ON THE JOIN OF VARIETIES OF GROUPS

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We introduce two pseudoabelian group varieties, each defined by one word in two variables, and prove that their join is not pseudoabelian. We also prove that every variety which is nilpotent of class two, is not join-inaccessible.

In most cases, given the sets of laws for two group varieties, it is very difficult to obtain a description of the set of laws for the join of the varieties. Therefore it is important to find connections between properties of varieties and of their join. In 1982, Kovács raised the following question [1, problem 8.22]. If G is a finite group contained in a join $\mathfrak{U} \vee \mathfrak{V}$ of two varieties $\mathfrak{U}$, $\mathfrak{V}$ of groups, must there exist finite groups $A \in \mathfrak{U}$, $B \in \mathfrak{V}$ such that G is a section of the direct product $A \times B$? In [3] a variety is called pseudoabelian if it is nonabelian but all its finite groups are abelian. Answering a question of H. Neumann [4, problem 5], Ol'shanskii [5] proved the existence of pseudoabelian varieties of groups. In 1973, Kovács and Newman [3] stated a question concerning the join of two pseudoabelian varieties: must the join of two pseudoabelian varieties be pseudoabelian? According to [2], a variety $\mathfrak{U}$ is called join-inaccessible if $\mathfrak{U} \subset \mathfrak{X} \vee \mathfrak{Y}$ implies $\mathfrak{U} \subset \mathfrak{X}$ or $\mathfrak{U} \subset \mathfrak{Y}$. In [2] Kovács raised the question as to whether the nilpotent just-nonabelian varieties ($\mathfrak{B}_4 \wedge \mathfrak{N}_2$ and $\mathfrak{B}_p \wedge \mathfrak{N}_2$ for odd primes p) are join-inaccessible.

In this paper we prove that there exist two pseudoabelian group varieties such that their join is not pseudoabelian and we also show that every variety which is nilpotent of class two, is not join-inaccessible. These results imply negative answers to all three questions cited above.

We put

$$v_0 = v_0(x,y) = \left(x^d y^d\right)^d x^d,$$
$$v_1 = v_1(x,y) = [v_0^d, x^d],$$
$$v_2 = v_2(x,y) = [v_1^d, x^d],$$
$$v_3 = v_3(x,y) = [v_2^d, x^d],$$
$$w_1(x,y) = [x,y]v_2^n[x,y]^{\varepsilon_1}v_2^{n+1}\cdots[x,y]^{\varepsilon_h}v_2^{n+h},$$
$$w_2(x,y) = [x,y]v_3^n[x,y]^{\varepsilon_1}v_3^{n+1}\cdots[x,y]^{\varepsilon_h}v_3^{n+h},$$

where $h \equiv 0 \pmod{10}$, $\varepsilon_{10k+1} = \varepsilon_{10k+2} = \varepsilon_{10k+3} = \varepsilon_{10k+5} = \varepsilon_{10k+6} = 1$, $\varepsilon_{10k+4} = \varepsilon_{10k+7} = \varepsilon_{10k+8} = \varepsilon_{10k+9} = \varepsilon_{10k+10} = -1$, $k = 0,1,\cdots,h/10-1$ and d,n,h are sufficiently large natural numbers.

Let W_i denote the verbal subgroup of an absolutely free group F of countable rank corresponding to the word $w_i(x,y)$ and let $\mathfrak{M}_i$ be the variety defined by the word $w_i(x,y)$, $i = 1,2$. Besides the varieties $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_1 \vee \mathfrak{M}_2, \mathfrak{M}_1 \wedge \mathfrak{M}_2$, we need to consider the variety $[\mathfrak{O}, \mathfrak{M}_1 \wedge \mathfrak{M}_2]$ which is defined by the set of words $[F, W_1W_2]$ and of which a free group is the free central extension of a free group of the variety $\mathfrak{M}_1 \wedge \mathfrak{M}_2$.

The construction and the study of the varieties $\mathfrak{M}_1$, $\mathfrak{M}_2, \mathfrak{M}_1 \wedge \mathfrak{M}_2$ and $[\mathfrak{O}, \mathfrak{M}_1 \wedge \mathfrak{M}_2]$ follow the patterns detailed in [6, 25.1], [6, 29.3] and [7]. In particular, the relatively free groups $G_1 = F/W_1$, $G_2 = F/W_2$, $G = F/W_1W_2$ are described in terms of their presentations with defining relations being introduced by induction.

LEMMA 1. *The varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are pseudoabelian varieties.*

PROOF: When substituting $[x,z]$ for every x in the word $w_i(x,y)$, $i = 1,2$, and taking into account the equation $\varepsilon_1 + \ldots + \varepsilon_h = 0$, we see that every metabelian group of the variety $\mathfrak{M}_i$ is nilpotent of class two. But it is also easily seen that in the variety $\mathfrak{M}_i$ all groups nilpotent of class two are abelian. So all metabelian groups in the varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are abelian. Hence all finite groups of the varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are abelian. The fact that the varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are nonabelian is proved in the same way as the corresponding part of [6, Theorem 30.1]. $\Box$

Now we consider the presentation $G = \langle \mathfrak{A} \mid R = 1, R \in \mathfrak{R} \rangle$ chosen in respect to the prescriptions of [6, 29.3] and [7] where $\mathfrak{A}$ is a basis of the group F.

LEMMA 2. *The presentation $G = G(\infty) = \langle \mathfrak{A} \mid R = 1, R \in \mathfrak{R} \rangle$ satisfies condition R.*

Lemma 2 is proved as Lemma 8 [7].

LEMMA 3. *Nontrivial values of the words $v_2(x,y)$ and $v_3(x,y)$ in G cannot be conjugate in G to powers of the same word that is a simple word or a period of the presentation $G(\infty)$.*

Lemma 3 can be proved in the same way as the corresponding parts of [7, Lemma 5] and [7, Lemma 10].

LEMMA 4. *Nontrivial values $w_1(X_1, Y_1)$ and $w_2(X_2, Y_2)$ cannot be equal in $\widetilde{G} = F/[F, W_1 W_2]$.*

PROOF: Suppose that $w_1(X_1, Y_1) = w_2(X_2, Y_2)$ in $\widetilde{G}$ where (X_1, Y_1) and (X_2, Y_2) are two pairs of words noncommutative in G. As in the proof of [6, Lemma 39.12], we can find defining words R_1 and R_2 which are conjugate to $w_1(X_1, Y_1)$ and $w_2(X_2, Y_2)$ respectively in rank less than the maximum of the ranks of R_1 and R_2. So there exists an annular diagram Δ such that the labels of its contours are graphically equal to R_1 and $R_2^{\pm 1}$ and the number of cells of ranks not less than the maximum of the ranks of R_1 and R_2 is even in Δ. Since $R_1 \not\equiv R_2$ by Lemma 3, the existence of the diagram Δ contradicts Lemma 2 and [6, Lemma 25.1]. ☐

THEOREM. $\mathfrak{M}_1 \vee \mathfrak{M}_2 \supset [\mathfrak{O}, \mathfrak{M}_1 \wedge \mathfrak{M}_2]$.

PROOF: Given a word $U \in W_1 \cap W_2$, we may consider equations

$$w_1(X_1, Y_1) w_1(X_2, Y_2) \ldots = U = w_2(X_1', Y_1') w_2(X_2', Y_2') \ldots$$

in $\widetilde{G}$. In the same way as in the proof of [6, Lemma 39.16] we obtain that all nontrivial values of the words $w_1(x, y)$ and $w_2(x, y)$ in $\widetilde{G}$ form a basis for the free abelian group $W_1 W_2 / [F, W_1 W_2]$. By Lemma 4, nontrivial values $w_1(X_i, Y_i)$ and $w_2(X_j', Y_j')$ are different in $\widetilde{G}$. Hence $U = 1$ in $\widetilde{G}$ whence $W_1 \cap W_2 \subset [F, W_1 W_2]$. ☐

COROLLARY 1. *The join $\mathfrak{M}_1 \vee \mathfrak{M}_2$ of two pseudoabelian varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is not pseudoabelian.*

PROOF: Any nonabelian finite group that is nilpotent of class two belongs to the variety $\mathfrak{M}_1 \vee \mathfrak{M}_2$ by the Theorem. Thus the variety $\mathfrak{M}_1 \vee \mathfrak{M}_2$ is not pseudoabelian but $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are both pseudoabelian by Lemma 1. ☐

COROLLARY 2. *Every variety which is nilpotent of class two, is not join-inaccessible.*

PROOF: Any variety that is nilpotent of class two is a subvariety of $\mathfrak{M}_1 \vee \mathfrak{M}_2$ by the Theorem. But as was noted in the proof of Lemma 1, every group, which is nilpotent of class two and belongs to $\mathfrak{M}_1$ or $\mathfrak{M}_2$, is abelian. Therefore, no nilpotent nonabelian variety is a subvariety of $\mathfrak{M}_1$ or $\mathfrak{M}_2$. ☐

REFERENCES

[1]   Kourovka Notebook, *Unsolved problems of the group theory*, Tenth Edition (Novosibirsk, 1986).

[2]   L.G. Kovács, 'Inaccessible varieties of groups', *Bull. Austral. Math. Soc.* **17** (1974), 506-510.

[3]   L.G. Kovács and M.F. Newman, 'Hanna Neumann's problems on varieties of groups', in *Proceedings of the Second International Conference on the Theory of Groups*, Lecture Notes in Mathematics **372** (Springer-Verlag, Berlin, Heidelberg, New York, 1974), **pp.** 417–431.

[4]   H. Neumann, *Varieties of groups* (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[5]   A.Yu. Ol'shanskii, 'Varieties in which all finite groups are abelian', *Mat. Sb.* **126** (1985), 59–82.

[6]   A.Yu. Ol'shanskii, *Geometry of defining relations in groups*, Mathematics and its application (Soviet Series) **70** (Kluwer Academic Publishers, Dordrecht, 1991).

[7]   A. Storozhev, 'On abelian subgroups of relatively free groups', *Comm. Algebra* **22** (1994), 2677–2701.

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