$

$$\operatorname{Ext}^i(\pi_\lambda M, \iota N) = \operatorname{Ext}^i(M, \pi\iota N) = \operatorname{Ext}^i(M, 0) = 0,$$

the first sequence implies that

$$\begin{aligned}\operatorname{Ext}^i(C, \iota N) = {} & \operatorname{Ext}^{i-1}(\iota\Delta \ker \eta(M), \iota N) \\ & \overset{\text{Proposition 3.11}}{=} \operatorname{Ext}^{i-1}(\Delta \ker \eta(M), N).\end{aligned}$$

Inserting this in the long exact sequence obtained from the second short exact sequence above, we obtain

$$\cdots \to \underbrace{\operatorname{Ext}^i(\iota\Delta \operatorname{cok} \eta(M), \iota N)}_{=\operatorname{Ext}^i(\Delta \operatorname{cok} \eta(M), N)} \to \operatorname{Ext}^i(\pi_\rho M, \iota N) \to \underbrace{\operatorname{Ext}^i(C, \iota N)}_{=\operatorname{Ext}^{i-1}(\Delta \ker \eta(M), N)} \to \ldots,$$

from which the proposition follows. ∎

THEOREM 3.14. *Suppose that* T *is cotilting in* $\mathcal{A}_\eta$ *and* U *is cotilting in* $\mathcal{A}$. *If*

- $\eta(U)$ *is injective, and*
- $\operatorname{Ext}^i_{\mathcal{A}_\eta}(\operatorname{cok} \eta(U), T) = 0$ *for all* $i > 0$,

then $\iota\delta(T) \oplus \pi_\rho U$ *is cotilting* $\mathcal{A}[\sqrt[p]{\eta}]$.

Proof. From Proposition 3.12 we require $\operatorname{Ext}^i(\pi_\rho U, \iota\delta T) = 0$. Now apply Proposition 3.13 with $M = U$ and $N = \delta(T)$. The first assumption of 3.12 holds since $\eta(U)$ is injective. For the second assumption, note that

$$\operatorname{Ext}^i(\Delta \operatorname{cok} \eta(U), \delta(T)) = \operatorname{Ext}^i(\operatorname{cok} \eta(U), \Delta_\rho\delta(T)) = \operatorname{Ext}^i_{\mathcal{A}_\eta}(\operatorname{cok} \eta(U), T). \quad ∎$$

§4. Cotilting in the general case

In this section, we turn our attention to the more general category

$$\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}].$$

We give a criterion for this category to have a cotilting object.

For each $I \subseteq \{1, \ldots, n\}$, we define the following full subcategory of $\mathcal{A}$:

$$\mathcal{A}_I := \{M \in \mathcal{A} \mid \eta_i(M) = 0 \text{ for all } i \in I\}\,.$$

Furthermore, assume that each such $\mathcal{A}_I$ has a cotilting object T_I. In particular, $T_\emptyset$ is a cotilting object in $\mathcal{A}$.

Before we proceed, we need to introduce several new categories, just as we did in Section 3, whose cotilting objects will be used to construct the cotilting object we are seeking.

For $H, I, J \subseteq \{1, \ldots, n\}$ with $H = \{a_1, \ldots, a_m\} \subseteq I$ and $J = \{b_1, \ldots, b_\ell\}$ with $J \cap I = \emptyset$, let

$$\mathcal{A}_I[\eta^J, 0^H] := \mathcal{A}_I[\sqrt[p_{b_1}]{\eta_{b_1}}, \ldots, \sqrt[p_{b_\ell}]{\eta_{b_\ell}}, \sqrt[p_{a_1}-1]{0}, \ldots, \sqrt[p_{a_m}-1]{0}].$$

We have, for any $K=\{c_1, \ldots, c_i\}\subseteq\{1, \ldots, n\}$ satisfying $K \cap (I \cup J) = \emptyset$, a restriction functor

$$|_K\colon \mathcal{A}_I[\eta^J, 0^H] \to \mathcal{A}_{I\cup K}[\eta^J, 0^H],$$
$$M|_K := \operatorname{cok} \eta_{c_1}(\operatorname{cok} \eta_{c_2}(\ldots \operatorname{cok} \eta_{c_i}(M))),$$

which is well defined since the F_i commute.

If either J or H is empty, we leave them out from the notation.

Construction 4.1. The category $\mathcal{A}_I[\eta^\emptyset, 0^H] = \mathcal{A}_I[0^H]$ has a special tilting object T_I^H constructed iteratively from T_I using the functor δ from Section 3. Explicitly, we start with $T_I^\emptyset = T_I$, and set $T_I^H = \delta T_I^{H\setminus\{h\}}$ for some $h \in H$, where

$$\delta : \mathcal{A}_I[0^{H\setminus\{h\}}] \to \mathcal{A}_I[0^{H\setminus\{h\}}][0^{1/(p_h-1)}] = \mathcal{A}_I[0^H]$$

is as in the beginning of Section 3. (One may convince oneself that the result of this iterative construction does not depend on the order.)

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Let H, I, J be as above, and let $a, b \in H$. We have the following diagram where every row and column is a recollement:

$$\begin{tikzcd}
\mathcal{A}_{I\setminus\{b\}}[\eta^J, 0^{H\setminus\{b\}}] \arrow[r, "\iota^a"] & \mathcal{A}_{I\setminus\{a,b\}}[\eta^{J\cup\{a\}}, 0^{H\setminus\{a,b\}}] \arrow[r, "\pi^a"] & \mathcal{A}_{I\setminus\{a,b\}}[\eta^J, 0^{H\setminus\{a,b\}}] \\
\mathcal{A}_{I\setminus\{b\}}[\eta^{J\cup\{b\}}, 0^{H\setminus\{b\}}] \arrow[r, "\iota^a"] \arrow[u, "\pi^b"] & \mathcal{A}_{I\setminus\{a,b\}}[\eta^{J\cup\{a,b\}}, 0^{H\setminus\{a,b\}}] \arrow[r, "\pi^a"] \arrow[u, "\pi^b"] & \mathcal{A}_{I\setminus\{a,b\}}[\eta^{J\cup\{b\}}, 0^{H\setminus\{a,b\}}] \arrow[u, "\pi^b"] \\
\mathcal{A}_I[\eta^J, 0^H] \arrow[r, "\iota^a"] \arrow[u, "\iota^b"] & \mathcal{A}_{I\setminus\{a\}}[\eta^{J\cup\{a\}}, 0^{H\setminus\{a\}}] \arrow[r, "\pi^a"] \arrow[u, "\iota^b"] & \mathcal{A}_{I\setminus\{a\}}[\eta^J, 0^{H\setminus\{a\}}] \arrow[u, "\iota^b"]
\end{tikzcd}$$

REMARK 4.2. Note that we have abused notation slightly by calling many different functors ι^a. However, no confusion should arise as they all have different domains and codomains, and the correct one is hence clear from context. The same applies to π^a and π^a_ρ as well.

LEMMA 4.3. *In the diagram of recollements above, all of the squares (including original functors and adjoint functors) commute, except ι_ρ and ι_λ. In particular, the following three equalities hold, which we will use later:*

(i) $\iota^a\iota^b = \iota^b\iota^a$ *(i.e. the left lower square of the diagram commutes);*
(ii) $\iota^a\pi^b_\rho = \pi^b_\rho\iota^a$*;*
(iii) $\pi^a_\rho\pi^b_\rho = \pi^b_\rho\pi^a_\rho$.

Proof. This is a simple, straightforward calculation. □

In light of this lemma, we define, for $H = \{a_1, \ldots, a_h\} \subseteq \{1, \ldots, n\}$ and an object M in an appropriate category,

$$\iota^H M := \iota^{a_1} \circ \cdots \circ \iota^{a_h}(M).$$

Similarly, we define π^H_ρ and π^H.

Similarly to the case with only one weight, we need to control how the adjoint pair $(\iota^a_\lambda, \iota^a)$ behaves with respect to Ext. We therefore need a more general version of the assumption used earlier.

Assumption 4.4. From now on, assume that for all $I \subseteq \{1, \ldots, n\}$ and $a \in \{1, \ldots, n\} \setminus I$, the category $\mathcal{A}_I$ has enough objects M such that $\eta_a(M)$ is a monomorphism.

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LEMMA 4.5. *Let $H, I, J \subseteq \{1, \ldots, n\}$, with $H \subseteq I$ and $J \cap I = \emptyset$. Suppose that $M, N \in \mathcal{A}_I[\eta^J, 0^H]$. Then, for all $a \in H$ and $i \geqslant 0$,*

$$\operatorname{Ext}^i(\iota^a M, \iota^a N) = \operatorname{Ext}^i(M, N).$$

Proof. By Assumption 4.4, we have that $\mathcal{A}_{I\setminus\{a\}}$ has enough objects on which η_a is monomorphism. Similarly to the proof of Lemma 3.9, one sees that this implies that also $\mathcal{A}_{I\setminus\{a\}}[\eta^J, 0^{H\setminus\{a\}}]$ has enough objects such that η_a is monomorphism. The result then follows from 3.11 and the observation that

$$\mathcal{A}_I[\eta^J, 0^H] = \left(\mathcal{A}_{I\setminus\{a\}}[\eta^J, 0^{H\setminus\{a\}}]\right)_{\eta_a}[\sqrt[p_a-1]{0}]. \qquad \square$$

LEMMA 4.6. *Let $J \subseteq I \subseteq \{1, \ldots, n\}$. For T_I^I as defined in Construction 4.1, we have*

$$\operatorname{Ext}^i(\pi_\rho^J M, \iota^J T_I^I) = 0$$

for all $i > 0$, if

- *$\eta_a(M|_{J'})$ is injective for all $J' \subset J$ and $a \in J \setminus J'$,*
- *$\operatorname{Ext}^i(M|_J, T_I^{I\setminus J}) = 0$.*

Proof. For all $a \in J$,

$$\operatorname{Ext}^i(\pi_\rho^J M, \iota^J T_I^I) = \operatorname{Ext}^i(\pi_\rho^a(\pi_\rho^{J\setminus\{a\}} M), \iota_a(\iota^{J\setminus\{a\}} T_I^I)).$$

Hence, using Proposition 3.13, we see that $\operatorname{Ext}^i(\pi_\rho^J M, \iota^J T_I^I) = 0$ for all $i > 0$, if

- $\operatorname{Ext}^i(\Delta \ker \eta_a(\pi_\rho^{J\setminus\{a\}} M), \iota^{J\setminus\{a\}} T_I^I) = 0$ for $i \geqslant 0$ and
- $\operatorname{Ext}^i(\Delta \operatorname{cok} \eta_a(\pi_\rho^{J\setminus\{a\}} M), \iota^{J\setminus\{a\}} T_I^I) = 0$ for $i > 0$.

Since

$$\ker \eta_a(\pi_\rho^{J\setminus\{a\}} M) = \pi_\rho^{J\setminus\{a\}} \ker \eta_a(M) \quad \text{and} \quad (\Delta_\rho)\iota^{J\setminus\{a\}} T_I^I = \iota^{J\setminus\{a\}} T_I^{I\setminus\{a\}},$$

the two conditions become

- $\operatorname{Ext}^i(\pi_\rho^{J\setminus\{a\}} \ker \eta_a M, \iota^{J\setminus\{a\}} T_I^{I\setminus\{a\}}) = 0$ for $i \geqslant 0$ and
- $\operatorname{Ext}^i(\pi_\rho^{J\setminus\{a\}} M|_{\{a\}}, \iota^{J\setminus\{a\}} T_I^{I\setminus\{a\}}) = 0$ for $i > 0$.

Now repeat this procedure $|J| - 1$ more times to get the result. $\square$

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THEOREM 4.7. *Let $\mathcal{A}$ be an abelian category equipped with endofunctors F_i and natural transformation η_i as in Section 2, satisfying Assumption 4.4. Assume that there are cotilting objects T_H in $\mathcal{A}_H$, such that for all $H \cap J = \emptyset$ and $a \notin H \cup J$,*

- $\eta_a(T_H|_J)$ *is injective,*
- $\mathrm{Ext}^i_{\mathcal{A}_{H\cup J}}(T_H|_J, T_{H\cup J}) = 0$ *for all $i > 0$.*

Then, with T_H^H as defined in Construction 4.1, and the functors $\pi_\rho^{[1,n]\setminus H}$ and ι^H as explained below Lemma 4.3, the object

$$T := \bigoplus_{H \subseteq \{1,\dots,n\}} \pi_\rho^{[1,n]\setminus H} \iota^H T_H^H$$

is a cotilting object in $\mathcal{A}[\sqrt[p_1]{\eta_1}, \dots, \sqrt[p_n]{\eta_n}]$.

Proof. First, we introduce the following notation: for $I \subseteq \{1, \dots, n\}$, let $\bar{I} := [1, n] \setminus I$.

Rigidity: we compute

$$\mathrm{Ext}^i(\pi_\rho^{\bar{H}} \iota^H T_H^H, \pi_\rho^{\bar{I}} \iota^I T_I^I)$$

for all $H, I \subseteq \{1, \dots, n\}$ and $i \geqslant 1$. If $H \cap \bar{I} \neq \emptyset$, then using Lemma 4.3,

$$\mathrm{Ext}^i(\pi_\rho^{\bar{H}} \iota^H T_H^H, \pi_\rho^{\bar{I}} \iota^I T_I^I) = \mathrm{Ext}^i(\pi^a \iota^a \iota^{H\setminus\{a\}} \pi_\rho^{\bar{H}} T_H^H, \pi_\rho^{\bar{I}\setminus\{a\}} \iota^I T_I^I) = 0,$$

where $a \in H \cap \bar{I}$, since $\pi^a \iota^a = 0$. Thus, we consider the case where $H \cap \bar{I} = \emptyset$ or, equivalently, $H \subseteq I$.

If $I = H$, then we use $\pi \pi_\rho = \iota_\lambda \iota = \mathsf{id}$ to obtain

$$\mathrm{Ext}^i(\pi_\rho^{\bar{H}} \iota^H T_H^H, \pi_\rho^{\bar{H}} \iota^H T_H^H) = \mathrm{Ext}^i(T_H^H, T_H^H) = 0,$$

since T_H^H is cotilting.

Finally, suppose that $H \subset I$, and let $J = I \setminus H$. Then, we have

$$\mathrm{Ext}^i(\pi_\rho^{\bar{H}} \iota^H T_H^H, \pi_\rho^{\bar{I}} \iota^I T_I^I) = \mathrm{Ext}^i(\pi_\rho^J T_H^H, \iota^J T_I^I).$$

By Lemma 4.6, this vanishes when

- $\eta_a(T_H^H|_{J'})$ is injective for all $J' \subset J$ and $a \in J \setminus J'$,
- $\mathrm{Ext}^i(T_H^H|_J, T_I^{I\setminus J}) = \mathrm{Ext}^i(T_H^H|_J, T_I^H) = 0$ for all $i > 0$.

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Thus, rigidity follows from the assumptions of the theorem and Lemma 3.3 applied $|H|$ times.

Cogeneration: suppose that $\mathrm{Ext}^i(M, T) = 0$ for all $i \geqslant 0$. We aim to show $M = 0$. We do so by proving that for all $I \subseteq \{1, \ldots, n\}$, we have $\pi^I M = 0$. Note first that

$$0 = \mathrm{Ext}^i(M, \pi_\rho^{[1,n]} T_\emptyset) = \mathrm{Ext}^i(\pi^{[1,n]} M, T_\emptyset),$$

and so $\pi^{[1,n]} M = 0$. We proceed by reverse induction on $|I|$.

Suppose that $\pi^J M = 0$ for all $|J| \geqslant k+1$. Let $I \subseteq \{1, \ldots, n\}$, such that $|I| = k$. Then, for all $a \in \bar{I}$, we have $\pi^a \pi^I M = 0$, so $\pi^I M = \iota^a N'$ for some N'. Hence, $\pi^I M = \iota^{\bar{I}} N$ for some N.

$$0 = \mathrm{Ext}^i(M, \pi_\rho^I \iota^{\bar{I}} T_{\bar{I}}^{\bar{I}}) = \mathrm{Ext}^i(\pi^I M, \iota^{\bar{I}} T_{\bar{I}}^{\bar{I}}) = \mathrm{Ext}^i(\iota^{\bar{I}} N, \iota^{\bar{I}} T_{\bar{I}}^{\bar{I}}) = \mathrm{Ext}^i(N, T_{\bar{I}}^{\bar{I}}),$$

and so $N = 0$, and hence $\pi^I M = 0$ for all π^I, with $|I| = k$. Therefore, $\pi^I M = 0$ for all $I \subseteq \{1, \ldots, n\}$, in particular $\pi^\emptyset M = M = 0$. □

§5. Global dimension

In this section, we study the global dimension of the categories $\mathcal{A}[\sqrt[p]{\eta}]$. The main aim is Theorem 5.7, showing that under certain assumptions (the most important of which is that F is an autoequivalence), the global dimension of $\mathcal{A}[\sqrt[p]{\eta}]$ equals that of $\mathcal{A}$.

We start by considering the categories on the left side of the recollement of Proposition 3.7. (The abelian category here is called $\mathcal{A}_\eta$ because these are the categories we want to apply this to. However, for this lemma this is just an arbitrary abelian category.)

Lemma 5.1. *Let $\mathcal{A}_\eta$ be abelian, and let $p \geqslant 2$. Then,*

- $\operatorname{gldim} \mathcal{A}_\eta[\sqrt[p-1]{0}] \leqslant \operatorname{gldim} \mathcal{A}_\eta + 1$ *(and in fact we have equality unless $p = 2$);*
- *if $M = [0 \to M_1 \xrightarrow{f_2^M} \cdots \xrightarrow{f_{p-1}^M} M_{p-1}] \in \mathcal{A}_\eta[\sqrt[p-1]{0}]$, such that all morphisms $f_2^M, \ldots, f_{p-1}^M$ are epi, then* $\operatorname{inj.dim} M \leqslant \operatorname{gldim} \mathcal{A}_\eta$.

We do not prove this lemma directly here, but the further discussion throughout the first half of this section results in an inductive proof. See Remark 5.5.

That means that for now, we assume that the lemma holds for a given p. Note that this is justified for $p = 2$. (In that case $\mathcal{A}_\eta[\sqrt[p-1]{0}] = \mathcal{A}_\eta$.) However, since our aim is not only the inductive proof, we now consider the general case, that is, categories $\mathscr{A}[\sqrt[p]{\eta}]$. Throughout, the following morphisms and resulting exact sequences will play a role.

Observation 5.2. Let $\mathcal{A}$, F and η be as in Proposition 3.7. Let $X \in \mathcal{A}[\sqrt[p]{\eta}]$.

(i) For the unit $\varepsilon_X : X \to \pi_\rho \pi X$, we have

$$\begin{aligned} \ker \varepsilon_X &= \iota\iota_\rho X \qquad \text{and} \\ \operatorname{cok} \varepsilon_X &= \iota(0 \to \operatorname{cok}(f_2^X \cdots f_p^X) \to \cdots \to \operatorname{cok} f_p^X). \end{aligned}$$

We note that all of the nonzero maps in $[0 \to \operatorname{cok}(f_2^X \cdots f_p^X) \to \cdots \to \operatorname{cok} f_p^X]$ are epimorphisms.

(ii) For the counit $\varphi_X : \pi_\lambda \pi X \to X$, we have

$$\begin{aligned} \ker \varphi_X &= \iota(0 \to \ker f_1^X \to \cdots \to \ker(f_1^X \cdots f_{p-1}^X)) \qquad \text{and} \\ \operatorname{cok} \varphi_X &= \iota\iota_\lambda X. \end{aligned}$$

We first study extensions in $\mathcal{A}[\sqrt[p]{\eta}]$, where the first term is in the image of the functor ι.

LEMMA 5.3. *Let $\mathcal{A}$, F and η be as in Section 3. For $X \in \mathcal{A}_\eta[\sqrt[p-1]{0}]$ and $Y \in \mathcal{A}[\sqrt[p]{\eta}]$, we have*

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota X, Y) = 0, \quad \forall n > \operatorname{gldim} \mathcal{A}_\eta + 1.$$

If, moreover, all of the maps $f_2^Y, \ldots, f_p^Y$ are epimorphisms, then the equality also holds for $n = \operatorname{gldim} \mathcal{A}_\eta + 1$.

Proof. We first observe that

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota X, \pi_\rho \pi Y) = \operatorname{Ext}^n_{\mathcal{A}}(\underbrace{\pi\iota X}_{=0}, \pi Y) = 0, \quad \forall n,$$

so the Ext space of the lemma vanishes provided that

(1) $\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota X, \ker \varepsilon_Y) = 0$ and
(2) $\operatorname{Ext}^{n-1}_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota X, \operatorname{cok} \varepsilon_Y) = 0$.

For the first space, we use Observation 5.2 to simplify

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota X, \ker \varepsilon_Y) = \operatorname{Ext}^n_{\mathcal{A}_\eta[\sqrt[p-1]{0}]}(X, \iota_\rho Y),$$

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so this space vanishes provided that $n > \operatorname{gldim} \mathcal{A}_\eta[\sqrt[p-1]{0}]$, and hence by Lemma 5.1 for $n > \operatorname{gldim} \mathcal{A}_\eta + 1$. Moreover, using the second part of Lemma 5.1, we see that this bound may be improved by 1 provided that all of the maps $f_2^{\iota_\rho Y}, \ldots f_{p-1}^{\iota_\rho Y}$ are epimorphisms. This holds provided that the corresponding maps $f_2^Y, \ldots, f_{p-1}^Y$ are epi.

For the second space, we use the remark in the first point of Observation 5.2. Note that this precisely tells us that we are in the situation of the second point of Lemma 5.1, whence

$$\operatorname{Ext}^{n-1}_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota X, \operatorname{cok} \varepsilon_Y) = 0, \quad \forall n-1 > \operatorname{gldim} \mathcal{A}_\eta.$$

Finally, we note that if all of the maps $f_2^Y, \ldots, f_p^Y$ are epi, then $\operatorname{cok} \varepsilon_Y = 0$, so the space in the second point vanishes. ∎

In the next step, we assume that the first object lies in the set $\mathcal{E}$, that is, that the map $f_1^X \cdots f_{p-1}^X$ is a monomorphism.

LEMMA 5.4. *Let $X \in \mathcal{E}$ and $Y \in \mathcal{A}[\sqrt[p]{\eta}]$. Then,*

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(X, Y) = 0, \quad \forall n > \max\{\operatorname{gldim} \mathcal{A}, \operatorname{gldim} \mathcal{A}_\eta + 1\}.$$

If, moreover, all of the maps $f_2^Y, \ldots, f_p^Y$ are epimorphisms, then

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(X, Y) = 0, \quad \forall n > \max\{\operatorname{gldim} \mathcal{A}, \operatorname{gldim} \mathcal{A}_\eta\}.$$

Proof. We start by observing that $X \in \mathcal{E}$ is equivalent to $\ker \varphi_X = 0$, whence we have the short exact sequence

$$0 \to \pi_\lambda \pi X \to X \to \iota\iota_\lambda X \to 0.$$

Therefore, it suffices to consider the two Ext spaces $\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\pi_\lambda \pi X, Y)$ and $\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota\iota_\lambda X, Y)$.

For the first of these, we note that

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\pi_\lambda \pi X, Y) = \operatorname{Ext}^n_{\mathcal{A}}(\pi X, \pi Y),$$

so this vanishes for $n > \operatorname{gldim} \mathcal{A}$.

For the second one, we use Lemma 5.3 above. ∎

REMARK 5.5. We observe that we have now completed an inductive proof of the upper bounds in Lemma 5.1. In fact, in the case $F = 0$ we have $\mathcal{E} = \mathcal{A}[\sqrt[p]{0}]$, so there is no restriction on X in the lemma above. The equality claimed in parenthesis in Lemma 5.1 follows from the following result.

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LEMMA 5.6. *Let $\mathcal{A}$, F and η be as in Section 3, and let $p \geqslant 2$. Then,*

$$\operatorname{gldim} \mathcal{A}[\sqrt[p]{\eta}] \geqslant \max\{\operatorname{gldim} \mathcal{A},\ \operatorname{gldim} \mathcal{A}_\eta + 1\}.$$

Proof. We have $\operatorname{gldim} \mathcal{A}[\sqrt[p]{\eta}] \geqslant \operatorname{gldim} \mathcal{A}$ because, for $X, Y \in \mathcal{A}$,

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\pi_\lambda X, \pi_\lambda Y) = \operatorname{Ext}^n_{\mathcal{A}}(X, \pi\pi_\lambda Y) = \operatorname{Ext}^n_{\mathcal{A}}(X, Y).$$

To see that $\mathcal{A}[\sqrt[p]{\eta}] \geqslant \operatorname{gldim} \mathcal{A}_\eta + 1$, we recall that for $Y \in \mathcal{A}_\eta$, we have

$$\Delta Y = [0 \to Y \to Y \to \cdots \to Y] \in \mathcal{A}_\eta[\sqrt[p-1]{0}].$$

Now, we observe that we have an epimorphism $f : \pi_\rho Y \to \iota \Delta Y$ in the category $\mathcal{A}[\sqrt[p]{\eta}]$. We may explicitly describe the kernel of f as

$$\ker f = [FY \to 0 \to \cdots \to 0 \to Y].$$

For any $X \in \mathcal{A}_\eta$, we have $\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota\Delta X, \pi_\rho Y) = \operatorname{Ext}^n_{\mathcal{A}}(\pi\iota\Delta X, Y) = 0$. Thus, the short exact sequence

$$0 \to \ker f \to \pi_\rho Y \to \iota\Delta Y \to 0$$

gives rise to isomorphisms

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota\Delta X, \ker f) = \operatorname{Ext}^{n-1}_{\mathcal{A}[\sqrt[p]{\eta}]}(\iota\Delta X, \iota\Delta Y) = \operatorname{Ext}^{n-1}_{\mathcal{A}_\eta}(X, Y)$$

for all n. It follows that

$$\operatorname{gldim} \mathcal{A}[\sqrt[p]{\eta}] \geqslant \operatorname{gldim} \mathcal{A}_\eta + 1. \qquad \square$$

We are now ready to prove the main result of this section, giving the precise value of the global dimension of the category $\mathcal{A}[\sqrt[p]{\eta}]$ under the assumption that F is an equivalence.

THEOREM 5.7. *Let $\mathcal{A}$, F and η be as in Section 3, and assume additionally that F is an equivalence. Then,*

$$\operatorname{gldim} \mathcal{A}[\sqrt[p]{\eta}] = \max\{\operatorname{gldim} \mathcal{A}, \operatorname{gldim} \mathcal{A}_\eta + 1\}.$$

One key ingredient for the proof is the following observation.

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Observation 5.8. The functor F induces an endofunctor $\sqrt[p]{F}$ of $\mathcal{A}[\sqrt[p]{\eta}]$, given by

$$\sqrt[p]{F}(FX_p \to X_1 \to \cdots \to X_p) = [FX_{p-1} \to FX_p \to X_1 \to \cdots \to X_{p-1}].$$

Moreover, if F is an autoequivalence of $\mathcal{A}$, then $\sqrt[p]{F}$ is an autoequivalence of $\mathcal{A}[\sqrt[p]{\eta}]$.

Proof. For $X \in \mathcal{A}[\sqrt[p]{\eta}]$, we consider the short exact sequence

$$0 \to \ker \varepsilon_X \to X \to \operatorname{im} \varepsilon_X \to 0.$$

We may explicitly describe the rightmost term by

$$\operatorname{im} \varepsilon_X = [FX_p \to \operatorname{im} f_2^X \cdots f_p^X \to \operatorname{im} f_3^X \cdots f_p^X \to \cdots \to X_p],$$

and, in particular, all of the maps $f_2^{\operatorname{im} \varepsilon_X}, \dots f_p^{\operatorname{im} \varepsilon_X}$ are monomorphisms. It follows that

$$(\sqrt[p]{F})^{-1}(\operatorname{im} \varepsilon_X) \in \mathcal{E}.$$

It follows that, for all $Y \in \mathcal{A}[\sqrt[p]{\eta}]$,

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\operatorname{im} \varepsilon_X, Y) = \operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}((\sqrt[p]{F})^{-1}(\operatorname{im} \varepsilon_X), (\sqrt[p]{F})^{-1}(Y)) = 0,$$

for $n > \max\{\operatorname{gldim} \mathcal{A}, \operatorname{gldim} \mathcal{A}_\eta + 1\}$. (The first equality holds because $\sqrt[p]{F}$ is an autoequivalence; the second is Lemma 5.4.)

On the other hand, since $\ker \varepsilon_X = \iota_\rho X$, we also have that

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(\ker \varepsilon_X, Y) = 0, \quad \forall n > \operatorname{gldim} \mathcal{A}_\eta + 1,$$

by Lemma 5.3.

Now, note that X is the middle term of a short exact sequence with $\ker \varepsilon_X$ and $\operatorname{im} \varepsilon_X$ as end terms. Therefore, we have

$$\operatorname{Ext}^n_{\mathcal{A}[\sqrt[p]{\eta}]}(X, Y) = 0, \quad \forall n > \max\{\operatorname{gldim} \mathcal{A}, \operatorname{gldim} \mathcal{A}_\eta + 1\}.$$

This is the desired upper bound for the global dimension of $\mathcal{A}[\sqrt[p]{\eta}]$. The fact that this is also a lower bound is seen in Lemma 5.6 above. □

REMARK 5.9. In the case that F is not an equivalence, one may extend the argument in the proof of Lemma 5.4 to the case where X is not necessarily in $\mathcal{E}$. In that case one has to account for a possible kernel of φ_X, resulting in the weaker upper bound

$$\operatorname{gldim} \mathcal{A}[\sqrt[p]{\eta}] \leqslant \max\{\operatorname{gldim} \mathcal{A},\ \operatorname{gldim} \mathcal{A}_\eta + 2\}.$$

However, we do not have any examples of Theorem 5.7 failing when F is not an equivalence.

Finally, we may apply Theorem 5.7 repeatedly to obtain the global dimension of categories of the form $\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}]$.

COROLLARY 5.10. *In the general situation of Theorem 4.7, and assuming further that all of the F_i are autoequivalences, we have*

$$\operatorname{gldim} \mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}] = \max_{I \subseteq \{1,\ldots,n\}} \operatorname{gldim} \mathcal{A}_I + |I|.$$

(Here we set $\operatorname{gldim} 0 = -\infty$*, or alternatively let the maximum run over all I such that $\mathcal{A}_I \neq 0$.)*

Proof. We can construct the category $\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}]$ iteratively, using the fact that

$$\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_i]{\eta_i}, \sqrt[p_{i+1}]{\eta_{i+1}}] = \mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_i]{\eta_i}][\sqrt[p_{i+1}]{\eta_{i+1}}],$$

where we have extended the action of F_{i+1} to $\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_i]{\eta_i}]$ component wise. Since the F_i commute, this construction is well defined and equals our original construction. Thus, the result follows from Theorem 5.7 applied repeatedly. ∎

§6. Applications to orders on projective varieties

In [IL], *GL orders* on $\mathbb{P}^d$ were used to study GL weighted projective spaces, which in turn were introduced in [HIMO]. We have already introduced GL orders in Definition 1.1: they are orders made up of tensor products of sheaves of algebras of the form

$$H_{p_i}(\mathcal{O}, \mathcal{O}(-L_i)) := \underbrace{\begin{bmatrix} \mathcal{O} & \mathcal{O}(-L_i) & \ldots & \mathcal{O}(-L_i) & \mathcal{O}(-L_i) \\ \mathcal{O} & \mathcal{O} & \ldots & \mathcal{O}(-L_i) & \mathcal{O}(-L_i) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \ldots & \mathcal{O} & \mathcal{O}(-L_i) \\ \mathcal{O} & \mathcal{O} & \ldots & \mathcal{O} & \mathcal{O} \end{bmatrix}}_{p_i}.$$

The connection that the category $\mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}]$ has to orders is described in the following proposition.

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PROPOSITION 6.1. *Let X be a projective variety over k, and let $L_1, \ldots, L_n$ be effective Cartier divisors on X. Let $\mathcal{A} = \mathsf{coh}\, X$, $F_i := - \otimes_X \mathcal{O}(-L_i)$ and η_i be given by tensoring with the natural inclusion $\mathcal{O}(-L_i) \hookrightarrow \mathcal{O}$. If*

$$A = \bigotimes_{i=1}^{n} A_i, \quad A_i = H_{p_i}(\mathcal{O}, \mathcal{O}(-L_i)),$$

then

$$\mathsf{mod}\, A \simeq \mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}].$$

Proof. Let $e_i \in H^0(X, A_j)$ be the global section with a 1 in the (i, i)-entry and 0 elsewhere.

We define the function $\Phi : \mathsf{mod}\, A \to \mathcal{A}[\sqrt[p_1]{\eta_1}, \ldots, \sqrt[p_n]{\eta_n}]$ as follows. For $M \in \mathsf{mod}\, A$ and $\mathbf{a} \in S = \{1, \ldots, p_1\} \times \cdots \times \{1, \ldots, p_n\}$, set

$$M_{\mathbf{a}} := (e_{a_1} \otimes \cdots \otimes e_{a_n})M.$$

As before, we extend this to allow the components of $\mathbf{a}$ to be 0, and treat them as functors F_i; that is, $M_{\mathbf{a}} := F_i M_{\mathbf{a}+p_i\mathbf{e}_i}$, if $a_i = 0$.

Now, set

$$f^i_{\mathbf{a}} : M_{\mathbf{a}-\mathbf{e}_i} \to M_{\mathbf{a}}$$

to be the natural map coming from the A_i-module structure. ∎

From now on, we assume the following.

Assumption 6.2. X is smooth and $D = \sum L_i$ is a simple normal crossing divisor; that is, for all $x \in \operatorname{Supp} D$, the local equations of L_i form a regular sequence in $\mathcal{O}_{X,x}$.

PROPOSITION 6.3. *The global dimension of A is equal to the dimension of X (and, in particular, finite).*

Proof. Let $d = \dim X$. The assumption on the L_i implies that the intersections $\cap_{i \in I} L_i$ are smooth of dimension $d - |I|$, or empty. Thus, we have

$$\operatorname{gldim} \mathcal{A}_I = \operatorname{gldim} \mathcal{O}_{\cap_{i \in I} L_i} \leqslant d - |I|.$$

The claim now follows from Corollary 5.10.

Alternatively, one may observe that the proof for the case $X = \mathbb{P}^d$ in [IL, Proposition 2.13] generalizes immediately to an arbitrary smooth X and L_i. ∎

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COROLLARY 6.4. ([IL], Proposition 5.2) *Let T be a tilting object in* $\mathsf{mod}\ A$.

- $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ A) = \mathsf{thick}\ T$.
- *There is a triangle equivalence* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ A) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\ \mathrm{End}_A(T))$.

PROPOSITION 6.5. (Serre duality) *Let A be a GL order, as before, and put $d = \dim X$. Let*

$$\omega_A := \mathcal{H}om_X(A, \omega_X),$$

which is an A-bimodule. Then, for any $M, N \in \mathsf{mod}\ A$, we have

$$\mathrm{Ext}^i_A(M, N) = D\,\mathrm{Ext}^{d-i}_A(N, \omega_A \otimes_A M),$$

where $D(-) = \mathrm{Hom}_k(-, k)$.

Proof. This proof is adapted from [AdJ]. Let $h^i : \mathsf{D}^b(\mathsf{mod}\ A) \to \mathsf{mod}\ A$ be the ith cohomology functor, let $\mathbf{R}\Gamma$ be the right derived functor of the global sections functor Γ, let $\mathbf{R}\mathcal{H}om_A(-, N)$ be the right derived functor of the sheaf hom functor $\mathcal{H}om_A(-, N)$, and let $- \otimes^{\mathbf{L}}_A N$ be the left derived functor of the tensor functor. For simplicity, we introduce some more notation. Let $\mathbb{H}^i := h^i \circ \mathbf{R}\Gamma$ be the hypercohomology functor, and for $M \in \mathsf{D}(\mathsf{mod}\ A)$, let $M^* := \mathbf{R}\mathcal{H}om_A(M, A)$. With this, we have

$$\begin{aligned}
\mathrm{Ext}^i_A(M, N) &= \mathbb{H}^i(\mathbf{R}\mathcal{H}om_A(M, N)) = \mathbb{H}^i(M^* \otimes^{\mathbf{L}}_A N)\\
&= \mathbb{H}^i(\mathbf{R}\mathcal{H}om(\mathcal{O}, M^* \otimes^{\mathbf{L}}_A N))\\
&\simeq D\mathbb{H}^{d-i}(\mathbf{R}\mathcal{H}om_{\mathcal{O}}(M^* \otimes^{\mathbf{L}}_A N, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}} \omega_X)\\
&= D\mathbb{H}^{d-i}(\mathbf{R}\mathcal{H}om_{\mathcal{O}}(M^* \otimes^{\mathbf{L}}_A N, \omega_X))\\
&= D\mathbb{H}^{d-i}(\mathbf{R}\mathcal{H}om_A(N, \mathbf{R}\mathcal{H}om_{\mathcal{O}}(M^*, \omega_X)))\\
&= D\mathbb{H}^{d-i}(\mathbf{R}\mathcal{H}om_A(N, \omega_A \otimes^{\mathbf{L}}_A M))\\
&= D\,\mathrm{Ext}^{d-i}_A(N, \omega_A \otimes_A M),
\end{aligned}$$

and we are done. □

COROLLARY 6.6. *$T \in \mathsf{mod}\ A$ is tilting if and only if it is cotilting.*

Proof. The proof follows immediately from 6.5. □

We now translate our results from Sections 3 and 4 to the category $\mathsf{mod}\ A$, but in light of the previous result we will say that T is tilting, as opposed to cotilting.

First, for each $i = 1, \ldots, n$, we consider a sheafified version of the standard recollement we presented in Example 3.6. It follows immediately from the explicit description of ι and π that this recollement coincides, via the equivalence of Proposition 6.1, with the recollement of Proposition 3.7.

Let e_i be the global idempotent of A_i with 1 in the bottom right entry and 0 elsewhere. We have the recollement

$$\mathsf{mod}\,\frac{A_i}{\langle e_i\rangle} \xrightarrow{\iota} \mathsf{mod}\,A_i \xrightarrow{\pi} \mathsf{mod}\,e_iA_ie_i,$$

where the terms explicitly are

$$\frac{A_i}{\langle e_i\rangle} = \underbrace{\begin{bmatrix} \mathcal{O}_{L_i} & 0 & \ldots & \ldots & 0 \\ \mathcal{O}_{L_i} & \mathcal{O}_{L_i} & 0 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ \mathcal{O}_{L_i} & \mathcal{O}_{L_i} & \ldots & \ldots & \mathcal{O}_{L_i} \end{bmatrix}}_{p_i-1},$$

$$A_i = \underbrace{\begin{bmatrix} \mathcal{O} & \mathcal{O}(-L_i) & \ldots & \ldots & \mathcal{O}(-L_i) \\ \mathcal{O} & \mathcal{O} & \mathcal{O}(-L_i) & \ldots & \mathcal{O}(-L_i) \\ \vdots & \vdots & & & \vdots \\ \mathcal{O} & \mathcal{O} & \ldots & \ldots & \mathcal{O} \end{bmatrix}}_{p_i}$$

$$\text{and} \quad \mathsf{mod}\,e_iA_ie_i = \mathsf{coh}\,X.$$

Here, the functor ι is natural inclusion, and π is given by $\pi(N) = e_iN$. In particular,

$$\pi_\rho(\mathcal{F}) = \mathcal{H}om(e_iA_i, \mathcal{F}) = \begin{bmatrix} \mathcal{F} \\ \vdots \\ \mathcal{F} \end{bmatrix}.$$

Furthermore, if T is a tilting object in $\mathsf{coh}\,L_i$, then the tilting object $\delta(T)$ in $\mathsf{mod}\,\frac{A_i}{\langle e_i\rangle}$ is given by

$$\delta(T) = \mathcal{H}om\left(\frac{A_i}{\langle e_i\rangle}, T\right) = \begin{bmatrix} T \\ T \\ \vdots \\ T \end{bmatrix} \oplus \begin{bmatrix} T \\ \vdots \\ T \\ 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} T \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(To see this, compare the definition of δ at the beginning of Section 3 with the equivalence of Proposition 6.1.)

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As a final step for translating our general setup from Section 4 to the explicit setup here, note that the restriction functors are just given by tensoring with the structure sheaf on the corresponding subvarieties. Thus, writing $\mathcal{O}_J$ for the structure sheaf on the intersection $\cap_{c\in J} L_j$, we obtain the following result directly by translating Theorem 4.7 to this setup.

THEOREM 6.7. *Let $\mathcal{A} = \mathsf{mod}\, A$ be as above. If, for all $I, J \subseteq \{1, \dots, n\}$ with $I \cap J = \emptyset$ and $j \notin I \cup J$,*

- $T_I \otimes \mathcal{O}_J \otimes \mathcal{O}(-L_j) \to T_I \otimes \mathcal{O}_J$ *is injective,*
- $\operatorname{Ext}^i_{\mathcal{O}_{I\cup J}}(T_I \otimes \mathcal{O}_J, T_{I\cup J}) = 0$, *for all $i > 0$,*

then

$$\bigoplus_{I\subseteq\{1,\dots,n\}} \mathcal{H}om\left(\bigotimes_{i\notin I} e_i A_i \otimes \bigotimes_{i\in I} \frac{A_i}{\langle e_i\rangle}, T_I\right)$$

is tilting in $\mathcal{A}$.

§7. Examples

We now apply Theorem 6.7 to various situations. Note that if T_I is in fact a tilting bundle, then the first condition of the theorem is automatically satisfied. Furthermore, by Serre vanishing, we can always twist the T_I so that the second condition is also satisfied. Inertly, assuming that all T_I with smaller index sets are already fixed, we twist $T_{I\cup J}$ in the second condition by a sufficiently high power of $\mathcal{O}_{I\cup J}(1)$ to guarantee the Ext-vanishing in that condition.

7.1 Geigle–Lenzing weighted projective lines

Let $X = \mathbb{P}^1_{X_0:X_1}$, and let $\mathcal{A} = \mathsf{coh}\, X$. For $i = 1, \dots, n$, choose points $L_i = (\lambda_{0,i} : \lambda_{1,i})$ and corresponding weights p_i. $T_\emptyset = \mathcal{O}_X \oplus \mathcal{O}_X(1)$ is a tilting object in $\mathsf{coh}\, X$ and $T_{\{i\}} = \mathcal{O}_{L_i}$ is a tilting object in $\mathsf{coh}\, L_i$. Then,

$$T = \mathcal{H}om\left(\bigotimes_{i=1}^n e_i A_i, T_\emptyset\right) \oplus \bigoplus_{i=1}^n \mathcal{H}om\left(\bigotimes_{j\neq i} e_j A_j \otimes \frac{A_i}{\langle e_i\rangle}, T_{\{i\}}\right)$$

is a tilting object in

$$\mathsf{mod} \bigotimes_{i=1}^n A_i := \mathsf{mod} \bigotimes_{i=1}^n H_{p_i}(\mathcal{O}, \mathcal{O}(-L_i)) \simeq \mathcal{A}[\sqrt[p_1]{\eta_1}, \dots, \sqrt[p_n]{\eta_n}]$$

with endomorphism algebra

$$\begin{tikzcd}
 & & \cdot \arrow[r] & \cdots \arrow[r] & \cdot \\
 & & \cdot \arrow[r] & \cdots \arrow[r] & \cdot \\
\cdot \arrow[r, shift left, "X_0"] \arrow[r, shift right, "X_1"'] & \cdot \arrow[uur, "y_1"] \arrow[ur, "y_2"'] \arrow[dr, "y_n"'] & \vdots & \vdots & \vdots \\
 & & \cdot \arrow[r] & \cdots \arrow[r] & \cdot
\end{tikzcd}$$

with relations

$$(\lambda_{1,i} X_0 - \lambda_{0,i} X_1) y_i = 0.$$

This algebra is known as the "squid". (The reader may check directly that this is indeed the quiver with relations for the endomorphism algebra, or check our discussion of squid algebras more generally in Section 7.4.)

7.2 Geigle–Lenzing weighted $\mathbb{P}^2$

Let $\mathcal{A} = \mathsf{coh}\,\mathbb{P}^2$. Note that both lines and smooth conics in $\mathbb{P}^2$ are isomorphic to $\mathbb{P}^1$, and hence have tilting bundles. Fix weights $p_1, \ldots, p_n$, and hyperplanes $L_1, \ldots, L_l$, as well as smooth conics $L_{l+1}, \ldots, L_n$. As before, we consider the category

$$\mathcal{A} \simeq \mathsf{mod} \bigotimes_{i=1}^{n} H_{p_i}(\mathcal{O}, \mathcal{O}(-L_i)).$$

$T_\emptyset = \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$ is a tilting bundle in $\mathsf{coh}\,\mathbb{P}^2$, $T_{\{i\}} = \mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1)$ is a tilting bundle in $\mathsf{coh}\, L_i$ (where $\mathcal{O}_{L_i}(1)$ denotes the degree-1 line bundle on L_i), and $T_{\{i,j\}} = \mathcal{O}_{L_i \cap L_j}$ is a tilting bundle in $L_i \cap L_j$. Thus, by Theorem 6.7,

$$T = \mathcal{H}om\left(\bigotimes_{i=1}^{n} e_i A_i, T_\emptyset\right) \oplus \bigoplus_{i=1}^{n} \mathcal{H}om\left(\bigotimes_{j \neq i} e_j A_j \otimes \frac{A_i}{\langle e_i \rangle}, T_{\{i\}}\right)$$
$$\oplus \bigoplus_{\substack{i,j=1 \\ i<j}}^{n} \mathcal{H}om\left(\bigotimes_{h \notin \{i,j\}} e_h A_h \otimes \frac{A_i}{\langle e_i \rangle} \otimes \frac{A_j}{\langle e_j \rangle}, T_{\{i,j\}}\right)$$

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is a tilting object provided that the two conditions of the theorem are satisfied. The first condition is automatically satisfied as all of the tilting objects are in fact vector bundles. Furthermore, the second condition is also satisfied as

$$\begin{aligned}&\mathrm{Ext}^1_{L_i}(T_\emptyset \otimes \mathcal{O}_{L_i}, T_{\{i\}})\\ &\quad = \begin{cases} \mathrm{Ext}^1_{L_i}(\mathcal{O}_{L_i}(-2) \oplus \mathcal{O}_{L_i}(-1) \oplus \mathcal{O}_{L_i}, \mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1)) = 0, \\ \qquad i = 0, \ldots, l, \\ \mathrm{Ext}^1_{L_i}(\mathcal{O}_{L_i}(-4) \oplus \mathcal{O}_{L_i}(-2) \oplus \mathcal{O}_{L_i}, \mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1)) = 0, \\ \qquad i = l+1, \ldots, n, \end{cases}\end{aligned}$$

and so T is indeed a tilting bundle in $\mathsf{mod}\, A$.

7.3 Geigle–Lenzing weighted Hirzebruch surfaces

In this section, we follow King's conventions from [K]. For $m \geqslant 0$, the Hirzebruch surface is defined as

$$\Sigma_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-m) \oplus \mathcal{O}_{\mathbb{P}^1}).$$

We embed such a surface in $\mathbb{P}^1_{x_0:x_1} \times \mathbb{P}^{m+1}_{y_0:\cdots:y_{m+1}}$ via

$$\{((x_0 : x_1), (y_0 : \cdots : y_{m+1})) \mid x_0 y_i = x_1 y_{i-1},\ \text{for } 1 \leqslant i \leqslant m\},$$

and so line bundles on Σ_m arise as pullbacks of line bundles on $\mathbb{P}^1$ and $\mathbb{P}^{m+1}$ via the stated embedding followed by the natural projections. Thus, $\mathsf{Pic}\, \Sigma_m = \mathbb{Z}^2$ with intersection form $\begin{bmatrix} 0 & 1 \\ 1 & m \end{bmatrix}$ and canonical bundle $\mathcal{O}(m-2, -2)$. Using the adjunction formula, which states that a smooth genus g curve C on a surface X with canonical divisor K satisfies

$$2g - 2 = C.(C + K),$$

we see that any curve of type $(a, 1)$ or $(1, 0)$ is rational and hence has a tilting bundle.

Until the end of this section, we let $\mathcal{O} = \mathcal{O}_{\Sigma_m}$. Similar to the GL weighted $\mathbb{P}^2$ case, for $i = 1, \ldots, l$ let L_i be a curve of type $(a_i, 1)$ and for $i = l+1, \ldots, n$ a $(1, 0)$ divisor. Note that since L_i is effective, $a_i \geqslant -m$ (see [K, Proposition 6.1]). As before, we consider the category

$$\mathcal{A} \simeq \mathsf{mod} \bigotimes_{i=1}^{n} H_{p_i}(\mathcal{O}, \mathcal{O}(-L_i)).$$

Let $T_\emptyset = \mathcal{O} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,1)$, which is a tilting bundle in $\mathsf{coh}\,\Sigma_m$. For $i = 1, \ldots, l$, let $T_{\{i\}} = \mathcal{O}_{L_i}(a_i + m) \oplus \mathcal{O}_{L_i}(a_i + m + 1)$, and let $T_{\{i\}} = \mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1)$ for $i = l+1, \ldots, m$. Finally, let $T_{\{i,j\}} = \mathcal{O}_{L_i \cap L_j}$. Thus, by Theorem 6.7,

$$T = \mathcal{H}om\left(\bigotimes_{i=1}^{n} e_i A_i, T_\emptyset\right) \oplus \bigoplus_{i=1}^{n} \mathcal{H}om\left(\bigotimes_{j \neq i} e_j A_j \otimes \frac{A_i}{\langle e_i \rangle}, T_{\{i\}}\right)$$
$$\oplus \bigoplus_{\substack{i,j=1 \\ i<j}}^{n} \mathcal{H}om\left(\bigotimes_{h \notin \{i,j\}} e_h A_h \otimes \frac{A_i}{\langle e_i \rangle} \otimes \frac{A_j}{\langle e_j \rangle}, T_{\{i,j\}}\right)$$

is a tilting object provided that the two conditions of the theorem are satisfied. Again, the first condition is automatically satisfied as all of the tilting objects are in fact vector bundles. Furthermore, the second condition is also satisfied as

$$\begin{aligned} &\mathrm{Ext}^1_{L_i}(T_\emptyset \otimes \mathcal{O}_{L_i}, T_{\{i\}}) \\ &= \begin{cases} \mathrm{Ext}^1_{L_i}(\!(\mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1) \oplus \mathcal{O}_{L_i}(a_i+m) \oplus \mathcal{O}_{L_i}(a_i+m+1), \\ \qquad \mathcal{O}_{L_i}(a_i+m) \oplus \mathcal{O}_{L_i}(a_i+m+1)\!) = 0, \quad i = 0, \ldots, l, \\ \mathrm{Ext}^1_{L_i}(\mathcal{O} \oplus \mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1) \oplus \mathcal{O}_{L_i}(1), \mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1)) = 0, \\ \qquad i = l+1, \ldots, n, \end{cases} \end{aligned}$$

since $a_i \geqslant -m$, and so T is indeed a tilting bundle in $\mathsf{mod}\,A$.

7.4 Squids

We have already seen the squid algebra that arose as the endomorphism algebra of tilting objects on GL weighted projective lines. We now generalize this to higher-dimensional GL weighted projective spaces.

Let $X = \mathbb{P}^d_{X_0:\cdots:X_d}$, and for $i = 1, \ldots, n$ let $L_i : \ell_i(X_0, \ldots, X_d) = 0$ be hyperplanes in general position. For $I \subseteq \{1, \ldots, n\}$ with $|I| \leqslant d$,

$$T_I = \mathcal{O}_I(|I|) \oplus \mathcal{O}_I(|I|+1) \oplus \cdots \oplus \mathcal{O}_I(d)$$

is a tilting bundle in $\mathsf{coh}\,L_I$, where

$$L_I = \bigcap_{i \in I} L_i.$$

Furthermore, as we have seen, the category $\mathsf{mod}\,A$, where

$$A = \bigotimes_{i=1}^{n} H_{p_i}(\mathcal{O}_X, \mathcal{O}_X(-L_i)),$$

has a tilting object as described in Theorem 6.7, since the first condition is trivial as all T_I are bundles, and the second condition is easy to verify with our choice of T_I.

We now describe $\mathrm{End}_A(T)$, where T is given by Theorem 6.7, presenting it as a quiver with relations. First, we describe the vertices, then the arrows, and finally the relations. For simplicity, we allow nonadmissible relations.

7.4.1 Vertices

The vertices correspond to the indecomposable summands of T. Writing

$$\mathcal{O}_I^{p/l} = \begin{bmatrix} \mathcal{O}_I \\ \vdots \\ \mathcal{O}_I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \subseteq \mathcal{O}_I^p$$

for the length p-vectors such that the last $l-1$ entries vanish, we may observe (see the discussion of functors above Theorem 6.7) that

$$\mathcal{H}om\left(\bigotimes_{i\notin I} e_i A_i \otimes \bigotimes_{i\in I} \frac{A_i}{\langle e_i\rangle}, -\right) = \left(\bigotimes_{i=1}^{n} \bigoplus_{a_i \in \begin{cases}\{1\} & i\notin I\\ \{2,\ldots,p_i\} & i\in I\end{cases}} \mathcal{O}_I^{p_i/a_i}\right) \otimes_{\mathcal{O}_I} -.$$

Thus,

$$T = \bigoplus_{I\subseteq\{1,\ldots,n\}} \bigoplus_{j=|I|}^{d} \left(\bigotimes_{i=1}^{n} \bigoplus_{a_i \in \begin{cases}\{1\} & i\notin I\\ \{2,\ldots,p_i\} & i\in I\end{cases}} \mathcal{O}_I^{p_i/a_i}\right) \otimes_{\mathcal{O}_I} \mathcal{O}_I(j).$$

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We note that, in total, the vector $\mathbf{a}$ runs over S, where $S = \times_{i=1}^{n}\{1, \ldots, p_i\}$, as before. Thus, setting $I_{\mathbf{a}} = \{i \mid a_i \neq 1\}$, we may reorganize the above to

$$T = \bigoplus_{\mathbf{a}\in S} \bigoplus_{j=|I_{\mathbf{a}}|}^{d} \underbrace{\left(\bigotimes_{i=1}^{n} \mathcal{O}_{I_{\mathbf{a}}}^{p_i/a_i}\right)}_{=:\mathcal{O}_{\mathbf{a}}}(j).$$

Thus, the vertices of the quiver are labeled $\mathcal{O}_{\mathbf{a}}(j)$ with $\mathbf{a} \in S$ and $|I_{\mathbf{a}}| \leqslant j \leqslant d$.

7.4.2 Arrows

- $(d+1)$ arrows labeled $X_{I_{\mathbf{a}}}^{0}, \ldots, X_{I_{\mathbf{a}}}^{d}$ between

$$\mathcal{O}_{\mathbf{a}}(j) \underset{X_{I_{\mathbf{a}}}^{d}}{\overset{X_{I_{\mathbf{a}}}^{0}}{\substack{\longrightarrow \\ \vdots \\ \longrightarrow}}} \mathcal{O}_{\mathbf{a}}(j+1)$$

  whenever $|I_{\mathbf{a}}| \leqslant j < d$.
- One arrow labeled y_i between

$$\mathcal{O}_{\mathbf{a}}(j) \to \mathcal{O}_{\mathbf{a}+\mathbf{e}_i}(j)$$

  when $I_{\mathbf{a}+\mathbf{e}_i} \leqslant j \leqslant d$.

7.4.3 Relations

- Commutativity relations: $X_{I_{\mathbf{a}}}^{i} X_{I_{\mathbf{a}}}^{j} = X_{I_{\mathbf{a}}}^{j} X_{I_{\mathbf{a}}}^{i}$, $X_{I_{\mathbf{a}}}^{i} y_j = y_j X_{I_{\mathbf{a}+\mathbf{e}_j}}^{i}$, $y_i y_j = y_j y_i$ whenever these compositions make sense.
- $\ell_i(X_{I_{\mathbf{a}}}^{0}, \ldots, X_{I_{\mathbf{a}}}^{d}) = 0$ for all $\mathbf{a} \in S$ and $i \in I_{\mathbf{a}}$.
- For all $\mathbf{a} \in S$ and i such that $a_i = 1$, for any subquivers of the form

$$\mathcal{O}_{\mathbf{a}}(|I_{\mathbf{a}}|) \underset{X_{I_{\mathbf{a}}}^{d}}{\overset{X_{I_{\mathbf{a}}}^{0}}{\substack{\longrightarrow \\ \vdots \\ \longrightarrow}}} \mathcal{O}_{\mathbf{a}}(|I_{\mathbf{a}}|+1) \xrightarrow{y_i} \mathcal{O}_{\mathbf{a}+\mathbf{e}_i}(|I_{\mathbf{a}+\mathbf{e}_i}|),$$

  we have the relation

$$\ell_j(X_{I_{\mathbf{a}}}^{0}, \ldots, X_{I_{\mathbf{a}}}^{d}) y_i = 0.$$

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EXAMPLE 7.1. (On $\mathbb{P}^2$ with two weights 3 and 3) Consider $\mathbb{P}^2_{X_0:X_1:X_2}$ and hyperplanes $L_i : \ell_i(X_0, X_1, X_2) = 0$ for $i = 1, 2$. Let

$$A = \begin{bmatrix} \mathcal{O} & \mathcal{O}(-L_1) & \mathcal{O}(-L_1) \\ \mathcal{O} & \mathcal{O} & \mathcal{O}(-L_1) \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix} \otimes \begin{bmatrix} \mathcal{O} & \mathcal{O}(-L_2) & \mathcal{O}(-L_2) \\ \mathcal{O} & \mathcal{O} & \mathcal{O}(-L_2) \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix} = A_1 \otimes A_2.$$

Then, $T_\emptyset = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$, $T_{\{i\}} = \mathcal{O}_{L_i}(1) \oplus \mathcal{O}_{L_i}(2)$ and $T_{\{1,2\}} = \mathcal{O}_{L_1 \cap L_2}$. Then, by Theorem 6.7,

$$T = (A_1 f_1 \otimes A_2 f_2 \otimes T_\emptyset) \oplus \left(\frac{A_1}{\langle e_1 \rangle} \otimes A_2 f_2 \otimes T_1 \right) \oplus \left(A_1 f_1 \otimes \frac{A_2}{\langle e_2 \rangle} \otimes T_2 \right)$$
$$\oplus \left(\frac{A_1}{\langle e_1 \rangle} \otimes \frac{A_2}{\langle e_2 \rangle} \otimes T_{1,2} \right)$$

is a tilting object in $\mathcal{A} = \mathsf{mod}\, A$. $\operatorname{End}_A(T)$ is given by the following quiver:

$$\begin{tikzcd}
 & & & \mathcal{O}_{(3,1)}(1) \arrow[r, Rrightarrow, "X_1^0{,}X_1^1{,}X_1^2"] & \mathcal{O}_{(3,1)}(2) \arrow[dr, "y_2"] & & \\
 & & \mathcal{O}_{(2,1)}(1) \arrow[ur, "y_1"] \arrow[r, Rrightarrow, "X_1^0{,}X_1^1{,}X_1^2"] & \mathcal{O}_{(2,1)}(2) \arrow[ur, "y_1"] \arrow[dr, "y_2"] & & \mathcal{O}_{(3,2)} \arrow[dr, "y_2"] & \\
\mathcal{O}_{(1,1)} \arrow[r, Rrightarrow, "X_\emptyset^0{,}X_\emptyset^1{,}X_\emptyset^2"] & \mathcal{O}_{(1,1)}(1) \arrow[ur, "y_1"] \arrow[dr, "y_2"] \arrow[r, Rrightarrow, "X_\emptyset^0{,}X_\emptyset^1{,}X_\emptyset^2"] & \mathcal{O}_{(1,1)}(2) \arrow[ur, "y_1"] \arrow[dr, "y_2"] & & \mathcal{O}_{(2,2)} \arrow[ur, "y_1"] \arrow[dr, "y_2"] & & \mathcal{O}_{(3,3)} \\
 & & \mathcal{O}_{(1,2)}(1) \arrow[r, Rrightarrow, "X_2^0{,}X_2^1{,}X_2^2"] \arrow[dr, "y_2"] & \mathcal{O}_{(1,2)}(2) \arrow[ur, "y_1"] \arrow[dr, "y_2"] & & \mathcal{O}_{(2,3)} \arrow[ur, "y_2"] & \\
 & & & \mathcal{O}_{(1,3)}(1) \arrow[r, Rrightarrow, "X_2^0{,}X_2^1{,}X_2^2"] & \mathcal{O}_{(1,3)}(2) \arrow[ur, "y_1"] & &
\end{tikzcd}$$

with commutativity relations as well as

$$\begin{array}{ll}
\ell_1(X_1^0, X_1^1, X_1^2), & \ell_2(X_2^0, X_2^1, X_2^2), \\
\ell_1(X_\emptyset^0, X_\emptyset^1, X_\emptyset^2)y_1, & \ell_2(X_\emptyset^0, X_\emptyset^1, X_\emptyset^2)y_2, \\
\ell_2(X_1^0, X_1^1, X_1^2)y_2, & \ell_1(X_2^0, X_2^1, X_2^2)y_1.
\end{array}$$

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