SOME CHARACTERIZATIONS OF GENERALIZED MANIFOLDS WITH BOUNDARIES

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In R. L. Wilder's book [**2**] the open and closed generalized manifolds are extensively studied. However, no study is made of the generalized manifold with boundary nor is a definition of such a space given except in the case of the generalized closed n-cell. A definition of a generalized manifold with boundary was given by the author in his paper [**1**]. Before undertaking the study of further properties of these manifolds it seems appropriate to characterize the manifolds with boundary in terms of the open and closed manifolds of Wilder. It is to that purpose that this paper is directed and in particular the generalized closed n-cell of Wilder is characterized as a special manifold with boundary.

The space M that we shall deal with will be a compact Hausdorff space and the homology theory used will be that of Čech in which the coefficient group for the chains will be an arbitrary field which we shall omit from the notation for a chain. We shall use small Roman letters for points and large Roman letters for sets of points. We shall use "$\cup$" for point set union or sum, "$\cap$" for intersection, reserving $+$ and $-$ for the group operations.

1. The generalized manifold with boundary; condition D.

Definition 1. If K is a closed subset of M, then we will say that *the local r-dimensional Betti number of M at x* mod K, denoted by $p_r(M \bmod K; x)$, *is the finite integer k* if k is the smallest positive integer with the property that corresponding to any open set P such that $x \in P$ there exists an open set Q such that $x \in Q$, $\bar{Q} \subset P$, and such that any $k+1$ Čech cycles of M mod

$$(M - (P - K)) = (M - P) \cup K$$

are linearly dependent with respect to homologies on M mod

$$M - (Q - K) = (M - Q) \cup K.$$

(Note that if $K = 0$ then this definition is equivalent to the definition of $p_r(M, x)$, the local Betti number of M at x. Also this is equivalent to Wilder's definition [**2**, p. 291] of the Betti number around a point.)

Definition 1.2. The compact space M will be called an *n-dimensional generalized manifold* (n-gm) *with boundary* if there exists a closed subset K of M such that:

(1) $M = K \cup A$ where A is open, $K = \bar{A} - A$, and dim $K < n$, and dim $M = n$ (in the sense of Lebesgue [**2**, p. 195],

Received October 31, 1950.

(2) $p_r(M \bmod K; x) = 0$ for all $x \in M$, $r \leqslant n-1$,

(3) $p_n(M \bmod K; x) = 1$ for all $x \in M$,

(4) $p_r(M, x) = 0$ for all $x \in K$, $r \leqslant n$.

(Note that this definition reduces to Wilder's definition of an n-dimensional generalized closed manifold (n-gcm) when $K = 0$ [**2**, p. 244] for (2) becomes $p_r(M, x) = 0$, $r < n$, which is equivalent to r-co-local connectedness $r < n$, and (4) no longer applies.)

THEOREM 1.1. *The boundary set K in Definition 1.2 is unique.*

Proof. Let K_1 and K_2 be two closed subsets of M satisfying the conditions of Definition 1.2. Suppose there is a point $x \in K_1$ but not $\in K_2$. By (3), $p_n(M \bmod K_2; x) = 1$, but since $x \notin K_2$, $p_n(M \bmod K_2; x) = p_n(M, x)$, hence $p_n(M, x) = 1$ which is a contradiction to $p_n(M, x) = 0$ by (4) since $x \in K_1$. Thus $K_1 = K_2$.

THEOREM 1.2. *A necessary and sufficient condition that M be a manifold with boundary is that there exist a closed subset K of M such that:*

(1) $M = K \cup A$, $K \cap A = 0$, $\dim M = n$,

(2) K is an $(n-1)$-gcm (see note after Definition 1.2),

(3) A is a non-compact n-gm (i.e., Definition 1.2 with $K = 0$ and M locally compact, but not compact),

(4) $p_r(M, x) = 0$, $r \leqslant n$ for all $x \in K$.

Proof of necessity. (1) This clearly follows from Definition 1.2 (1). (2) We first note that K is compact since $K = \bar{A} - A$ is a closed subset of the compact set M. We next show that $p_{n-1}(K, x) = 1$ for all $x \in K$ and that K is colc^{n-2} (that is, r-co-locally connected $r \leqslant n-2$, which is equivalent to $p_r(K, x) = 0$, $r \leqslant n-2$ for all $x \in K$). Since by Definition 1.2 (4) $p_{r-1}(M, x) = p_r(M, x) = 0$ for $r \leqslant n$ and all $x \in K$, it follows that $p_{r-1}(K, x) = p_r(M \bmod K; x)$ which [**2**, p. 291, Theorem 1.4] gives us the result by referring to (2) and (3) of the definition. Finally, $\dim K = n-1$, for $\dim K < n$, but if $\dim K \leqslant n-2$, then [**2**, p. 196, Theorem 7.7] we conclude that $p_{n-1}(K, x) = 0$ contrary to what we have just proved above.

(3) We first observe that $K \neq 0$ since $\dim K = n-1$ and, therefore, A is locally compact but not compact, since it is an open subset of a compact space M and has a non-vacuous boundary K. Also $\dim A = n$ by Definition 1.2 (3). A is clearly colc^{n-1} and $p_n(A, x) = 1$ for $x \in A$ by (2) and (3) since $p_r(M \bmod K; x) = p_r(M, x) = p_r(A, x)$ for $x \in A$.

(4) This is the same as Definition 1.2 (4).

Proof of sufficiency. By (1) $\bar{A} - A \subset K$ and if $x \in K$, but $x \notin \bar{A} - A$ then

$$p_{n-1}(M, x) = p_{n-1}(K, x),$$

but $p_{n-1}(M, x) = 0$ by (4) which contradicts $p_{n-1}(K, x) = 1$ by part of (2); therefore $\bar{A} - A = K$. Also $\dim K = M - 1 < M$ by (2).

By [**2**, p. 292, Theorem 1.4] again together with $p_{r-1}(M, x) = p_r(M, x) = 0$ for $x \in K$, $r \leqslant n$, we conclude that $p_{r-1}(K, x) = p_r(M \bmod K; x)$ for all $x \in K$.

By (2) and the above equality, we have $p_r(M \bmod K;\, x) = p_{r-1}(K, x) = 0$ for all $x \in K$, $r \leqslant n - 1$, and for $x \in A$ this follows from (3).

By (2) and the above equality we have $p_n(M \bmod K; x) = p_{n-1}(K, x) = 1$ for all $x \in K$, which follows from (3) for $x \in A$.

This is the same as (4).

Definition 1.3. The n-gm M with boundary will be said to satisfy condition D if $p^n(M_1, K) = 0$ ($p^n(M_1, K)$ denotes the number of n-cycles on M_1 mod K linearly independent with respect to homologies on M_1 mod K), where M_1 is a proper closed subset of M. We will denote this by saying that M is an n-D-gm with boundary. (This corresponds to condition D [**2**, p. 250].)

THEOREM 1.3. *A necessary and sufficient condition that M be an n-*D-gm *with boundary is that there exists a closed subset K of M such that*:

(1) $M = K \cup A$, $K \cap A = 0$, dim $M = n$.

(2) *K is an $(n - 1)$*-gcm.

(3) *A is a non-compact n*-gm *satisfying* D′ (i.e., if z^n is an infinite cycle of A on a subset A_1 of A closed relative to A such that the closure of $A - A_1$ relative to A is compact, then $z^n \backsim 0$ on A. See [**2**, p. 254]).

(4) *$p_r(M, x) = 0$, $r \leqslant n$, for all $x \in K$.*

Before proving this theorem we remark that in the case of a closed subset K of a compact space M, such as we have here, the cycles M mod K and the infinite cycles on $A = M - K$ are related in a one to one fashion so that if z^n is a cycle on M mod K and $\mathfrak{z}^n$ is the related infinite cycle on A, then $z^n \backsim 0$ on M mod K if and only if $\mathfrak{z}^n \backsim 0$ on A. This result has been verified by Wilder in connection with some of his work that is not yet published. We now proceed to the proof.

Proof of necessity. All the conditions (1), (2), (3), and (4) follow from Theorem 1.2, except that A satisfies D′. To verify this let $A_1 \subset A$ be closed relative to A, such that $\overline{A - A_1}$ relative to A is compact. Let $\mathfrak{z}^n$ be an infinite cycle on A_1, then by the above remark there is a corresponding cycle z^n on M mod K and z^n will be on $M_1 = A_1 \cup K_1$ which is a proper closed subset of M. By hypothesis $p^n(M_1, K) = 0$ implies that $z^n \backsim 0$ on M_1, hence on M_1 mod K, and by the remark $\mathfrak{z}^n \backsim 0$ on A.

Proof of sufficiency. That M is an n-gm with boundary follows from Theorem 1.2. To show that M satisfies D, let M_1 be a proper closed subset of M. Now $A \cap (M - M_1) \neq 0$, for otherwise $A \subset M_1$ and $\bar{A} = M \subset M$; also $A \cap (M - M_1)$ is open and therefore contains an open set U such that $\bar{U} \subset A \cap (M - M_1)$. If z^n is a cycle on M_1 mod K_1 then the corresponding infinite cycle $\mathfrak{z}^n$ of A is on $A - U$, which is proper closed relative to A, and

$$\overline{A - (A - U)} = \bar{U} \subset A$$

is compact. Thus $\mathfrak{z}^n \backsim 0$ on A and by the remark $z^n \backsim 0$ on M mod K, but this implies that $z^n = 0$ mod K (with respect to the n-dimensional coverings of M); hence $z^n \backsim 0$ on M_1 mod K (with respect to all coverings).

THEOREM 1.4. *If M is an n-D-gm with boundary K, then $p^n(M_1, K_1) = 0$, where M_1 and K_1 are closed subsets of M and K, respectively, and at least one subset is proper.*

Proof. It is sufficient to prove the theorem for cycles whose coordinates are restricted to a complete family of coverings of M, [**2**, p. 130], and since M is n-dimensional we can suppose this to be the family of n-dimensional coverings of M [**2**, p. 195]. First consider the case where K_1 is a proper closed subset of K. Let $p \in K - K_1$, then $p_n(M, x) = 0$ and we can choose open sets U, V such that $x \in V$, $\bar{V} \subset U$, $U \cap K_1 = 0$, and such that any n-cycle on M mod $(M - U)$ is $\backsim 0$ on M mod $(M - V)$. In particular if z^n is a cycle on M_1 mod K_1, then it is a cycle mod $(M - U)$; hence $\backsim 0$ on M mod $(M - V)$. Since only n-dimensional coverings are being used, this means that

$$z^n = 0 \quad \text{mod } (M - V),$$

that is, z^n is on $M - V$. Now $M - V$ is a proper closed subset of M; therefore $p^n(M - V, K) = 0$, and $z^n \backsim 0$ on $M - V$ mod K, but as before this implies that z^n is on K; hence $\backsim 0$ on M_1 mod K, that is, $p^n(M_1, K_1) = 0$. The only remaining case would be where M_1 is a proper subset of M and $K_1 = K$, then $p^n(M_1, K_1) = 0$ by hypothesis.

Remark. The condition D is actually stronger than the similar condition $p^n(M, K_1) = 0$, for proper closed subsets K_1 of K, which it implies, as is shown by letting M consist of the union of a bounded 2-cell and a disjoint projective plane.

THEOREM 1.5. *If M is an n-*D-gm *with boundary K, then $p^n(M, \mathrm{K}) \leqslant 1$.*

Proof. Suppose $C^n{}_1$ and $C^n{}_2$ are cycles on M mod K linearly independent with respect to homologies on M mod K. Let $x \in M - K$, then $p_n(M, x) = p_n(M \text{ mod } K; x) = 1$ (by Definition 1.2 (3)); therefore we can find open sets V, U such that $x \in V$, $\bar{V} \subset U$, $U \cap K = 0$ and such that any two cycles mod $(M - U)$ are linearly dependent with respect to homologies mod $(M - V)$. In particular $C^n{}_1$ and $C^n{}_2$ are cycles mod $(M - U)$; hence there exist elements a_1, a_2 of the coefficient field, not both zero, such that

$$a_1 C^n{}_1 + a_2 C^n{}_2 \sim 0 \quad \text{mod } (M - V).$$

Again we can restrict our cycles to n-dimensional coverings of M, which implies that

$$z^n = a_1 C^n{}_1 + a_2 C^n{}_2$$

is on $M - V$. Since $M - V$ is a proper closed subset of M, and M satisfies D, we have $z^n \backsim 0$ on $M - V$ mod K contrary to the assumption that $C^n{}_1$ and $C^n{}_2$ were linearly independent mod K; thus $p^n(M, K) \leqslant 1$.

2. Orientability.

Definition 2.1. An n-gm M with boundary K is called *orientable* if M is the carrier of a cycle z^n mod K such that $z^n \not\sim \Gamma^n$ mod K on M where Γ^n is a cycle mod K on a proper closed subset of M. (If $K = 0$, this becomes the definition of an orientable n-gcm.)

THEOREM 2.1. *A necessary and sufficient condition that M be an orientable* n-gm *with boundary is that there exist a closed subset K of M such that*:

(1) $M = K \cup A$, $K \cap A = 0$, dim $M = n$.

(2) *K is an* $(n-1)$-gcm.

(3) *A is a non-compact* n-gm *orientable in the sense that there is an infinite cycle on A not homologous on A to any infinite cycle on a proper closed subset of A.*

(4) $p_r(M, x) = 0$, $r \leqslant n$, *for all* $x \in K$.

(See [**2**, p. 254] for definition of infinite cycle on a proper closed subset.)

From Theorem 1.2 it follows that we need only show that the orientability of A is a necessary and sufficient condition for the orientability of M.

Proof of necessity. Let z^n be the cycle on M mod K given in the definition of orientability and let $\mathfrak{z}^n$ be the infinite cycle on A corresponding to z^n according to the remark after the statement of Theorem 1.3. If $z^n \sim \mathfrak{z}^n{}_1$ on A where $\mathfrak{z}^n{}_1$ is an infinite cycle on a proper closed subset A_1 of A, then $\mathfrak{z}^n{}_1$ determines a cycle $z^n{}_1$ on $A_1 \cup K = M_1$ which is a proper closed subset of M such that $z^n \sim z^n{}_1$ mod K on M, contrary to the hypotheses on z^n. Thus $\mathfrak{z}^n$ is the required infinite cycle on A.

Proof of sufficiency. Let $\mathfrak{z}^n$ be the infinite cycle on A in the definition of the orientability of A and let z^n be the cycle on M mod K corresponding to it. Suppose $z^n \sim z^n{}_1$ on M mod K, where $z^n{}_1$ is a cycle on M_1 mod K, and M_1 is a proper closed subset of M. Then $z^n{}_1$ would correspond to an infinite cycle $\mathfrak{z}^n{}_1$ on a proper closed subset A_1 of A as in the proof of Theorem 1.3, and z^n would be $\sim \mathfrak{z}^n{}_1$ on A, contrary to the hypotheses on $\mathfrak{z}^n$. Thus z^n is the required cycle on M mod K in the definition of orientability.

THEOREM 2.2. *If M is an orientable* n-gm *with boundary K, then K is an orientable* $(n-1)$-gcm.

Proof. K is an $(n-1)$-gcm by Theorem 1.2. To show that K is orientable, let z^n be the cycle on M mod K according to the definition of orientability; then $\partial z^n = z^{n-1}$ is a cycle on K. Suppose $z^{n-1} \sim z^{n-1}{}_1$ on K where $z^{n-1}{}_1$ is a cycle on a proper closed subset K_1 of K. Thus $z^{n-1}{}_1 \sim 0$ on M and by [**2**, p. 201, Lemma 1.4] there exists a cycle C^n mod K_1 on M such that $\partial C^n \sim z^{n-1}{}_1$ on K_1. Choose $x \in K - K_1$, and since $p_n(M, x) = 0$, there exist open sets U, V, such that $x \in V$, $\bar{V} \subset U$, $U \cap K_1 = 0$, and such that any n-cycle mod $(M - U)$ is ~ 0 mod $(M - V)$. In particular C^n is a cycle mod $(M - U)$; therefore $C^n = 0$ on V (when restricted to the complete family of n-dimensional coverings). Thus C^n

https://doi.org/10.4153/CJM-1952-029-x Published online by Cambridge University Press

is on a proper closed subset $M_1 = M - V$ of M. Now

$$\partial C^n - \partial z^n \sim {z^{n-1}}_1 - z^{n-1} \sim 0 \text{ on } K;$$

therefore, there exists by [**2**, p. 201, Lemma 1.6] a cycle Γ^n on M such that $(C^n - z^n) \backsim \Gamma^n \bmod K$ on M. Again Γ^n is a cycle on M (mod K_1); therefore Γ^n (when restricted to n-dimensional coverings) is on M_1 as before. Now $z^n \backsim C^n - \Gamma^n \bmod K$ on M, where $C^n - \Gamma^n$ is on M_1. It follows that z^n is $\backsim \bmod K$ to a cycle on a proper closed subset of M. This contradicts the orientability assumption; hence z^{n-1} is not homologous to a cycle on a proper closed subset of K, which is the orientability condition for K.

The orientability condition for the n-D-gm can be more simply stated as is indicated in the following theorem.

THEOREM 2.3. *A necessary and sufficient condition that an n-D-gm M with boundary K be orientable is that $p^n(M, K) = 1$.*

Proof. By Theorem 1.5, $p^n(M, K) \leqslant 1$; thus the orientability assumption, which implies $p^n(M, K) \geqslant 1$, implies $p^n(M, K) = 1$. Conversely, $p^n(M, K) = 1$ implies the existence of a cycle z^n on $M \bmod K$ which is not $\backsim 0 \bmod K$ and, therefore, is not homologous to a cycle ${z^n}_1$ on a proper closed subset M_1 of M, for any such cycle z_1 is $\backsim 0$ by property D.

In connection with the orientable n-D-gm it turns out that if condition D had been stated, "$p^n(M, K_1) = 0$, where M_1, K_1 are closed subsets of M and K, respectively, such that one but not both inclusions are proper," then the n-dimensional part of Definition 1.2 (4) follows from the other hypotheses. This is embodied in the following theorem and corollary.

THEOREM 2.4. *For an orientable n-gm M with boundary K satisfying the condition $p^n(M, K_1) = 0$ where K_1 is a proper closed subset of K, the condition $p_n(M, x_1) = 0$ for $x \in K$ in the definition of a manifold follows from the other conditions in the definition.*

Proof. It will be sufficient to prove the proposition for the complete family of n-dimensional coverings of M. Let x be any point of K and U any open set, $x \in U$, and let γ^n be an arbitrary n-cycle mod $(M - U)$, hence mod $[(M - U) \cup K]$. Since $p_n(M \bmod K, x) = 1$, there is an open set $V, \bar{V} \subset U, x \in V$, such that there is only one cycle mod $[(M - U) \cup K]$ linearly independent with respect to homologies on M mod $[(M - V) \cup K]$. Let z^n be the cycle on $M \bmod K$ from the definition of orientability, then z^n is a cycle mod $[(M - U) \cup K]$. Also

$$z^n \not\sim 0 \quad \text{mod } [(M - V) \cup K],$$

for otherwise it would be on $(M - V) \cup K$ (since only n-dimensional coverings are being used), contrary to the orientability assumption which says that z^n is not $\backsim$ to a cycle mod K on a proper closed subset of M. Now suppose $\gamma^n \not\sim 0$ mod $[(M - V) \cup K]$ on M, then there exist elements $a_1 \neq 0$ and $a_2 \neq 0$ of the coefficient field such that $a_1\gamma^n + a_2 z^n \backsim 0$ mod $[(M - V) \cup K]$ on M, but this means that $a_1\gamma^n + a_2 z^n = 0$ on $V - K$, hence $= 0$ on

$$\overline{V-K} = \bar{V}.$$

Then $a_1\partial\gamma^n + a_2\partial z^n = 0$ on $\bar{V}$, but $\partial\gamma^n$ is on $M - U$, hence $= 0$ on $\bar{V} \subset U$; thus $\partial z^n = 0$ on $\bar{V}$. Now $K_1 = K - V$ is a proper closed subset of K and by the above z^n is a cycle mod K_1; therefore by the hypothesis $p^n(M, K_1) = 0$, we have $z^n \backsim 0$ on M mod K_1, contrary to the orientability assumption. Thus we conclude that

$$\gamma^n \sim 0 \quad \text{mod } [(M - V) \cup K] \text{ on } M;$$

hence $\gamma^n = 0$ on $V - K$ (since only n-dimensional coverings are considered). This, however, implies $\gamma^n = 0$ on

$$\overline{V-K} = \bar{V};$$

therefore $\gamma^n \backsim 0$ mod $(M - V)$, and $p_n(M, x) = 0$ for all $x \in K$.

COROLLARY 2.4.1. *For an orientable* n-D-gm *M with boundary K the condition $p^n(M, K_1) = 0$, where K_1 is a proper closed subset of K, is equivalent to the condition $p_n(M, x) = 0$ for all $x \in K$.*

Proof. The proof follows by combining Theorems 1.4 and 2.4

The following example shows that condition (4) of Definition 1.2 is necessary for $r < n$ even in the case of an orientable n-D-gm.

Example. Let M be a solid pinched sphere, i.e., a 2-sphere plus its interior in which all points on some fixed diameter are identified. Let K equal the boundary 2-sphere with the pinched points, then M satisfies conditions D and (1), (2), (3), and (4) (for $r = n = 3$) of Definition 1.2, but $p_2(M, x) = 1$ where x is the pinched point.

The next theorem and its corollaries clarify the role of the n-D-gm in connection with the orientable n-gm.

3. The orientable manifold satisfying condition D.

THEOREM 3.1. *If M is an* n-gm *with boundary K, then M has only a finite number of components $M_1 \cup M_2 \cup \ldots \cup M_k$, and each component M_i is an* n-gm *with boundary $K_i = K \cap M_i$; and if M is orientable, then each M_i is an orientable* n-D-gm *with boundary K_i.*

Proof. M has only a finite number of components since it is compact and locally-0-connected. Let $A_i = M_i - K_i$, then clearly (1) $\bar{A}_i - A_i = K_i$. Conditions (2), $p_r(M_i \text{ mod } K_i, x) = 0$ for all $x \in M_i$, $r \leqslant n - 1$; (3) $p_n(M_i \text{ mod } K_i, x) = 1$ for all $x \in K_i$; and (4), $p_r(M_i, x) = 0$ for all $x \in K_i$, $r \leqslant n$ follow immediately from the corresponding conditions on M and K since $K_i \subset K$ and the M_i are separated. The condition dim $M_i = n$ follows since $M_i \subset M$ and dim $M = n$ implies that dim $M_i \leqslant n$, and condition (3) for M_i requires that dim $M_i \geqslant n$. Also if M is orientable, then each M_i is orientable, for the cycle

z^n in the definition of orientability of M can be written in the form

$$z^n = z^n_1 + \ldots + z^n_k,$$

where each z_i is the part of z^n on M_i and is clearly a cycle mod K_i with the properties required for the orientability of M_i.

Finally we will show that M_i satisfies condition D. To this end consider M'_i a proper closed subset of M_i. We must show $p^n(M'_i, K_i) = 0$, and it will be sufficient to consider only the complete family of n-dimensional coverings of M. Let C^n be a cycle on M'_i mod K_i and let $M''_i \subset M'_i$ be a minimal locus of concentration for the cycle C^n, that is, M''_i is a closed set such that every open set $\supset M''_i$ is a carrier of C^n and M''_i is minimal with respect to that property. The existence of such a minimal locus of concentration is guaranteed by [**2**, p. 205, 2.2]. Let x be a point on the boundary of M''_i relative to M_i; such points exist since M_i is connected and M''_i is a proper closed subset, and $x \in M''_i$. Since $p_n(M_i \bmod K_i, x) = 1$, there exist open sets V, U such that $x \in V$, $\bar{V} \subset U$, and such that there is only one n-cycle on M_i mod $[(M_i - U) \cup K_i]$ linearly independent with respect to homologies mod $[(M_i - V) \cup K_i]$. Now both C^n and z^n_i are cycles mod K_i, hence, mod $[(M_i - U) \cup K_i]$; therefore, there exist elements a_1 and a_2 of the coefficient field, not both zero, such that

$$a_1 z^n_i + a_2 C^n \sim 0 \quad \text{mod } [(M_i - V) \cup K_i].$$

Now z^n_i is not $\backsim 0$ mod $[(M_i - V) \cup K_i]$, for if it were, then it would be equal to zero on $V - K_i$ (since only n-dimensional coverings are being used), and z^n_i would be on the proper closed subset $(M - V) \cup K_i$ of M_i, contrary to the orientability of M_i. Also if $M''_i \not\subset K_i$ then C^n is not $\backsim 0$ mod $[(M_i - V) \cup K_i]$ for if it were, then, as above,

$$[(M_i - V) \cup K_i] \cap M''_i$$

would be a proper closed subset of M''_i and a locus of concentration for C^n, contrary to the minimal property of M''_i. We therefore conclude that $a_1 \neq 0$ and $a_2 \neq 0$ in the preceding homology, and that

$$C^n \sim -(a_1/a_2) z^n_i \quad \text{mod } [(M_i - V) \cup K_i] \text{ on } M_i;$$

hence $z^n = -(a_1/a_2) z^n_i$ on $V - K_i$. Since x is a boundary point of M''_i and $K_i = \bar{A}_i - A_i$, there is a point $y \in V - K_i - M''_i$. Let W be an open set such that

$$x \in W, \quad \bar{W} \cap M''_i = 0, \quad W \subset V - K_i - M''_i.$$

Now just as before $z^n_i \neq 0$ on W, but $C^n = 0$ on W, since M''_i is a locus of concentration and this requires the open set $M_i - \bar{W} \supset M''_i$ to carry C^n. This is, however, contrary to $C^n = -(a_1/a_2) z^n_i$ on $V - K_i$; therefore we conclude that $M''_i \subset K_i$, and that $C^n = 0$ mod K_i.

COROLLARY 3.1.1. *A necessary and sufficient condition that the n-gm M with boundary K be orientable is that $p^n(M_i, K_i) = 1$ for each component M_i of M where $K_i = K \cap M_i$, and that each M_i satisfy condition* D; *in particular, a* *necessary and sufficient condition that a connected* n-gm M *with boundary* K *be orientable is that* $p^n(M, K) = 1$, *and* M *satisfy* D.

Proof. The necessity follows from Theorems 2.4 and 3.1. The sufficiency follows from Theorem 2.4 which requires each M_i to be orientable, and from the fact that M is clearly orientable if each component is.

COROLLARY 3.1.2. *If* M *is an orientable* n-gm *with boundary* K, *then* $p^n(M, K)$ *is the number of components of* M.

Proof. By Corollary 3.1.1, $p^n(M_i, K_i) = 1$ for each component; therefore $p^n(M, K)$ is the number of components, since $H^n(M, K)$ is isomorphic to the direct sum of the groups $H^n(M_i, K_i)$.

COROLLARY 3.1.3. *A necessary and sufficient condition that an orientable* n-gm M with boundary K *be an* n-D-gm *is that* M *be connected.*

Proof. The necessity follows from Corollary 3.1.2 and Theorem 2.4. The sufficiency follows directly from the Theorem.

The above theorem and corollaries allow us to restrict our attention in the orientable case to the n-D-gm.

THEOREM 3.2. *If* M *is an orientable* n-D-gm (*or equivalently connected orientable* n-gm) *with boundary* K, *then* M *is an irreducible membrane relative ro* z^{n-1}, *the cycle referred to in the Definition* 2.1. (See Definition [**2**, p. 209].)

Proof. In the proof of Theorem 2.2 it was shown that if z^n is the cycle on M mod K in Definition 2.1, then $\partial z^n = z^{n-1}$ satisfies the definition of orientability for the $(n-1)$-gcm K. Clearly $z^{n-1} \backsim 0$ on M; suppose also that $z^{n-1} \backsim 0$ on M_1, a proper closed subset of M. By [**2**, p. 201, Lemma 1.4] there exists a cycle $z^n{}_1$ on M_1 mod K such that $\partial z^n{}_1 \backsim z^{n-1}$ on K. Thus

$$\partial(z^n - z^n{}_1) = z^{n-1} - \partial z^n{}_1 \sim 0 \text{ on } K$$

and by [**2**, p. 201, Lemma 1.6] there exists a cycle C^n on M such that $z^n - z^n{}_1 \backsim C^n$ on M mod K. By property D, $C^n \backsim 0$; hence $z^n \backsim z^n{}_1$ mod K on M contrary to the orientability hypothesis of M. Thus M is an irreducible membrane for the homology $z^{n-1} \backsim 0$ on M.

Definition 3.1. If K is a closed subset of the compact space M then $g^r(M; K, 0)$ is the maximum number of r-cycles on $K \backsim 0$ on M and linearly independent with respect to homologies on K [**2**, p. 211].

THEOREM 3.3. *If* M *is an orientable* n-D-gm *with boundary* K, *then* $g^{n-1}(M; K, 0) = 1$ *irreducibly* (that is, $g^{n-1}(M; K, 0) = 1$ *and* $g^{n-1}(M; K_1, 0) = 0$ *where* K_1 *is a proper closed subset of* K).

Proof. Theorem 3.2 yields a cycle z^{n-1} on $K \backsim 0$ on M such that $z^{n-1} \not\sim 0$ on K (since $z^{n-1} \not\sim 0$ on any proper closed subset of M). Thus $g^{n-1}(M; K, 0) \geqslant 1$. Now consider two cycles $z^{n-1}{}_1$ and $z^{n-1}{}_2$ on K such that $z^{n-1}{}_i \backsim 0$ on M $(i = 1, 2)$.

By [**2**, p. 201, Lemma 1.4], there exist cycles $C^n{}_i$ on M mod K such that $\partial C^n{}_i \backsim z^{n-1}{}_i$ on K $(i = 1, 2)$. By Theorem 8, $p^n(M, K) = 1$; hence there exist elements a_1, a_2 of the coefficient field, not both zero, such that $a_1 C^n{}_1 + a_2 C^n{}_2 \backsim 0$ on M mod K. By [**2**, p. 201, Lemma 1.3],

$$\partial(a_1 C^n_1 + a_2 C^n_2) = a_1 \partial C^n_1 + a_2 \partial C^n_2 \sim 0 \text{ on } K;$$

hence $a_1 z^{n-1}{}_1 + a_2 z^{n-1}{}_2 \backsim 0$ on K, that is, $g^{n-1}(M; K, 0) \leqslant 1$. Thus $g^{n-1}(M; K, 0) = 1$.

Next consider the proper closed subset K_1 of K and a cycle $z^{n-1}{}_1$ on K_1 such that $z^{n-1}{}_1 \backsim 0$ on M. As before, there exists a cycle $C^n{}_1$ on M mod K_1 such that $\partial C^n{}_1 \backsim z^{n-1}{}_1$ on K_1. By Theorem 4, $p^n(M, K_1) = 0$; hence $C^n{}_1 \backsim 0$ on M mod K_1, which implies that $z^{n-1}{}_1 \backsim 0$ on K_1. Thus $g^{n-1}(M; K_1, 0) = 0$.

4. The generalized n-cell. Before proving the next theorem we prove three lemmas needed later.

LEMMA 1. *If M is an n-gm with boundary K, then $A = M - K$ is* ulc^{n-1} (uniformly r-locally connected $r \leqslant n - 1$).

Proof. Let A' be homeomorphic with A such that $A' \cap A = 0$ and such that $A' \cup K$ is an n-gm M' with boundary K; then by Theorem 2.4 of (1) $M \cup M' = S$ is an n-gcm. By [**2**, p. 292, Theorem 1.7] we have $S - K = A' \cup A$ is $(n - r - 1)$ $-$ ulc for $0 \leqslant n - r - 1 \leqslant n - 1$, since

$$p_r(K, x) = p^r(K, x) = 0, \qquad 0 \leqslant r \leqslant n - 1,$$

by Theorem 1.2. Since A' and A are separate, this implies that A is ulc^{n-1}.

LEMMA 2. *If M is an orientable n-gm with boundary K and γ^r, $r \leqslant n - 1$, is a cycle on K, then there exists a compact cycle z^r in $A = M - K$ such that $\gamma^r \backsim z^r$ on M.*

Proof. As in the preceding proof an n-gcm S can be constructed with A as an open ulc^{n-1} subset of S. Furthermore it follows from Theorem 3.2 of (1) that S is orientable. The conclusion now follows from [**2**, p. 301, Theorem 5.9].

LEMMA 3. *If M is an orientable n-gm with boundary K such that $A = M - K$ is an F_σ, then $h^{n-r}(A) \approx H^r(M, K)$ $r \leqslant n$ if either group has finite dimension,* where $h^s(A)$ denotes the s-dimensional (unaugmented) homology group of A with respect to compact cycles.

Proof. As in the preceding lemmas, A can be considered as an open subset of an n-gcm. By Theorem 2.1, A is an orientable non-compact n-gm. By [**2**, p. 258, Theorems 5.13 and 5.14], $H^r(A) \approx h^{n-r}(A)$, $r \leqslant n$, if either group has finite dimension, where $H^r(A)$ is the r-dimensional homology group of A with respect to infinite cycles. By the remark after Theorem 1.3, $H^r(A) \approx H^r(M, K)$; thus $h^{n-r}(A) \approx H^r(M, K)$, $r \leqslant n$.

Definition 4.1. *A generalized closed n-cell* is an orientable n-D-gm with a non-vacuous boundary K such that $p^r(M, K) = 0$, $0 < r < n$.

https://doi.org/10.4153/CJM-1952-029-x Published online by Cambridge University Press

THEOREM 4.1. *If K is a closed subset of M such that $M - K$ is an F_σ, then a necessary and sufficient condition that M be a generalized closed n-cell with boundary K is that*:

(1) $M = K \cup A$, $K \cap A = 0$, dim $M = n$.

(2) *K is an* $(n-1)$-gcm.

(3) *A is a generalized* (open) *n-cell* (i.e., a non-compact orientable n-gm satisfying D′ which is cell-like in the sense that its compact (augmented) homology groups of dimensional $< n$ reduce to the identity).

(4) $p_r(M, x) = 0$, $r \leqslant n$, *for all* $x \in K$.

Proof of necessity. By Theorems 1.3 and 2.1, it follows that (1), (2), (4) are satisfied, and that A is an orientable n-gm satisfying D′. By Theorem 2.3, $p^n(M, K) = 1$, and by hypothesis $p^r(M, K) = 0$, $0 < r < n$; therefore by Lemma 13.3,

$$h^{n-r}(A) \approx H^r(M, K), \qquad 0 < r \leqslant n.$$

Thus for $0 < r < n$, $h^{n-r}(A) \approx 0$ and $h^0(A)$ has dimension 1, or if the augmented homology groups are used the 0-dimensional group also reduces to the identity, as required in condition (3).

Proof of sufficiency. M is an orientable n-D-gm by Theorems 1.3 and 2.1. By Lemma 13.3 $h^{n-r}(A) \approx H^r(M, K)$ for $0 \leqslant n - r < n$, i.e., for $0 < r \leqslant n$ since the dimension of the left-hand group is finite by property (3). In particular, $h^{n-r}(A) \approx 0$ for $0 < n - r < n$; therefore $p^r(M, K) = 0$ for $0 < r < n$.

THEOREM 4.2. *If M is a generalized closed n-cell with boundary K, such that $A = M - K$ is an F_σ, then $p^r(M) = 0$ for all r* (where augmented theory is used).

Proof. By Corollary 3.1.3, M is connected and $p^r(M) = 0$ (using augmented theory). Next let z^r be a cycle on M, $0 < r < n$. Since z^r is also a cycle mod K and $p^r(M, K) = 0$, $0 < r < n$, it follows that $z^r \backsim 0$ mod K on M. By [**2**, p. 203, Lemma 1.13], there is a cycle γ^r on K such that $\gamma^r \backsim z^r$ on M. By Lemma 2 above, there is a compact cycle C^r of A such that $\gamma^r \backsim C^r$ on M. Since by Theorem 4.1, A is a generalized n-cell, $C^r \backsim 0$ on A; hence, $z^r \backsim 0$ on M. Since M is an n-D-gm,

$$p^n(M, 0) = p^n(M) = 0$$

by Theorem 4. Finally, $p^r(M) = 0$ for all $r > n$ since M is n-dimensional; thus $p^r(M) = 0$ for all r.

THEOREM 4.3. *If M is a generalized closed n-cell with boundary K, such that $A = M - K$ is an F_σ, then K is a sphere-like* $(n-1)$-gcm. (The $(n-1)$-gcm is sphere-like if its homology groups are isomorphic to those of the $(n-1)$-sphere.)

Proof. Let z^r, $0 \leqslant r \leqslant n-2$, be a cycle on K, then by Lemma 2 above, there is a compact cycle γ^r in A such that $z^r \backsim \gamma^r$ on M. By Theorem 4.1, A is a generalized n-cell, and it follows that $\gamma^r \backsim 0$ in A; hence $z^r \backsim 0$ on M. By

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[**2**, p. 201, Lemma 1.4], there is a cycle C^{r+1} mod K on M such that $\partial C^{r+1} \backsim z^r$ on K. Now $1 \leqslant r+1 \leqslant n-1$; hence $p^{r+1}(M, K) = 0$, by hypothesis. Thus $C^{r+1} \backsim 0$ mod K on M, and by [**2**, p. 201, Lemma 1.3], $z^r \backsim 0$ on K. By Theorem 2.2, K is orientable; therefore, there is at least one cycle z^{n-1} on K, not $\backsim 0$ on K; hence $p^{n-1}(K) > 0$. Consider two cycles $z^{n-1}{}_1$ and $z^{n-1}{}_2$ on K, then by the same argument used above in the lower dimensions, we have two cycles $C^n{}_1$ and $C^n{}_2$ mod K on M such that $\partial C^n{}_i \backsim z^{n-1}{}_i$ on K $(i = 1, 2)$. Since M is an orientable n-D-gm, we have $p^n(M, K) = 1$ by Theorem 2.3; therefore, there exist elements a_1 and a_2 of the coefficient group, not both 0, such that

$$a_1 C^n{}_1 + a_2 C^n{}_2 \sim 0 \quad \text{mod } K \text{ on } M.$$

Thus as before $a_1 \partial C^n{}_1 + a_2 \partial C^n{}_2 \backsim 0$ on K; hence,

$$a_1 z^{n-1}{}_1 + a_2 z^{n-1}{}_2 \sim 0 \text{ on } K,$$

which proves that $p^{n-1}(K) = 1$.

THEOREM 4.4. *If M is a generalized closed n-cell with boundary K, such that $A = M - K$ is an F_σ, and if z^{n-1} is a non-bounding $(n-1)$-cycle on K, then M is an irreducible membrane relative to z^{n-1}.*

Proof. By Theorem 3.2 there is one cycle z^{n-1} on K satisfying the conclusion of the theorem. By Theorem 4.3 any two non-bounding $(n-1)$-cycle on K are linearly dependent; therefore, any such cycle satisfies the conclusion of the theorem.

Remark. Wilder has defined a generalized closed n-cell [**2**, p. 287] as a compact space M satisfying Conditions (1), (2), (3), and (4) of Theorem 4.1, and in addition the properties in the conclusions of Theorems 4.3 and 4.4. Thus we have proved (at least in the case where $M - K$ is on F_σ), that the weakened conditions (1), (2), (3), and (4) are equivalent to Wilder's apparently stronger conditions. It should also be noted that if (1), (2), (3), and (4) are taken as the definition of a closed n-cell, then an examination of the proofs of Theorems 4.2, 4.3, and 4.4 shows that the conclusions of those theorems hold without the hypothesis that $M - K$ be an F_σ if the additional assumption $p^r(M, K) = 0$, $0 < r < n$, is added. Thus we have proved the following theorem.

THEOREM 4.5. *A necessary and sufficient condition that the n-gm M with boundary K, such that $A = M - K$ is an F_σ, be a generalized closed n-cell in the sense of Wilder is that $p^r(M, K) = 0$, $0 < r < n$, or that A be a generalized (open) n-cell.*

The next theorem, which is a summary of the necessary and sufficient conditions contained in Theorems 1.2, 1.3, 2.1, and 4.1, shows that in each case the condition imposed on M is equivalent to a similar condition on $A = M - K$ in the presence of three other conditions that do not change.

THEOREM 4.6. *A necessary and sufficient condition that M be*:

(a) *an* n-gm *with boundary,*
(b) *an* n-D-gm *with boundary,*
(c) *an orientable* n-gm *with boundary,*
(d) *a generalized closed n-cell,*

is that there exists a closed subset K of M, such that:

(1) $M = K \cup A$, $K \cap A = 0$, $\dim M = n$.
(2) *K is an* $(n-1)$-gcm.
(3) *A is an open, non-compact* $\begin{cases} \text{(a) } n\text{-gm,} \\ \text{(b) } n\text{-gm } \textit{satisfying } \mathrm{D}', \\ \text{(c) } \textit{orientable } n\text{-gm,} \\ \text{(d) } \textit{generalized } n\textit{-cell.} \end{cases}$
(4) $p_r(M, x) = 0$, $r \leqslant n$, *for all* $x \in K$.

(In case (d) the additional hypothesis that A be an F_σ must be included.)

5. Classical manifolds. We close with two theorems which show that the generalized manifolds with boundary reduce to the classical ones in the one- and two-dimensional separable cases.

THEOREM 5.1. *If M is a connected, separable* 1-gm *with boundary K and* $K = 0$, *then M is a 1-sphere and if* $K \neq 0$, *then M is an arc with end points a and b such that* $K = a \cup b$.

Proof. By Theorem 1.2, K is a 0-gcm, that is, a finite set of points; and by Conditions (2) and (4), M is a Peano continuum. Suppose M is not a 1-sphere; then either M is acyclic or contains a 1-sphere J. If $M \supset J$, but $M \neq J$, then the argument in the remarks on [**2**, p. 271] yields a neighbourhood P of a point $x \in J$ and three arcs xx', xy', and xy such that each lies entirely in P except their end points x', y', and y which lie on the boundary of P, and each pair of arcs has only the point x in common. We can also suppose P is chosen so that it contains no points of $K - x$. Then the arcs $(x'x \cup xy)$ and $(x'x \cup xy')$ carry 1-cycles mod $[(M - \mathrm{P}) \cup K]$ (actually mod $(M - P)$) which, because M is one-dimensional, are linearly independent with respect to homologies mod $[(S - Q) \cup K]$ for every $Q, \bar{Q} \subset P$, contrary to $p_1(M \bmod K; x) = 1$. If M is acylic, then $M \supset$ a (maximal) non-degenerate arc J. If $M \neq J$, the argument above again yields a neighbourhood P of an interior point $x \in J$ and three arcs xx', xy', and xy with the above properties, and leads to a contradiction as before. Thus M is either a 1-sphere or an arc. If M is a 1-sphere, then $K = 0$; for if $x \in K$, then

$$p_1(M \bmod K; x) = p_1(M \bmod x; x) = 2,$$

contrary to property (3). If M is an arc with end points a and b, then $K \subset a \cup b$ by the argument just applied above for the 1-sphere. Finally a, $b \in K$, for if $a \notin K$ then

$$p_1(M \bmod K, a) = p_1(M, a) = 0,$$

contrary to property (3). This shows that the remainder of the theorem holds.

COROLLARY 5.1. *If M is a separable* 1-gm *with boundary K, then M consists of a finite number of components each of which is a* 1*-sphere or an arc.*

Proof. This follows directly from Theorem 5.1 and the fiirst part of Theorem 3.1.

THEOREM 5.2. *If M is a connected separable* 2-gm *with boundary K, then M is a classical* 2-gm *from which a finite number of open* 2*-cells whose closures are disjoint have been deleted. K consists of the union of the* 1*-spheres that form the boundaries of the deleted* 2*-cells.*

Proof. By Theorem 1.2, K is a 1-gcm, hence $K = J_1 \cup J_2 \cup \ldots \cup J_k$, where the (J_i) are pairwise disjoint 1-spheres. Let

$$M' = (A_1 \cup J_1) \cup (A_2 \cup J_2) \cup \ldots \cup (A_k \cup J_k)$$

where each A_i is an open 2-cell with boundary J_i, $A_i \cap A_j = 0$, $i \neq j$, and $A_i \cap M = 0$ for all i. Now M' is a 2-gm with boundary K and by (1) $M' \cup M$ is a 2-gm. By [**2**, p. 272, Theorem 2.3], $M' \cup M$ is a classical 2-gcm and our theorem follows. (If M is not assumed connected then as before M consists of a finite number of components each of which has the property of Theorem 5.2.)

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