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REGULARITY CRITERION AND CLASSIFICATION FOR ALGEBRAS OF JORDAN TYPE

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Abstract. We show that Artin–Schelter regularity of a $\mathbb{Z}$-graded algebra can be examined by its associated $\mathbb{Z}^r$-graded algebra. We prove that there is exactly one class of four-dimensional Artin–Schelter regular algebras with two generators of degree one in the Jordan type. This class is strongly noetherian, Auslander regular, and Cohen–Macaulay. Their automorphisms and point modules are described.

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Introduction The classification of Artin–Schelter regular algebras, or classification of quantum projective spaces, is one of important questions in noncommutative projective algebraic geometry. Many researchers have been interested in Artin–Schelter regular algebras and many have made great contributions on the subject. In the case of global dimension 4, plenty of Artin–Schelter regular algebras have been discovered in recent years [**9**, **13**, **17**, **18**], and most of them are endowed with an appropriate $\mathbb{Z}^2$-grading. It is not the case for four-dimensional Artin–Schelter regular algebras of Jordan type (see the subsection 3.1 below). This motivates us to study this kind of algebras.

The idea used here is to link a $\mathbb{Z}$-graded algebra with an appropriate $\mathbb{Z}^r$-graded algebra for some positive integer $r (> 1)$. By means of the *leading homogeneous polynomials* $\mathrm{LH}(\mathcal{G})$ (see the subsection 2.2) of some Gröbner basis $\mathcal{G}$ of an ideal, we show it is available for those $\mathbb{Z}$-graded algebras without $\mathbb{Z}^2$-grading on them. Our first result is a regularity criterion for a connected graded algebra.

THEOREM 0.1. *Let $A = k\langle X\rangle/I$ be a connected graded algebra. Then, A is Artin–Schelter regular in case there is an appropriate $\mathbb{Z}^r$-grading on $k\langle X\rangle$ such that $\mathbb{Z}^r$-graded algebra $k\langle X\rangle/(\mathrm{LH}(\mathcal{G}))$ is Artin–Schelter regular, where $\mathcal{G}$ is the reduced Gröbner basis of I with respect to an admissible order $\prec_{\mathbb{Z}^r}$ on the free monoid generated by X including* 1.

An application of the criterion in this paper is the connected Artin–Schelter regular algebras of dimension 4 with two generators whose Frobenius data is of Jordan type. The generic constraints condition (see [9]) in this case turns out to be invalid. Using the A_∞-algebra theory and applying the criterion to the Jordan type, we get a classification result:

THEOREM 0.2. *The algebra* $\mathcal{J} = \mathcal{J}(u, v, w) = k\langle x, y\rangle/(f_1, f_2)$ *is an Artin–Schelter regular algebra of global dimension* 4, *where*

$$\begin{aligned} f_1 &= xy^2 - 2yxy + y^2x, \\ f_2 &= x^3y - 3x^2yx + 3xyx^2 - yx^3 + (1-u)xyxy + uyx^2y \\ &\quad + (u-3)yxyx + (2-u)y^2x^2 - vy^2xy + vy^3x + wy^4, \end{aligned}$$

and $u, v, w \in k$.

If k *is algebraically closed of characteristic* 0, *then it is, up to isomorphism, the unique Artin–Schelter regular algebra of global dimension* 4 *which is generated by two elements whose Frobenius data is of Jordan type.*

As the criterion for Artin–Schelter regularity, we provide a similar method to recognize the ring-theoretic and homological properties of an Artin–Schelter regular algebra from the known one.

THEOREM 0.3. *Let* $\mathcal{J}$ *be the Artin–Schelter regular algebra showed in the theorem above. Then*

(a) $\mathcal{J}$ *is strongly noetherian, Auslander regular and Cohen–Macaulay;*

(b) The automorphism group of $\mathcal{J}$ *is* $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a \in k\backslash\{0\},\ b \in k\right\}$;

(c) $\mathcal{J}$ *has two classes of point modules up to isomorphism.*

Here is an outline of the paper. In Section 1, we review some basic definitions of Artin–Schelter regular algebras, A_∞-algebras and $\mathbb{Z}^r$-filtered algebras. The links about regularity between $\mathbb{Z}^r$-filtered algebras and associated $\mathbb{Z}^r$-graded algebras are considered in Section 2. The next two sections are devoted to an application of the criterion to the classification of Artin–Schelter regular algebras of Jordan type. Properties of the classified result $\mathcal{J}$ of Jordan type in Theorem 0.2 are presented in Section 5.

Throughout the paper, let k be a commutative based field; in Sections 3–5, we will assume that k is algebraically closed with characteristic 0. Unless otherwise stated, graded means $\mathbb{Z}$-graded, the tensor product $\otimes$ means $\otimes_k$. For simplicity, we only consider graded algebras that are generated in degree 1. The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. We denote $\mathbb{Z}^r$ the set of r-tuples of $\mathbb{Z}$ with the standard basis $\varepsilon_i = (0, \dots, 1, \dots, 0)$ for $i = 1, 2, \dots, r$.

1. Preliminaries. In this section, we recall the definitions of Artin–Schelter regular algebras, A_∞-algebras, and $\mathbb{Z}^r$-filtered algebras as well as some fundamental consequents in preparation for the classification.

By a *norm map* on $\mathbb{Z}^r$ we mean the map $\|\cdot\| : \mathbb{Z}^r \to \mathbb{Z}$ which sends $\alpha = (a_1, \dots, a_r)$ to $\sum_{i=1}^r a_i$. An *admissible ordering* $<$ related to the norm map is a total ordering such that $\|\alpha_1\| < \|\alpha_2\|$ implies $\alpha_1 < \alpha_2$, and $\alpha_1 < \alpha_2$ implies $\alpha_1 + \alpha_3 < \alpha_2 + \alpha_3$ for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}^r$.

Let $A = \bigoplus_{\alpha \in \mathbb{Z}^r} A_\alpha$ be a $\mathbb{Z}^r$-graded algebra. For a homogeneous element $a \in A_\alpha$, we call α and $\|\alpha\|$ the *degree* and the *total degree* of a, denoted by $\deg a$ and $\operatorname{tdeg} a$, respectively. We say A is *connected* if $A_\alpha = 0$ for all $\alpha \notin \mathbb{N}^r$ and $A_0 = k$. A connected $\mathbb{Z}^r$-graded algebra A is called *proper* if A is generated by $\bigoplus_{i=1}^r A_{\varepsilon_i}$ with $A_{\varepsilon_i} \neq 0$ for $i = 1, 2, \dots, r$. We denote by $\operatorname{GrMod} A$ the category of $\mathbb{Z}^r$-graded left A-modules with morphisms of A-homomorphisms preserving degrees, and by $\operatorname{grmod} A$ the full subcategory of GrMod A consisting of finitely generated $\mathbb{Z}^r$-graded left A-modules. The categories of $\mathbb{Z}^r$-graded right A-modules, denoted by GrMod A^o and grmod A^o, respectively, are defined analogously. When $r = 1$, it goes back to the usual graded situation.

Given a $\mathbb{Z}^r$-graded A-module $M = \bigoplus_{\alpha\in\mathbb{Z}^r} M_\alpha$ and $\beta \in \mathbb{Z}^r$, its *shift* is $M(\beta) \in \mathrm{GrMod}\,A$ defined by $M(\beta)_\alpha = M_{\alpha+\beta}$ for any $\alpha \in \mathbb{Z}^r$. For $M, N \in \mathrm{GrMod}\,A$, we write $\underline{\mathrm{Hom}}_A(M, N) = \bigoplus_{\alpha\in\mathbb{Z}^r} \mathrm{Hom}_{\mathrm{GrMod}\,A}(M, N(\alpha))$ and $\underline{\mathrm{Ext}}^i_A(M, N) = \bigoplus_{\alpha\in\mathbb{Z}^r} \mathrm{Ext}^i_{\mathrm{GrMod}\,A}(M, N(\alpha))$.

1.1. Artin–Schelter regular algebras. The following definition is originally due to Artin and Schelter [**2**].

DEFINITION 1.1. A connected $\mathbb{Z}^r$-graded algebra A is called *Artin–Schelter regular* (AS-regular, for short) of dimension d if the following three conditions hold:

(AS1) A has finite global dimension d;
(AS2) A has finite Gelfand–Kirillov dimension;
(AS3) A is Gorenstein; that is, for some $l \in \mathbb{Z}^r$,

$$\underline{\mathrm{Ext}}^i_A(k, A) = \begin{cases} k(l) & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$$

where l is called *Gorenstein parameter*.

The following proposition was originally proved for $\mathbb{Z}$-graded algebras, and holds true in our $\mathbb{Z}^r$-setting.

PROPOSITION 1.2 ([**14**, Proposition 3.1]). *Let A be a $\mathbb{Z}^r$-graded AS-regular algebra of dimension d with Gorenstein parameter l, then the minimal projective resolution of ${}_A k$ is*

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to A \to_A k \to 0,$$

where $P_j = \bigoplus_{i=1}^{s_j} Ae_i^j$ is a finitely generated free module on basis $\{e_i^j\}_{i=1}^{s_j}$ with $\deg e_i^j \in \mathbb{N}^r$ for all $1 \leq j \leq d-1$, and $P_d = Ae_d$ is a free module on e_d with $\deg e_d = l$.

Artin and Schelter in [**2**] conjecture that all AS-regular algebras are noetherian. The examples of AS-regular algebras found so far are in their guess. With the assumption of noetherian on AS-regular algebras, some abstract properties have been proved in small dimensions. For example, any noetherian connected graded AS-regular algebra of global dimension 4 and GKdim $A = 4$ is a domain (see [**4**, Theorem 3.9]). Hence, in the following we assume that AS-regular algebras are domains which is a fundamental assumption in the classification of AS-regular algebras of Jordan type.

As the paper [**9**] observes, the AS-regular algebras of global dimension 4 which are domains have three resolution types as they named (14641), (13431) and (12221) according to the number of generators in degree 1. In the Sections 3–5, we will focus on the AS-regular algebra A of type (12221) whose minimal resolution of trivial module ${}_A k$ is

$$0 \to A(-7) \to A(-6)^{\oplus 2} \to A(-4) \oplus A(-3) \to A(-1)^{\oplus 2} \to A \to_A k \to 0, \qquad (*)$$

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and the Hilbert series is

$$H_A(t) = \frac{1}{(1-t)^3(1-t^2)(1-t^3)}.$$

In some papers, the condition (AS2) is not required in the definition of Artin–Schelter regular algebra. The following theorem is a pivotal point in the classification which was proved by using A_∞-algebra method.

THEOREM 1.3 ([**8**, Theorem 12.5], [**11**, Corollary D]). *Let A be a connected graded algebra, and let $E(A) := \mathrm{Ext}_A^*(k,k)$ be the Yoneda algebra of A. Then, A satisfies* (AS1) *and* (AS3) *if and only if $E(A)$ is Frobenius.*

1.2. A_∞-algebras. The definition and notation of the A_∞-algebra are introduced in this subsection briefly. We refer to [**6**, **9**] for the details.

DEFINITION 1.4. Let $E = \bigoplus_{i\in\mathbb{Z}} E^i$ be a $\mathbb{Z}$-graded k-vector space. The vector space E is an A_∞-*algebra* if it is endowed with a family of graded k-linear maps

$$m_n : \ E^{\otimes n} \to E, \quad n \geq 1,$$

of degree $2-n$ satisfying the following *Stasheff identities* $\mathsf{SI(n)}$:

$$\sum_{\substack{r+s+t=n;\\ s\geq 1;\ r,t\geq 0}} (-1)^{r+st} m_{r+1+t}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0$$

for all $n \geq 1$.

Note that when the formulas are applied to elements, additional signs appear due to the Koszul sign rule. We assume every A_∞-algebra in this article contains an identity element $1 \in E^0$ which satisfies *strictly unital condition*; that is,

(a) $m_2(1,x) = x$ and $m_2(x,1) = x$, for every $x \in E$;

(b) $m_n(x_1,\ldots,x_n) = 0$, if $x_i = 1$ for some i and $n \neq 2$.

Now let A be a connected graded algebra, then the *Yoneda algebra* $E(A)$ is bi-graded naturally: one is the *homological degree*, written as superscript, and the other one is *Adams degree*, written as subscript, the latter is induced by the grading of A. It is a basic fact that $E(A)$ can be viewed as the cohomology algebra of some differential graded algebra.

For any differential graded algebra D, there is a canonical A_∞-algebra structure on its cohomology algebra $\mathrm{H}(D)$ which is unique in the sense of A_∞-isomorphisms. This is a key result in the A_∞-world named "*Minimal Model Theorem*" (see [**6**]). A concrete method of constructing the minimal model is provided in [**12**]. As a consequence, $E(A)$ is equipped with a natural A_∞-algebra structure. We adopt that the A_∞-algebra structure in this paper are bi-graded and all multiplications $\{m_n\}$ preserve Adams degree; that is, $\deg(m_n) = (2-n, 0)$. It is a nontrivial hypothesis since such an A_∞-algebra structure exists (see [**10**]).

We use $E(A)$ to denote both the usual associative Yoneda algebra and the A_∞-Ext-algebra with any choice of its A_∞-structure. There is such a graded algebra A that its associative Yoneda algebra $E(A)$ does not contain enough information to recover the original algebra A; on the other hand, the information from the A_∞-algebra $E(A)$ is sufficient to recover A. This is the point of the following theorem:

THEOREM 1.5 ([**10**, Corollary B]). *Let A be a connected graded algebra which is finitely generated in degree 1 and E be the A_∞-algebra $\mathrm{Ext}^*_A(k_A, k_A)$. Let $R = \bigoplus_{n\geq 2} R_n$ be a minimal graded space of relations of A such that $R_n \subset A_1 \otimes A_{n-1} \subset A_1^{\otimes n}$. Let $i : R_n \to A_1^{\otimes n}$ be the inclusion map, and $i^\sharp$ be its k-linear dual. Then, the multiplication m_n of E restricted to $(E^1)^{\otimes n}$ is equal to the map*

$$i^\sharp : (E^1)^{\otimes n} = (A_1^\sharp)^{\otimes n} \to R_n^\sharp \subset E^2,$$

where $R^\sharp$ and $A_1^\sharp$ is the graded k-linear dual of R and A_1, respectively.

1.3. $\mathbb{Z}^r$-filtered algebras and modules. The basic definitions and notations of $\mathbb{Z}^r$-filtered algebras and $\mathbb{Z}^r$-filtered modules are given in this subsection. We refer to [**7**] for the details. Let $<$ be a fixed admissible ordering on $\mathbb{Z}^r$.

DEFINITION 1.6. A k-algebra B is called a $\mathbb{Z}^r$-*filtered algebra* if there is a family $\{F_\alpha(B)\}_{\alpha\in\mathbb{Z}^r}$ of k-subspaces of B such that

(a) $F_\alpha(B) \subseteq F_{\alpha'}(B)$ if $\alpha < \alpha'$;
(b) $F_\alpha(B)F_{\alpha'}(B) \subseteq F_{\alpha+\alpha'}(B)$, for any $\alpha, \alpha' \in \mathbb{Z}^r$;
(c) $B = \bigcup_{\alpha\in\mathbb{Z}^r} F_\alpha(B)$, and $1 \in F_0(B)$.

In the definition above, the family $\{F_\alpha(B)\}_{\alpha\in\mathbb{Z}^r}$ is called a $\mathbb{Z}^r$-*filtration* of B.

An *associated $\mathbb{Z}^r$-graded algebra* of $\mathbb{Z}^r$-filtered algebra B is defined by

$$G^r(B) = \bigoplus_{\alpha\in\mathbb{Z}^r} \frac{F_\alpha(B)}{F_{<\alpha}(B)},$$

where $F_{<\alpha}(B) = \bigcup_{\alpha'<\alpha} F_{\alpha'}(B)$.

DEFINITION 1.7. Let B be a $\mathbb{Z}^r$-filtered algebra and M a B-module. We say M is a $\mathbb{Z}^r$-*filtered* B-module if there exists a $\mathbb{Z}^r$-filtration on it; that is, a family $\{F_\alpha(M)\}_{\alpha\in\mathbb{Z}^r}$ of k-subspaces of M such that

(a) $F_\alpha(M) \subseteq F_{\alpha'}(M)$ if $\alpha < \alpha'$;
(b) $F_\alpha(B)F_{\alpha'}(M) \subseteq F_{\alpha+\alpha'}(M)$, for any $\alpha, \alpha' \in \mathbb{Z}^r$;
(c) $M = \bigcup_{\alpha\in\mathbb{Z}^r} F_\alpha(M)$.

Also there is an *associated $\mathbb{Z}^r$-graded module* of M

$$G^r(M) = \bigoplus_{\alpha\in\mathbb{Z}^r} \frac{F_\alpha(M)}{F_{<\alpha}(M)},$$

where $F_{<\alpha}(M) = \bigcup_{\alpha'<\alpha} F_{\alpha'}(M)$. Clearly, $G^r(M)$ is a $\mathbb{Z}^r$-graded $G^r(B)$-module.

Let M be a $\mathbb{Z}^r$-filtered B-module. If L is a submodule of M, there is an *induced $\mathbb{Z}^r$-filtration* $\{F_\alpha(L)\}_{\alpha\in\mathbb{Z}^r}$ on L, where $F_\alpha(L) = L \bigcap F_\alpha(M)$. And an *induced $\mathbb{Z}^r$-filtration* $\{F_\alpha(M/L)\}_{\alpha\in\mathbb{Z}^r}$ on M/L is defined by $F_\alpha(M/L) = (F_\alpha(M) + L)/L$. We assume that the $\mathbb{Z}^r$-filtration on submodules and quotient modules is always the induced one in this paper.

DEFINITION 1.8. For two $\mathbb{Z}^r$-filtered $M, N \in \mathrm{GrMod}\,B$, let ϕ be a B-homomorphism from M to N. We say ϕ is $\mathbb{Z}^r$-*filtered* if $\phi(F_\alpha(M)) \subseteq F_\alpha(N)$ for any $\alpha \in \mathbb{Z}^r$.

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Moreover, ϕ induces a $G^r(B)$-homomorphism from $G^r(M)$ to $G^r(N)$ denoted by $G^r(\phi)$.

Furthermore, ϕ is called *strict* if $\phi(F_\alpha(M)) = \phi(M) \bigcap F_\alpha(N)$ for all $\alpha \in \mathbb{Z}^r$. The strictness also yields $\phi(F_{<\alpha}(M)) = \phi(M) \cap F_{<\alpha}(N)$ for all α. When $r = 1$, it coincides with the usual situation.

2. $\mathbb{Z}^r$-filtered algebras and associated $\mathbb{Z}^r$-graded algebras. This section is devoted to set up a link between the connected graded algebra and its associated $\mathbb{Z}^r$-graded algebra. We define a $\mathbb{Z}^r$-filtration on a connected graded algebra related to a partition of the generator set, and discuss relevant homological properties. Using Gröbner basis theory, we prove two criterions for examining regularity and ring-properties from known algebras.

In [**17**, **18**], the authors proved that the regularity and some other ring-properties of $\mathbb{N}$-filtered algebras can be examined from their associated graded algebras. However, $\mathbb{N}$-filtration is not enough to reduce the complexity in general. Torrecillas and Lobillo studied GK-dimension and global dimension of $\mathbb{N}^r$-filtered algebras in [**15**] and [**16**]. To deal with general cases, $\mathbb{Z}^r$-filtration should be much more selective. This leads to the following natural question, do the conclusions in [**17**, **18**] still hold for some $\mathbb{Z}^r$-filtration?

We consider two-dimensional AS-regular algebras firstly. It only has two types:

$$\mathfrak{A}(q) := k\langle x, y\rangle/(xy - qyx), \qquad \mathfrak{A}' := k\langle x, y\rangle/(yx - xy - x^2).$$

The $\mathbb{Z}^2$-graded part $yx - xy$ in the relation of $\mathfrak{A}'$ is a special case of the relation of $\mathfrak{A}(q)$. In fact, $\mathfrak{A}(1)$ is an associated $\mathbb{Z}^2$-graded algebra of $\mathfrak{A}'$ for some $\mathbb{Z}^2$-filtration. Similar phenomenon exists in S_2 and S_2' of three-dimensional AS-regular algebras (see [**2**]). Those evidences inspire us to find a criterion in general.

2.1. A $\mathbb{Z}^r$-filtration arising from a partition on the generator set. In order to guarantee that morphisms preserve the degrees, we need a special $\mathbb{Z}^r$-filtration. Some natural homological properties depending on this $\mathbb{Z}^r$-filtration are collected in this subsection.

First, we give an admissible ordering on the group $\mathbb{Z}^r$ as follows. Let $\alpha = (a_1, \ldots, a_r)$ and $\beta = (b_1, \ldots, b_r)$ be two arbitrary elements of $\mathbb{Z}^r$. We define $\alpha < \beta$ if one of the following two cases being satisfied

(a) $\|\alpha\| < \|\beta\|$, or

(b) $\|\alpha\| = \|\beta\|$ and there exists a $t\,(1 \leq t \leq r)$ such that $a_i = b_i$ for $i < t$ but $a_t < b_t$.

Now let $A = k\langle X\rangle/I$ be a connected graded algebra, where X is the minimal set of generators of A. Denote X^* the free monoid generated by X including 1. There is a canonical projection

$$\pi : k\langle X\rangle \to A.$$

Given a positive integer r $(1 < r \leq \#(X))$, we introduce a $\mathbb{Z}^r$-grading $\deg^r$ on $k\langle X\rangle$ as follows: let $\{X_1, X_2, \ldots, X_r\}$ be a partition of X, define $\deg^r x := (\delta_{1i}, \delta_{2i}, \ldots, \delta_{ri})$ if $x \in X_i$ $(i = 1, 2, \ldots, r)$, where δ_{ij} is the Kronecker symbol. Using this grading, we can get a $\mathbb{Z}^r$-filtration on $k\langle X\rangle$ defined by: $F_\alpha(k\langle X\rangle) = 0$ if $\alpha < 0$ and $F_\alpha(k\langle X\rangle) = \mathrm{Span}_k\{u \in X^* \mid \deg^r u \leq \alpha\}$ if $\alpha \geq 0$. That induces a $\mathbb{Z}^r$-filtration on A

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defined by

$$F_\alpha(A) = \pi(F_\alpha(k\langle X\rangle)), \quad \text{for any } \alpha \in \mathbb{Z}^r.$$

Convention: From now on, we fix this $\mathbb{Z}^r$-filtration on A, and denote by $G^r(A)$ the associated $\mathbb{Z}^r$-graded algebra of A.

Note that since A is generated in degree 1, the following is obvious.

LEMMA 2.1. *The $\mathbb{Z}^r$-filtration on A defined as above satisfies*

(a) $F_\alpha(A) \subseteq \bigoplus_{i\le\|\alpha\|} A_i$ and $F_\alpha(A) \subseteq A_{\|\alpha\|} + F_{<\alpha}(A)$ for all $\alpha \in \mathbb{Z}^r$.

(b) If $\alpha \notin \mathbb{N}^r$, then $F_\alpha(A) = F_{<\alpha}(A)$.

(c) $G^r(A)$ is a connected properly $\mathbb{Z}^r$-graded algebra.

Let $P = \bigoplus_{i=1}^s Ae_i$ be a finitely generated free A-module on the basis $\{e_i\}_{i=1}^s$. Take $\alpha_i \in \mathbb{Z}^r$ $(i = 1, 2, \ldots, s)$ such that $\|\alpha_i\| = \deg e_i$. We define a $\mathbb{Z}^r$-filtration on P by

$$F_\alpha(P) = \bigoplus_{i=1}^{s} \Big(\sum_{\gamma+\alpha_i\le\alpha} F_\gamma(A) \Big) e_i, \quad \alpha \in \mathbb{Z}^r. \tag{F1}$$

It is easy to check that $G^r(P)$, the associated $\mathbb{Z}^r$-graded module of P, is finitely generated and free as $G^r(A)$-module. We call $(P, \{\alpha_i\}_{i=1}^s)$ a *$\mathbb{Z}^r$-filtered pair of the free module P*.

Conversely, let $\underline{P} = \bigoplus_{i=1}^s G^r(A)\underline{e}_i$ be a finitely generated free $G^r(A)$-module on the basis $\{\underline{e}_i\}_{i=1}^s$ with $\deg \underline{e}_i = \alpha_i$ for $i = 1, 2, \ldots, s$. Then, there exists a $\mathbb{Z}^r$-filtered pair $(P, \{\alpha_i\}_{i=1}^s)$ of a free module P such that $G^r(P) = \underline{P}$, where $P = \bigoplus_{i=1}^s Ae_i$ is a finitely generated free A-module. Moreover, we set $\deg e_i = \operatorname{tdeg} \underline{e}_i$ for $i = 1, 2, \ldots, s$.

For other modules $M \in \operatorname{grmod} A$, some extra hypotheses of $\mathbb{Z}^r$-filtration on M is required. For convenience, we introduce a $\mathbb{Z}^r$-filtration on M by

$$F_\alpha(M) = \sum_{i=1}^{s} \Big(\sum_{\gamma+\beta_i\le\alpha} F_\gamma(A) \Big) \xi_i, \quad \text{for all } \alpha \in \mathbb{Z}^r, \tag{F2}$$

where $M = \sum_{i=1}^s A\xi_i$ and $\beta_i = (0, \ldots, 0, \deg \xi_i)$ for any $i = 1, 2, \ldots, s$. This $\mathbb{Z}^r$-filtration on finitely generated modules is a so-called "good" $\mathbb{Z}^r$-filtration. It also assures that $G^r(M) \neq 0$ if $M \neq 0$. Using Lemma 2.1 on this $\mathbb{Z}^r$-filtration of the finitely generated module M we obtain:

LEMMA 2.2. *Let M be a finitely generated A-module.*

(a) The associated $\mathbb{Z}^r$-graded $G^r(A)$-module $G^r(M)$ is finitely generated.

(b) $F_\alpha(M) \subseteq \bigoplus_{i\le\|\alpha\|} M_i$ and $F_\alpha(M) \subseteq M_{\|\alpha\|} + F_{<\alpha}(M)$ for all $\alpha \in \mathbb{Z}^r$.

(c) Let $\alpha = (a_1, \ldots, a_{r-1}, a_r) \in \mathbb{Z}^r$ such that $(a_1, \ldots, a_{r-1}) \notin \mathbb{N}^{r-1}$, then $F_\alpha(M) = F_{<\alpha}(M)$.

(d) For every $i \in \mathbb{Z}$, there are only finite $\alpha \in \mathbb{Z}^r$ such that $\|\alpha\| = i$ and $F_\alpha(M) \neq F_{<\alpha}(M)$.

Lemma 2.2 implies that the $\mathbb{Z}^r$-filtration $\{F_\alpha(M)\}_{\alpha\in\mathbb{Z}^r}$ on the finitely generated module M is well-ordering with respect to the order by inclusion. Furthermore, for any $m \in M$, there exists $\alpha \in \mathbb{Z}^r$ such that $F_\alpha(M) \neq F_{<\alpha}(M)$ and $m \in F_\alpha(M)$. The $\mathbb{Z}^r$-filtration on modules in $\operatorname{grmod} A^o$ can be defined in a similar way.

In the sequel, all $\mathbb{Z}^r$-filtration on free modules and finitely generated modules is considered to be defined as (F1) and (F2), respectively. With these preparations, we turn to consider the homological aspect of them.

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LEMMA 2.3. *Suppose $\mathbb{Z}^r$-filtered modules $M_1, M_2, M_3 \in \operatorname{GrMod} A$. Consider $\mathbb{Z}^r$-filtered sequence*

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3, \tag{$\natural$}$$

with $\varphi_2\varphi_1 = 0$ and the associated $\mathbb{Z}^r$-graded sequence:

$$G^r(M_1) \xrightarrow{G^r(\varphi_1)} G^r(M_2) \xrightarrow{G^r(\varphi_2)} G^r(M_3). \tag{$G^r(\natural)$}$$

(a) If sequence $(\natural)$ is exact and φ_1, φ_2 are strict, then sequence $(G^r(\natural))$ is exact.
(b) Suppose $M_i \in \operatorname{grmod} A$ (i=1, 2, 3), then sequence $(G^r(\natural))$ is exact if and only if $(\natural)$ is exact and φ_1, φ_2 are strict .

Proof. (a) Clearly $G^r(\varphi_2)G^r(\varphi_1) = 0$. Let $m_2 \in F_\alpha(M_2)\backslash F_{<\alpha}(M_2)$, and $0 \neq \overline{m}_2 \in G^r(M_2)$. Suppose $G^r(\varphi_2)(\overline{m}_2) = 0$. This implies $\varphi_2(m_2) \in F_{<\alpha}(M_3)$. However, $\varphi_2(m_2) \in \varphi_2(F_{<\alpha}(M_2))$ by the strictness of φ_2. There exists $m_2' \in F_{\alpha'}(M_2)$ such that $m_2 - m_2' \in \operatorname{Ker}\varphi_2$ where $\alpha' < \alpha$. Exactness of $(\natural)$ and strictness of φ_1 yield

$$\varphi_1(F_\alpha(M_1)) = \operatorname{Im}(\varphi_1) \cap F_\alpha(M_2) = \operatorname{Ker}(\varphi_2) \cap F_\alpha(M_2).$$

Thus, $\varphi_1(m_1) = m_2 - m_2'$ for some $m_1 \in F_\alpha(M_1)$. Then, $G^r(\varphi_1)(\overline{m}_1) = \overline{\varphi_1(m_1)} = \overline{m}_2$. This shows $\operatorname{Ker} G^r(\varphi_2) \subseteq \operatorname{Im} G^r(\varphi_1)$.

(b) The sufficiency is a special case of (a). To get the necessity, we proceed in two steps. The first step is to show the strictness. We need only to prove the strictness of φ_2 since a similar argument is valid for φ_1.

Choose $m_3 \in F_\alpha(M_3) \cap \operatorname{Im}(\varphi_2)$ and $m_3 \notin F_{<\alpha}(M_3)$. There exists $m_2 \in M_2$ with degree $\|\alpha\|$ which belongs to $F_{\alpha'}(M_2)$ such that $\varphi_2(m_2) = m_3$. If $\alpha' \leq \alpha$, the strictness is clear since $F_{\alpha'}(M_2) \subseteq F_\alpha(M_2)$. We assume $\alpha' > \alpha$, then $G^r(\varphi_2)(\overline{m}_2) = \overline{\varphi_2(m_2)} = 0$. By the exactness, there exists $m_1 \in F_{\alpha'}(M_1)$ with degree $\|\alpha\|$ such that $G^r(\varphi_1)(\overline{m}_1) = \overline{\varphi_1(m_1)} = \overline{m}_2$. Thus, $m_2' = m_2 - \varphi_1(m_1) \in F_{<\alpha'}(M_2)$ such that $\varphi_2(m_2') = m_3$. The proof is completed if $m_2' \in F_\alpha(M_2)$. Otherwise, there is $\alpha'' > \alpha$ such that $m_2' \in F_{\alpha''}(M_2)$ and $F_{\alpha''} \neq F_{<\alpha''}$. Repeat this procedure, by Lemma 2.2(d), it stops in finite steps. Finally, we get $\widetilde{m}_2 \in F_\alpha(M_2)$ such that $\varphi_2(\widetilde{m}_2) = m_3$.

The second step is exactness. Let $m_2 \in F_\alpha(M_2)\backslash F_{<\alpha}(M_2)$ such that $\varphi_2(m_2) = 0$. Then, $G^r(\varphi_2)(\overline{m}_2) = 0$. Hence $\overline{m}_2 \in \operatorname{Ker} G^r(\varphi_2)$. There exists $m_1^{(1)} \in F_\alpha(M_1)\backslash F_{<\alpha}(M_1)$ satisfying $G^r(\varphi_1)(\overline{m_1^{(1)}}) = \overline{\varphi_1(m_1^{(1)})} = \overline{m}_2$. Thus, $m_2' = m_2 - \varphi_1(m_1^{(1)}) \in F_{<\alpha}(M_2) \cap \operatorname{Ker}\varphi_2$. There is $\alpha' < \alpha$ such that $m_2' \in F_{\alpha'}(M_2)$ and $F_{\alpha'}(M_2) \neq F_{<\alpha'}(M_2)$. Repeat this procedure, by Lemma 2.2(d) and M_2 is bounded below, there exist finite number of $m_1^{(1)}, m_1^{(2)}, \ldots, m_1^{(t)}$ such that $m_2 = \varphi_1(\sum_{i=1}^t m_1^{(i)})$. Therefore, $\operatorname{Ker}\varphi_2 \subseteq \operatorname{Im}\varphi_1$. □

The following corollary tells that the properties of submodules can also be obtained from its associated $\mathbb{Z}^r$-graded version.

COROLLARY 2.4. *Let $M_1, M_2 \in \operatorname{GrMod} A$ be $\mathbb{Z}^r$-filtered modules.*
(a) Suppose $\phi : M_1 \to M_2$ is a strict $\mathbb{Z}^r$-filtered homomorphism. Then, $\operatorname{Im} G^r(\phi) \cong G^r(\operatorname{Im}\phi)$ and $\operatorname{Ker} G^r(\phi) \cong G^r(\operatorname{Ker}\phi)$.
(b) If L is a submodule of M_1, then $G^r(M_1/L) \cong G^r(M_1)/G^r(L)$.
(c) Suppose $M_1 \in \operatorname{grmod} A$. If L_1, L_2 are two submodules of M_1 and $L_1 \subseteq L_2$ such that $G^r(L_1) = G^r(L_2)$, then $L_1 = L_2$.

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Proof. (a) and (b) are immediate consequences of Lemma 2.3(a).

(c) Notice that the $\mathbb{Z}^r$-filtration on L_1, L_2 and L_2/L_1, induced from the one on M_1, is also a well-ordering. Then, $L_2/L_1 = 0$ follows from $G^r(L_2/L_1) = 0$, and the latter follows from (b). $\square$

Next lemma is a key step for constructing a resolution. This lemma is similar to [**7**, Chapter 2, Proposition 2.3]. However, one shall notice that the homomorphism constructed in next lemma preserves degrees. In other words, the conclusion holds in the graded module category GrMod A.

LEMMA 2.5. *Let M be a finitely generated A-module, and $\underline{P} = \bigoplus_{i=1}^{s} G^r(A)\underline{e}_i$ a finitely generated free $G^r(A)$-module with basis $\{\underline{e}_i\}_{i=1}^{s}$. Assume $\underline{\phi} : \underline{P} \to G^r(M)$ is a surjective morphism in* GrMod $G^r(A)$. *Then, there exist a finitely generated free A-module P and a strict $\mathbb{Z}^r$-filtered surjection $\phi : P \to M$ in* GrMod A *such that $G^r(\phi) = \underline{\phi}$.*

Proof. As mentioned above, set $P = \bigoplus_{i=1}^{s} Ae_i$ on a basis of $\{e_i\}_{i=1}^{s}$ with $\deg e_i = \operatorname{tdeg} \underline{e}_i$ for $i = 1, 2, \ldots, s$. Let $(P, \{\alpha_i\}_{i=1}^{s})$ be a $\mathbb{Z}^r$-filtered pair of the free module P such that $G^r(P) = \underline{P}$ where $\alpha_i = \deg \underline{e}_i$ $(i = 1, 2, \ldots, s)$.

Assume $\underline{\phi}(\underline{e}_i) = \overline{m}_i$, where $\overline{m}_i$ is a homogeneous element in $G^r(M)_{\alpha_i}$ represented by m_i. Here, $m_i \in F_{\alpha_i}(M)\backslash F_{<\alpha_i}(M)$ and $m_i \in M_{\|\alpha_i\|}$ for $i = 1, 2, \ldots, s$. The existence of m_i is guaranteed by Lemma 2.2.

We define the morphism $\phi : P \to M$ in GrMod A by $\phi(e_i) = m_i$. It is easy to see that $G^r(\phi) = \underline{\phi}$. Since $\underline{\phi}$ is surjective, ϕ is a strict $\mathbb{Z}^r$-filtered surjection by Lemma 2.3(b). $\square$

The following lemma exhibits a construction of a free resolution for a $\mathbb{Z}^r$-filtered graded module from a free resolution of its associative $\mathbb{Z}^r$-graded module.

LEMMA 2.6. *Let $\mathbb{Z}^r$-filtered M be a finitely generated A-module. Suppose that $G^r(M)$ has a finite free resolution (the length is finite and each term in it is finitely generated):*

$$0 \longrightarrow \underline{P}_m \xrightarrow{d_m} \underline{P}_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_2} \underline{P}_1 \xrightarrow{d_1} \underline{P}_0 \xrightarrow{d_0} G^r(M) \longrightarrow 0,$$

where $\underline{P}_j = \bigoplus_{i=1}^{s_j} G^r(A)\underline{e}_i^j$ for $0 \leq j \leq m$. Then, there exists a finite free resolution of M in GrMod A*:*

$$0 \longrightarrow P_m \xrightarrow{d_m} P_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

where $P_j = \bigoplus_{i=1}^{s_j} Ae_i^j$ with $\deg e_i^j = \operatorname{tdeg} \underline{e}_i^j$ and $(P_j, \{\deg \underline{e}_i^j\}_{i=1}^{s_j})$ is a $\mathbb{Z}^r$-filtered pair of the free module P_j such that $G^r(P_j) \cong \underline{P}_j$ and $G^r(d_j) = \underline{d}_j$ for all $1 \leq i \leq s_j, 0 \leq j \leq m$.

As a consequence, $\operatorname{gldim} A \leq \operatorname{gldim} G^r(A)$.

Proof. The proof is similar to [**16**, Theorem 2.7] and [**7**, Chapter 2, Proposition 2.5]. But we need to notice that d_i is constructed as in Lemma 2.5 which preserves degrees for all $i = 1, 2, \ldots, m$. $\square$

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In the sequel, we use the following notation if there is no confusion:

$$(-)^\vee := \underline{\mathrm{Hom}}_A(-, A);$$
$$(-)^\vee := \underline{\mathrm{Hom}}_{G^r(A)}(-, G^r(A)).$$

As usual, the $\mathbb{Z}^r$-filtration on the module $P^\vee$ for a finitely generated free A-module $P = \bigoplus_{i=1}^s Ae_i$ with basis $\{e_i\}_{i=1}^s$ is defined as

$$F_\alpha(P^\vee) = \{f \in P^\vee \mid f(F_{\alpha'}(P)) \subseteq F_{\alpha'+\alpha}(A) \text{ for all } \alpha' \in \mathbb{Z}^r\}.$$

However, there is an isomorphism $\theta : P^\vee \cong P'$ in grmod A^o , where $P' = \bigoplus_{i=1}^s e_i' A$ is a free A^o-module on a basis $\{e_i'\}_{i=1}^s$ with $\deg e_i' = -\deg e_i$ for $i = 1, 2, \ldots, s$. It is easy to check that θ is a strict $\mathbb{Z}^r$-filtered isomorphism. Thus, the $\mathbb{Z}^r$-filtration above also satisfies Lemma 2.2.

LEMMA 2.7. *Let $(\underline{P}_i, \underline{d}_i)$ and (P_i, d_i) be defined as in Lemma 2.6 for $i = 1, 2, \ldots, m$. Then, $G^r(P_i^\vee) \cong \underline{P}_i^\vee$ and $G^r(d_i^\vee) = \underline{d}_i^\vee$ for $i = 1, 2, \ldots, m$.*

Proof. Since $P_i^\vee$ is a finitely generated free module, $\underline{P}_i^\vee \cong G^r(P_i^\vee)$ is an easy result by Lemma 2.2. And the other one can be verified straightforwardly by Lemma 2.5 and the isomorphism θ. □

2.2. The regularity. Now, we give the regularity criterion for a connected graded algebra.

Let $A = k\langle X\rangle/I$ be a connected graded algebra. We keep the $\mathbb{Z}^r$-filtration on A defined in last subsection. Actually, this $\mathbb{Z}^r$-filtration is equivalent to a $\mathbb{Z}^r$-grading on $k\langle X\rangle$ such that $G^r(A)$ is a $\mathbb{Z}^r$-graded algebra. Provided an appropriate $\mathbb{Z}^r$-grading, one may derive some properties of A from $G^r(A)$.

THEOREM 2.8. *Let $A = k\langle X\rangle/I$ be a connected graded algebra. If $G^r(A)$ is AS-regular for an appropriate $\mathbb{Z}^r$-grading on $k\langle X\rangle$, then A is AS-regular.*

Proof. By Lemma 2.6, we know gldim $A \le$ gldim $G^r(A)$ is finite.

Notice that $G^r(A)$ can be seen as a $\mathbb{N}^r$-graded algebra which does not change the GK-dimension. Thus, GKdim $A =$ GKdim $G^r(A)$ is finite by [**15**, Theorem 2.8].

It remains to show that A is Gorenstein. Since $G^r(A)$ is AS-regular, there exists a minimal free resolution of ${}_{G^r(A)}k$:

$$0 \longrightarrow \underline{P}_n \xrightarrow{\underline{d}_n} \underline{P}_{n-1} \xrightarrow{\underline{d}_{n-1}} \cdots \xrightarrow{\underline{d}_2} \underline{P}_1 \xrightarrow{\underline{d}_1} \underline{P}_0 \xrightarrow{\underline{d}_0} {}_{G^r(A)}k \longrightarrow 0,$$

where $\underline{P}_j = \bigoplus_{i=1}^{s_j} G^r(A)\underline{e}_i^j$ for all $0 \le j \le n$.

It is easy to know $G^r(k) \cong k$ as $G^r(A)$-modules. By Lemma 2.6, there exists a finite free resolution of ${}_Ak$ in GrMod A:

$$0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} {}_Ak \longrightarrow 0,$$

where $P_j = \bigoplus_{i=1}^{s_j} Ae_i^j$ with $\deg e_i^j = \operatorname{tdeg} \underline{e}_i^j$ and $(P_j, \{\deg \underline{e}_i^j\}_{i=1}^{s_j})$ is a $\mathbb{Z}^r$-filtered pair of the free module P_j such that $G^r(P_j) \cong \underline{P}_j$ and $G^r(d_j) = \underline{d}_j$ for all $1 \le i \le s_j$, $0 \le j \le n$.

The regularity of $G^r(A)$ implies $\mathrm{Im}(\underline{d}_{i-1}^\vee) = \mathrm{Ker}(\underline{d}_i^\vee)$. Note that $G^r(d_i^\vee) = \underline{d}_i^\vee$ and $G^r(P_i^\vee) \cong \underline{P}_i^\vee$ by Lemma 2.7 for $i = 1, 2, \ldots, n$. Thus, $\underline{\mathrm{Ext}}_A^i(k, A) = 0$ for all $i < n$

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by Lemma 2.3(b). Moreover, $d_n^\vee$ is strict. Now we turn to compute $\underline{\mathrm{Ext}}_A^n(k, A)$. By definition, we have

$$\underline{\mathrm{Ext}}^n_{G^r(A)}(k, G^r(A)) = \underline{P}_n^\vee / \operatorname{Im}(\underline{d}_n^\vee) \cong G^r(P_n^\vee)/\operatorname{Im}(G^r(d_n^\vee)) \cong k(l)$$

for some $l \in \mathbb{Z}^r$. By Corollary 2.4, we obtain

$$G^r(\underline{\mathrm{Ext}}_A^n(k, A)) = G^r(P_n^\vee/\operatorname{Im}(d_n^\vee)) \cong G^r(P_n^\vee)/\operatorname{Im}(G^r(d_n^\vee)) \cong k(l),$$

where the $\mathbb{Z}^r$-filtration on $\underline{\mathrm{Ext}}_A^n(k, A)$ is induced by the one on $P_n^\vee$.

Hence, $F_{-l}(\underline{\mathrm{Ext}}_A^n(k, A))/F_{<-l}(\underline{\mathrm{Ext}}_A^n(k, A)) \cong k$ and $F_\alpha(\underline{\mathrm{Ext}}_A^n(k, A)) = F_{<\alpha}(\underline{\mathrm{Ext}}_A^n(k, A))$ for all $\alpha \in \mathbb{Z}^r$ except for $\alpha = -l$. Since the $\mathbb{Z}^r$-filtration on $P_n^\vee$ satisfies Lemma 2.2, we know $\underline{\mathrm{Ext}}_A^n(k, A) \cong k(\|l\|)$. In addition, $\operatorname{gldim} A = \operatorname{gldim} G^r(A)$. $\square$

To make the regularity criterion theorem above available in practice, a good way is to use Gröbner theory. We review noncommutative Gröbner basis theory briefly, a detailed treatment can be found in [7]. We firstly choose an arbitrary monomial ordering $\prec$ on X^*. This induces a $\mathbb{Z}^r$-graded admissible ordering $\prec_{\mathbb{Z}^r}$ on X^*: for $u, v \in X^*$, $u \prec_{\mathbb{Z}^r} v$ is defined by

(a) $\deg^r(u) < \deg^r(v)$, or

(b) $\deg^r(u) = \deg^r(v)$ and $u \prec v$.

For a nonzero polynomial $f \in k\langle X\rangle$, we can write $f = \sum_{i=1}^q f_i$, where each nonzero f_i is $\mathbb{Z}^r$-homogeneous with $\deg^r f_i = \alpha_i$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_q$. The element f_q is called the *leading homogeneous polynomial* of f, and it is denoted by $\mathrm{LH}(f)$. Let $\mathcal{G}$ be the reduced monic Gröbner basis of I under admissible ordering $\prec_{\mathbb{Z}^r}$, and let $\mathrm{LH}(\mathcal{G}) = \{\mathrm{LH}(f) \mid f \in \mathcal{G}\}$.

With the above preparations, now we are in position to prove Theorem 0.1.

Proof of Theorem 0.1 Due to the observation above Theorem 2.8, there exists a partition on generator set X corresponding to the $\mathbb{Z}^r$-grading on $k\langle X\rangle$. This partition induces a $\mathbb{Z}^r$-filtration on A as defined in Section 2.1, and $G^r(A)$ is the associated $\mathbb{Z}^r$-graded algebra. Here, the ordering $<$ on $\mathbb{Z}^r$ is the top priority in $\prec_{\mathbb{Z}^r}$. And $\mathcal{G}$ is the reduced Gröbner basis of I with respect to $\prec_{\mathbb{Z}^r}$. From [7, Chapter 4, Theorem 2.3], we know

$$G^r(A) \cong k\langle X\rangle/(\mathrm{LH}(\mathcal{G}))$$

as $\mathbb{Z}^r$-graded algebras. Thus, A is AS-regular by Theorem 2.8. $\square$

REMARK 2.9. (1) Theorem 0.1 provides a possible generalized deformation from known $\mathbb{Z}^r$-graded AS-regular algebras; that is, by adding some appropriate low-terms to the relations, one may produce some new classes of AS-regular algebras.

(2) The regularity criterion might not work for some AS-regular algebras. For example, the three-dimensional Sklyanin algebra

$$A = k\langle x, y, z\rangle/(axz + bzx + cy^2, ayx + bxy + cz^2, azy + byz + cx^2),$$

where scalars $a, b, c \in k$. It is AS-regular except for some special values of scalars. When $c \neq 0$, $G^r(A)$ are not domain for all $\mathbb{Z}^r$-grading on $k\langle x, y, z\rangle$ where $r = 2, 3$. However, all AS-regular algebras of global dimension 3 are noetherian domains (see [4]). So all of them are not AS-regular. Nevertheless, it is interesting to find a class of

AS-regular algebras $A = k\langle X\rangle/I$ such that there exists an appropriate $\mathbb{Z}^r$-grading on $k\langle X\rangle$ and make $k\langle X\rangle/(\mathrm{LH}(\mathcal{G}))$ to be AS-regular where $\mathcal{G}$ is the Gröbner basis of I.

2.3. Ring-theoretic and homological properties. AS-regular algebras obtained so far all have nice ring-theoretic and homological properties, such as noetherian, strongly noetherian and Auslander regular. In this subsection, we show that those properties also hold if their associated $\mathbb{Z}^r$-graded algebras have them.

THEOREM 2.10. *Let $A = k\langle X\rangle/I$ be a connected graded algebra. If $G^r(A)$ is strongly noetherian and Auslander regular for an appropriate $\mathbb{Z}^r$-grading on $k\langle X\rangle$, then so is A.*

Before proving this theorem, we need some lemmas.

First, we set the definition of $\mathbb{Z}^r$-filtration on tensor product. Let A_1 and A_2 be two algebras with A_1 being a $\mathbb{Z}^r$-filtered algebra. We introduce a $\mathbb{Z}^r$-filtration on $A_1 \otimes A_2$ by

$$F_\alpha(A_1 \otimes A_2) = F_\alpha(A_1) \otimes A_2 \quad \text{for all } \alpha \in \mathbb{Z}^r.$$

REMARK 2.11. Suppose A_1 is a connected graded algebra and A_2 is regard as a graded algebra concentrated in degree 0. If the $\mathbb{Z}^r$-filtration on A_1 is the one defined in Subsection 2.1, then the $\mathbb{Z}^r$-filtration on $A_1 \otimes A_2$ satisfies Lemma 2.1(a,b). If the $\mathbb{Z}^r$-filtration on modules in grmod $A_1 \otimes A_2$ is defined as in (F2), then Lemmas 2.2, 2.3 and Corollary 2.4 still hold in the category grmod $(A_1 \otimes A_2)$.

LEMMA 2.12. *Let A_1 and A_2 be two algebras where A_1 is a $\mathbb{Z}^r$-filtered algebras, then $G^r(A_1 \otimes A_2) \cong G^r(A_1) \otimes A_2$.*

Proof. For any $\alpha \in \mathbb{Z}^r$, there exists an exact sequence as vector space,

$$0 \to F_{<\alpha}(A_1) \to F_\alpha(A_1) \to G^r(A_1)_\alpha \to 0.$$

Note that A_2 is flat as k-module. Hence, acting $- \otimes A_2$ on that sequence,

$$0 \to F_{<\alpha}(A_1) \otimes A_2 \to F_\alpha(A_1) \otimes A_2 \to G^r(A_1)_\alpha \otimes A_2 \to 0$$

is still exact, which implies

$$G^r(A_1)_\alpha \otimes A_2 \cong \frac{F_\alpha(A_1) \otimes A_2}{F_{<\alpha}(A_1) \otimes A_2} = G^r(A_1 \otimes A_2)_\alpha.$$

It is easy to check that $G^r(A_1 \otimes A_2) \cong G^r(A_1) \otimes A_2$ as $\mathbb{Z}^r$-graded algebras. □

LEMMA 2.13. *Let A be a connected graded algebra. If a $\mathbb{Z}^r$-filtered A-module M has a finite free resolution, then $G^r(\underline{\mathrm{Ext}}^i_A(M, A))$ is a subquotient of $\underline{\mathrm{Ext}}^i_{G^r(A)}(G^r(M), G^r(A))$ for any $i \geq 0$.*

Proof. We claim that for any $\mathbb{Z}^r$-filtered homomorphism $\phi : N_1 \to N_2$, we have $G^r(\mathrm{Ker}\,\phi) \subseteq \mathrm{Ker}(G^r(\phi))$ and $\mathrm{Im}(G^r(\phi)) \subseteq G^r(\mathrm{Im}\,\phi)$, where N_1, N_2 are two $\mathbb{Z}^r$-filtered A-modules. If it is the case, the conclusion follows from Lemma 2.6.

Now we verify the claim. For any $\alpha \in \mathbb{Z}^r$,

$$G^r(\mathrm{Ker}\,\phi)_\alpha = \frac{\mathrm{Ker}\,\phi \bigcap F_\alpha(N_1)}{\mathrm{Ker}\,\phi \bigcap F_{<\alpha}(N_1)} \cong \frac{\mathrm{Ker}\,\phi \bigcap F_\alpha(N_1) + F_{<\alpha}(N_1)}{F_{<\alpha}(N_1)}.$$

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However,

$$\mathrm{Ker}(G^r(\phi))_\alpha = \frac{\mathrm{Ker}\,\phi \bigcap F_\alpha(N_1) + \phi^{-1}(F_{<\alpha}(N_2)) \bigcap F_\alpha(N_1) + F_{<\alpha}(N_1)}{F_{<\alpha}(N_1)},$$

where $\phi^{-1}(F_{<\alpha}(N_2)) = \{n_1 \in N_1 \mid \phi(n_1) \in F_{<\alpha}(N_2)\}$. Obviously, $G^r(\mathrm{Ker}\,\phi) \subseteq \mathrm{Ker}(G^r(\phi))$.

The proof for $\mathrm{Im}\,\phi$ is similar. □

We now recall the definition of j-number of modules. Let A be a $\mathbb{Z}^r$-graded algebra and $M \in \mathrm{GrMod}\,A$,

$$j_A(M) = \inf\{i \mid \underline{\mathrm{Ext}}^i_A(M, A) \neq 0\}.$$

LEMMA 2.14. *Let A be a connected graded algebra. If $G^r(M)$ has a finite free resolution for $\mathbb{Z}^r$-filtered $M \in \mathrm{grmod}\,A$ (resp. $\mathrm{grmod}\,A^o$), then $j_A(M) \geq j_{G^r(A)}(G^r(M))$ (resp. $j_{A^o}(M) \geq j_{G^r(A)^o}(G^r(M))$).*

Proof. Assume $G^r(M)$ has a finite free resolution

$$0 \longrightarrow \underline{P}_m \xrightarrow{\underline{d}_m} \underline{P}_{m-1} \xrightarrow{\underline{d}_{m-1}} \cdots \xrightarrow{\underline{d}_2} \underline{P}_1 \xrightarrow{\underline{d}_1} \underline{P}_0 \xrightarrow{\underline{d}_0} G^r(M) \longrightarrow 0,$$

where $\underline{P}_j = \bigoplus_{i=1}^{s_j} G^r(A)\underline{e}_i^j$ for $0 \leq j \leq m$. By Lemma 2.6, M has a free resolution

$$0 \longrightarrow P_m \xrightarrow{d_m} P_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

where $P_j = \bigoplus_{i=1}^{s_j} Ae_i^j$ with $\deg e_i^j = \mathrm{tdeg}\,\underline{e}_i^j$ and $(P_j, \{\deg \underline{e}_i^j\}_{i=1}^{s_j})$ is a $\mathbb{Z}^r$-filtered pair of the free module P_j such that $G^r(P_j) \cong \underline{P}_j$ and $G^r(d_j) = \underline{d}_j$ for $1 \leq i \leq s_j$, $0 \leq j \leq m$.

Put $t = j_{G^r(A)}(G^r(M))$; that is, the following sequence is exact

$$\underline{P}_t^\vee \xleftarrow{\underline{d}_t^\vee} \underline{P}_{t-1}^\vee \xleftarrow{\underline{d}_{t-1}^\vee} \cdots\cdots \xleftarrow{\underline{d}_0^\vee} \underline{P}_0^\vee \longleftarrow 0.$$

By Lemmas 2.7 and 2.3(b), the sequence

$$P_t^\vee \xleftarrow{d_t^\vee} P_{t-1}^\vee \xleftarrow{d_{t-1}^\vee} \cdots\cdots \xleftarrow{d_0^\vee} P_0^\vee \longleftarrow 0$$

is also exact, which implies $j_A(M) \geq t$. □

Proof of Theorem 2.10 Assume $G^r(A)$ is strongly noetherian. For every commutative noetherian algebra B, $G^r(A \otimes B)$ is noetherian by Lemma 2.12. It follows immediately from Remark 2.11 and Corollary 2.4 that $A \otimes B$ is noetherian.

If $G^r(A)$ is Auslander regular, so A is also noetherian with finite global dimension. For any $M \in \mathrm{grmod}\,A$ and $i \in \mathbb{N}$, let N be an A^o-submodule of $\underline{\mathrm{Ext}}^i_A(M, A)$. We need to show $j_{A^o}(N) \geq i$.

From Lemma 2.13, we know that $G^r(N)$ is a subquotient of $\underline{\mathrm{Ext}}^i_{G^r(A)}(G^r(M), G^r(A))$. Therefore, the Auslander condition implies $j_{G^r(A)^o}(G^r(N)) \geq i$. However, $j_{A^o}(N) \geq j_{G^r(A)^o}(G^r(N)) \geq i$ by Lemma 2.14.

The right ones can be verified similarly. □

Analogous to the criterion of the regularity, we also have a corollary to examine some ring-theoretic properties by means of Gröbner basis.

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COROLLARY 2.15. *Let $A = k\langle X\rangle/I$ be a connected graded algebra. Suppose $\mathcal{G}$ is the reduced Gröbner basis of I with respect to an admissible ordering $\prec_{\mathbb{Z}^r}$ for some $\mathbb{Z}^r$-grading on $k\langle X\rangle$. If the $\mathbb{Z}^r$-graded algebra $k\langle X\rangle/(\mathrm{LH}(\mathcal{G}))$ is strongly noetherian and Auslander regular, then so is A.*

REMARK 2.16. We fail to prove that A is Cohen–Macaulay if $G^r(A)$ is Cohen–Macaulay. It is equivalent to prove $j_{G^r(A)}(G^r(M)) = j_A(M)$ for any $M \in \mathrm{grmod}\, A$. We conjecture it is true. For the class of AS-regular algebras $\mathcal{J}$, we will prove it directly in Section 5.

3. A_∞-algebra structure of Jordan type. From this section, we turn to the AS-regular algebras of type (12221). As mentioned in the introduction, we hope to classify the AS-regular algebras whose Frobenius data is of Jordan type. We first review the A_∞-algebra structures on the Ext-algebra of the type (12221), the readers may find the details in **[9]**. After that, we concentrate on analysing and solving the equations gotten from the Stasheff identities in Jordan case.

In our case, A is generated by two elements x_1 and x_2 with two relations r_3 and r_4 whose degrees are 3 and 4, respectively. Denote by $E := E(A)$ the A_∞-Ext-algebra of A.

3.1. A_∞-Ext-algebras of type (12221). Notice that our A_∞-algebra structures satisfy the strictly unital condition, all multiplications and Stasheff identities can be described without $E^0 = k$.

3.1.1. Multiplications. According to the minimal resolution $(*)$ of trivial module ${}_Ak$, we know

$$E \cong k \oplus E^1_{-1} \oplus E^2_{-3} \oplus E^2_{-4} \oplus E^3_{-6} \oplus E^4_{-7},$$

where $\dim E^1_{-1} = \dim E^3_{-6} = 2,\ \ \dim E^2_{-3} = \dim E^2_{-4} = \dim E^4_{-7} = 1.$

As stated above, all m_n preserves Adams degree. After straightforward computation, we have $m_n = 0$ except for $n = 2, 3, 4$. The non-trivial multiplications m_2, m_3, m_4 are described explicitly in **[9]**. For the sake of computation, we copy them below.

• m_2 : The possible non-trivial actions of m_2 on $E^{\otimes 2}$ are

$$\begin{aligned} &E^1_{-1} \otimes E^3_{-6} \to E^4_{-7}, \quad E^3_{-6} \otimes E^1_{-1} \to E^4_{-7},\\ &E^2_{-3} \otimes E^2_{-4} \to E^4_{-7}, \quad E^2_{-4} \otimes E^2_{-3} \to E^4_{-7}. \end{aligned}$$

By Lemma 1.3, the algebra E is Frobenius. The Frobenius structure on E can be described as follows. There exists a basis $\{\beta_1, \beta_2\}$ of E^1_{-1}, a basis $\{\gamma_1\}$ of E^2_{-3}, a basis $\{\gamma_2\}$ of E^2_{-4}, a basis $\{\xi_1, \xi_2\}$ of E^3_{-6}, and a basis $\{\eta\}$ of E^4_{-7} such that

$$\begin{aligned} &\gamma_1\gamma_2 = \eta, \quad \gamma_2\gamma_1 = t\eta, \quad t \in k,\\ &\beta_i\xi_j = \delta_{ij}\eta, \quad \xi_i\beta_j = r_{ij}\eta, \quad r_{ij} \in k, \end{aligned}$$

where $t \neq 0$, $\mathcal{R} = (r_{ij})$ is nonsingular. The pair $(\mathcal{R}, t)$ is called the *Frobenius data* of E.

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Since k is algebraically closed, $\mathcal{R}$ is similar to a diagonal matrix or a Jordan block; that is,

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} g & 1 \\ 0 & g \end{pmatrix}.$$

We will focus on the latter case which is called *Jordan type*.

• m_3 : Possible nonzero components of m_3 on $E^{\otimes 3}$ are

$$\begin{gathered}
(E^1_{-1})^{\otimes 3} \to E^2_{-3}, \\
(E^1_{-1})^{\otimes 2} \otimes E^2_{-4} \to E^3_{-6},\ E^1_{-1} \otimes E^2_{-4} \otimes E^1_{-1} \to E^3_{-6},\ E^2_{-4} \otimes (E^1_{-1})^{\otimes 2} \to E^3_{-6}, \\
E^1_{-1} \otimes (E^2_{-3})^{\otimes 2} \to E^4_{-7},\ E^2_{-3} \otimes E^1_{-1} \otimes E^2_{-3} \to E^4_{-7},\ (E^2_{-3})^{\otimes 2} \otimes E^1_{-1} \to E^4_{-7}.
\end{gathered}$$

For $1 \le i, j, k \le 2$, we have

$$\begin{gathered}
m_3(\beta_i, \beta_j, \beta_k) = a_{ijk}\gamma_1, \\
m_3(\beta_i, \beta_j, \gamma_2) = b_{13ij}\xi_1 + b_{23ij}\xi_2,\ m_3(\beta_i, \gamma_1, \gamma_1) = c_{1i}\eta, \\
m_3(\beta_i, \gamma_2, \beta_j) = b_{12ij}\xi_1 + b_{22ij}\xi_2,\ m_3(\gamma_1, \beta_i, \gamma_1) = c_{2i}\eta, \\
m_3(\gamma_2, \beta_i, \beta_j) = b_{11ij}\xi_1 + b_{21ij}\xi_2,\ m_3(\gamma_1, \gamma_1, \beta_i) = c_{3i}\eta,
\end{gathered}$$

where the coefficients are scalars in k.

• m_4 : The possible non-trivial actions of m_4 on $E^{\otimes 4}$ are

$$\begin{gathered}
(E^1_{-1})^{\otimes 4} \to E^2_{-4}, \\
(E^1_{-1})^{\otimes 3} \otimes E^2_{-3} \to E^3_{-6}, \qquad E^2_{-3} \otimes (E^1_{-1})^{\otimes 3} \to E^3_{-6}, \\
E^1_{-1} \otimes E^2_{-3} \otimes (E^1_{-1})^{\otimes 2} \to E^3_{-6},\ (E^1_{-1})^{\otimes 2} \otimes E^2_{-3} \otimes E^1_{-1} \to E^3_{-6}.
\end{gathered}$$

For $1 \le i, j, k, h \le 2$, we have

$$\begin{gathered}
m_4(\beta_i, \beta_j, \beta_k, \beta_h) = v_{ijkh}\gamma_2, \\
m_4(\beta_i, \beta_j, \beta_k, \gamma_1) = u_{14ijk}\xi_1 + u_{24ijk}\xi_2,\ m_4(\beta_i, \beta_j, \gamma_1, \beta_k) = u_{13ijk}\xi_1 + u_{23ijk}\xi_2, \\
m_4(\beta_i, \gamma_1, \beta_j, \beta_k) = u_{12ijk}\xi_1 + u_{22ijk}\xi_2,\ m_4(\gamma_1, \beta_i, \beta_j, \beta_k) = u_{11ijk}\xi_1 + u_{21ijk}\xi_2,
\end{gathered}$$

where the coefficients are scalars in k.

3.1.2. Stasheff identities for the A_∞-algebra E. The nontrivial Stasheff identities are just SI(4), SI(5), SI(6).

• SI(4): Since $m_1 = 0$, SI(4) becomes

$$m_3(m_2 \otimes \mathrm{id}^{\otimes 2} - \mathrm{id} \otimes m_2 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_2) - m_2(m_3 \otimes \mathrm{id} + \mathrm{id} \otimes m_3) = 0.$$

Applying it to the basis of E, the non-trivial ones give the relationships between the coefficients.

$$\begin{array}{ll} a_{ijk} = b_{i3jk}, & b_{i2jk} = \sum_{s=1}^2 r_{sk} b_{s3ij}, \\ b_{i1jk} = \sum_{s=1}^2 r_{sk} b_{s2ij}, & -t a_{ijk} = \sum_{s=1}^2 r_{sk} b_{s1ij}, \end{array} \quad \text{for } 1 \le i, j, k \le 2. \tag{SI(4a)}$$

Immediately, we have

$$-t a_{ijk} = \sum_{s,t,u=1}^{2} r_{sk} r_{tj} r_{ui} a_{uts}, \quad \text{for } 1 \le i, j, k \le 2. \tag{SI(4b)}$$

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• SI(5): The Stasheff identity SI(5) is equivalent to

$$m_4(m_2 \otimes \mathrm{id}^{\otimes 3} - \mathrm{id} \otimes m_2 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id}^{\otimes 2} \otimes m_2 \otimes \mathrm{id} - \mathrm{id}^{\otimes 3} \otimes m_2)$$
$$+m_3(m_3 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes m_3 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_3) + m_2(m_4 \otimes \mathrm{id} - \mathrm{id} \otimes m_4) = 0.$$

Then, it follows that SI(5) holds if and only if for $1 \le i, j, k, h \le 2$,

$$\begin{aligned}
&a_{ijk}c_{2h} - a_{jkh}c_{1i} + tv_{ijkh} - u_{i4jkh} = 0,\\
&a_{ijk}c_{3h} + r_{1h}u_{14ijk} + r_{2h}u_{24ijk} - u_{i3jkh} = 0,\\
&r_{1h}u_{13ijk} + r_{2h}u_{23ijk} - u_{i2jkh} = 0,\\
&c_{1i}a_{jkh} - r_{1h}u_{12ijk} - r_{2h}u_{22ijk} + u_{i1jkh} = 0,\\
&a_{jkh}c_{2i} - a_{ijk}c_{3h} - r_{1h}u_{11ijk} - r_{2h}u_{21ijk} + v_{ijkh} = 0.
\end{aligned} \tag{SI(5a)}$$

• SI(6): The Stasheff identity SI(6) becomes

$$m_4(-m_3 \otimes \mathrm{id}^{\otimes 3} - \mathrm{id} \otimes m_3 \otimes \mathrm{id}^{\otimes 2} - \mathrm{id}^{\otimes 3} \otimes m_3 \otimes \mathrm{id} - \mathrm{id}^{\otimes 3} \otimes m_3)$$
$$+m_3(m_4 \otimes \mathrm{id}^{\otimes 2} - \mathrm{id} \otimes m_4 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_4) = 0.$$

Applying it to the basis of E, all are trivial except for $(\beta_i, \beta_j, \beta_k, \beta_h, \beta_m, \beta_n)$. We obtain

$$\begin{aligned}
&-a_{ijk}u_{s1hmn} + a_{jkh}u_{s2imn} - a_{khm}u_{s3ijn} + a_{hmn}u_{s4ijk}\\
&\quad +b_{s1mn}v_{ijkh} - b_{s2in}v_{jkhm} + b_{s3ij}v_{khmn} = 0,
\end{aligned} \quad \text{for } 1 \le i, j, k, h, m, n, s \le 2. \tag{SI(6a)}$$

Since $E^2 = E^2_{-3} \oplus E^2_{-4}$, the relations $R = \{r_3, r_4\}$ where $\deg r_3 = 3$ and $\deg r_4 = 4$. By Lemma 1.5 and the A_∞-algebra structure on E described above, we can write

$$r_3 = \sum_{1 \le i,j,k \le 2} a_{ijk}x_ix_jx_k,$$
$$r_4 = \sum_{1 \le i,j,k,h \le 2} v_{ijkh}x_ix_jx_kx_h.$$

Furthermore, r_3 and r_4 are neither zero nor a product of lower-degree polynomials since A is a domain.

3.2. Jordan type. We now concentrate on the Jordan type. We write

$$\mathcal{R} = \begin{pmatrix} -g & 1 \\ 0 & -g \end{pmatrix}.$$

Next, we work with m_3 by considering SI(4) to describe r_3. By SI(4b), we have

$$\begin{aligned}
&(t - g^3)a_{111} = 0,\\
&(t - g^3)a_{112} + g^2a_{111} = 0, \quad (t - g^3)a_{212} + g^2(a_{211} + a_{112}) - ga_{111} = 0,\\
&(t - g^3)a_{121} + g^2a_{111} = 0, \quad (t - g^3)a_{221} + g^2(a_{211} + a_{121}) - ga_{111} = 0,\\
&(t - g^3)a_{211} + g^2a_{111} = 0, \quad (t - g^3)a_{122} + g^2(a_{112} + a_{121}) - ga_{111} = 0,\\
&(t - g^3)a_{222} + g^2(a_{221} + a_{212} + a_{122}) - g(a_{112} + a_{121} + a_{211}) + a_{111} = 0.
\end{aligned} \tag{SI(4c)}$$

If $t - g^3 \ne 0$, all $a_{ijk} = 0$ which implies $r_3 = 0$. Therefore

$$t = g^3.$$

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From (SI(4c)), it is easy to obtain

$$\begin{cases} a_{111} = a_{112} = a_{121} = a_{211} = 0, \\ a_{221} + a_{212} + a_{122} = 0. \end{cases}$$

Hence, $r_3 = a_{122}x_1x_2^2 + a_{212}x_2x_1x_2 + a_{221}x_2^2x_1 + a_{222}x_2^3$. Moreover, it is easy to see $a_{122}a_{221} \neq 0$ since A is a domain, we write $a_{122} = 1$, $a_{221} = p \neq 0$ and $a_{222} = w$, $a_{212} = -(1+p)$. So

$$r_3 = x_1x_2^2 - (1+p)x_2x_1x_2 + px_2^2x_1 + wx_2^3.$$

We get the solutions for $\{b_{isjk}\}$ from (SI(4a)),

$$\begin{array}{llll} b_{1322} = 1, & b_{2312} = -(1+p), & b_{2321} = p, & b_{2322} = w, \\ b_{1222} = g(1+p), & b_{2212} = -gp, & b_{2221} = -g, & b_{2222} = 1 - gw, \\ b_{1122} = g^2p, & b_{2112} = g^2, & b_{2121} = -g^2(1+p), & b_{2122} = gp + g^2w, \end{array}$$

the other of b_{ijkh} are zero.

Then, consider SI(5) to describe r_4. By replacing r_4 with the equivalent relation

$$r_4 - v_{1122}x_1r_3 - v_{2122}x_2r_3 - v_{1221}r_3x_1 - v_{1222}r_3x_2,$$

we may assume that

$$v_{1122} = v_{2122} = v_{1221} = v_{1222} = 0.$$

Using (SI(5a)) recursively to eliminate $\{u_{isjkh}\}$, we obtain equations:

$$(1-g^4t)v_{1111} = 0, \tag{3.0.1}$$

$$(1-g^4t)v_{1112} = -g^3tv_{1111}, \tag{3.0.2}$$

$$(1-g^4t)v_{1121} = -g^3tv_{1111}, \tag{3.0.3}$$

$$(1-g^4t)v_{1122} = -g^3t(v_{1112}+v_{1121}) + g^2tv_{1111} - (g^4c_{11} + c_{21} + g^3c_{31}) = 0, \tag{3.0.4}$$

$$(1-g^4t)v_{1211} = -g^3tv_{1111}, \tag{3.0.5}$$

$$(1-g^4t)v_{1212} = -g^3t(v_{1112}+v_{1211}) + g^2tv_{1111} + (1+p)(g^4c_{11} + c_{21} + g^3c_{31}), \tag{3.0.6}$$

$$\begin{aligned}(1-g^4t)v_{1221} = &-g^3t(v_{1121}+v_{1211}) + g^2tv_{1111} \\ &-p(g^4c_{11} + c_{21} + g^3c_{31}) + gc_{11} + g^4c_{21} + c_{31} = 0,\end{aligned} \tag{3.0.7}$$

$$\begin{aligned}(1-g^4t)v_{1222} = &-g^3tv_{1212} + g^2t(v_{1112}+v_{1121}+v_{1211}) - gtv_{1111} - c_{11} - g^3c_{21} \\ &-w(g^4c_{11} + c_{21} + g^3c_{31}) + gc_{12} + g^4c_{22} + c_{32} = 0,\end{aligned} \tag{3.0.8}$$

$$(1-g^4t)v_{2111} = -g^3tv_{1111}, \tag{3.0.9}$$

$$(1-g^4t)v_{2112} = -g^3t(v_{1112}+v_{2111}) + g^2tv_{1111}, \tag{3.0.10}$$

$$(1-g^4t)v_{2121} = -g^3t(v_{1121}+v_{2111}) + g^2tv_{1111} - (1+p)(gc_{11} + g^4c_{21} + c_{31}), \tag{3.0.11}$$

$$\begin{aligned}(1-g^4t)v_{2122} = &-g^3t(v_{2112}+v_{2121}) + g^2t(v_{1112}+v_{1121}+v_{2111}) - gtv_{1111} \\ &+(1+p+g^3)c_{11} + g^3(1+p)c_{21} - (1+p+g^3)gc_{12} \\ &-(g^4(1+p)+1)c_{22} - (g^3+p+1)c_{32} = 0,\end{aligned} \tag{3.0.12}$$

$$(1-g^4t)v_{2211} = -g^3t(v_{1211}+v_{2111}) + g^2tv_{1111} + p(gc_{11} + g^4c_{21} + c_{31}), \tag{3.0.13}$$

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$$\begin{aligned}(1-g^4t)v_{2212} &= -g^3t(v_{1212}+v_{2112}+v_{2211})+g^2t(v_{1112}+v_{1211}+v_{2111})-gtv_{1111}\\ &\quad +(-g^3(1+p)-p)c_{11}-g^3pc_{21}+(g^4(1+p)+gp)c_{12}\\ &\quad +(g^4p+1+p)c_{22}+(p+g^3(1+p))c_{32}, \qquad (3.0.14)\\ (1-g^4t)v_{2221} &= -g^3t(v_{2121}+v_{2211})+g^2t(v_{1121}+v_{1211}+v_{2111})-gtv_{1111}\\ &\quad +w(gc_{11}+g^4c_{21}+c_{31})+g^3pc_{11}-p(g^4c_{12}+c_{22}+g^3c_{32}), \qquad (3.0.15)\\ (1-g^4t)v_{2222} &= -g^3t(v_{2212}+v_{2221})+g^2t(v_{1212}+v_{2112}+v_{2121}+v_{2211})\\ &\quad -gt(v_{1112}+v_{1121}+v_{1211}+v_{2111})+tv_{1111}\\ &\quad +w(g^3c_{11}-c_{11}-g^3c_{21})\\ &\quad +w((-g^4+g)c_{12}+(g^4-1)c_{22}+(1-g^3)c_{32}). \qquad (3.0.16)\end{aligned}$$

If $1-g^4t\neq 0$, then all $v_{1ijk}=0$, which implies x_2 is a zero divisor, contrary to our assumption. Hence,

$$g^4t=1, \qquad g^7=1.$$

From (3.0.2), we get $v_{1111}=0$. Hence, (3.0.4), (3.0.6), (3.0.7) become

$$\begin{aligned} v_{1112}+v_{1121} &= -gM, && (3.0.17)\\ v_{1112}+v_{1211} &= (1+p)gM, && (3.0.18)\\ v_{1121}+v_{1211} &= (g^4-p)gM, && (3.0.19)\end{aligned}$$

where $M:=g^4c_{11}+c_{21}+g^3c_{31}$. The equations (3.0.10), (3.0.11), (3.0.13) become

$$\begin{aligned} v_{1112}+v_{2111} &= 0, && (3.0.20)\\ v_{1121}+v_{2111} &= -(1+p)g^5M, && (3.0.21)\\ v_{1211}+v_{2111} &= pg^5M. && (3.0.22)\end{aligned}$$

Following from the equations (3.0.17)-(3.0.22), we obtain

$$\begin{aligned} -gM &= (g^4-p-pg^4)gM,\\ (1+p)gM &= (2g^4-p+pg^4)gM.\end{aligned}$$

Note that $g^7=1$, we have two cases:

Case 1 $M=0$.
Case 2 $M\neq 0,\ g=1,\ p=1$.

4. Regular algebras of Jordan type. We continue to analyse the A_∞-algebra structures in this section. We solve all the algebras corresponding to Case 1 and Case 2, and prove that there is one class of AS-regular algebras in Case 2, and no AS-regular algebra in Case 1.

PROPOSITION 4.1. *Suppose that A is an AS-regular algebra of type* (12221) *which is Jordan type, then Case 1 gives no AS-regular algebras and Case 2 gives exactly one class of AS-regular algebras.*

4.1. Case 1: non-AS-regular algebras. If $M = 0$, the equations (3.0.17)–(3.0.22) tell that

$$v_{1112} = v_{1121} = v_{1211} = v_{2111} = 0.$$

Then,

$$\begin{aligned} r_4 =& v_{1212}x_1x_2x_1x_2 + v_{2112}x_2x_1^2x_2 + v_{2121}x_2x_1x_2x_1 + v_{2211}x_2^2x_1^2 \\ &+ v_{2212}x_2^2x_1x_2 + v_{2221}x_2^3x_1 + v_{2222}x_2^4. \end{aligned}$$

Since A is a domain, v_{1212} must be nonzero. Hence, we can assume $v_{1212} = 1$. Now

$$\begin{aligned} r_4 =& x_1x_2x_1x_2 + v_{2112}x_2x_1^2x_2 + v_{2121}x_2x_1x_2x_1 + v_{2211}x_2^2x_1^2 \\ &+ v_{2212}x_2^2x_1x_2 + v_{2221}x_2^3x_1 + v_{2222}x_2^4. \end{aligned}$$

Next, we start to perform the computations to find all solutions of $\{a_{ijk}, v_{ijkh}\}$ satisfying all Stasheff identities $\mathsf{SI}(4)$, $\mathsf{SI}(5)$ and $\mathsf{SI}(6)$. Find expressions of $\{v_{1212}, v_{2112}, v_{2121}, v_{2211}, v_{2212}\}$ from (3.0.1)–(3.0.16) which are represented by $\{g, p, w, c_{11}, c_{21}, c_{12}, c_{22}, c_{32}, v_{2221}\}$, and formulas of $\{u_{isjkh}\}$ from SI(5a). We omit them because of its length. Then, input the expressions of $\{a_{ijk}, b_{stmn}, u_{isjkh}, v_{ijkh}\}$ into SI(6a). This produces 2^7 equations involving the variables $\{g, p, w, c_{11}, c_{21}, c_{12}, c_{22}, c_{32}, v_{2221}, v_{2222}\}$. We compute those and solve the equations by Maple.

After deleting useless solutions, we have five different solutions in total. Input them into the coefficients of r_3, r_4 as listed below.

Solution 1

$$\begin{aligned} &g = 1, \quad p = 1, \quad w = 0, \\ &v_{1212} = 1, \quad v_{2112} = -1, \quad v_{2121} = -1, \quad v_{2211} = 1, \\ &v_{2212} = c_{22}, \quad v_{2221} = -c_{22}, \quad v_{2222} = v_{2222}. \end{aligned} \tag{S1}$$

Solution 2

$$\begin{aligned} &g = 1, \quad p = -1, \quad w = w, \\ &v_{1212} = 1, \quad v_{2112} = 1, \quad v_{2121} = 1, \quad v_{2211} = -3, \\ &v_{2212} = 1 - \frac{w}{2}, \quad v_{2221} = \frac{7w}{2} - 1, \quad v_{2222} = -\frac{3w^2}{2} + \frac{w}{2}. \end{aligned} \tag{S2}$$

Solution 3

$$\begin{aligned} &g = 1, \quad p = -1, \quad w = \frac{2}{7}, \\ &v_{1212} = 1, \quad v_{2112} = 1, \quad v_{2121} = 1, \quad v_{2211} = -3, \\ &v_{2212} = \frac{6}{7}, \quad v_{2221} = 0, \quad v_{2222} = v_{2222}. \end{aligned} \tag{S3}$$

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Solution 4

$$
\begin{aligned}
&g = j, \quad p = -j^3, \quad w = w,\\
&v_{1212} = 1, \quad v_{2112} = j, \quad v_{2121} = -j^6 - j^2 - 2j - 2,\\
&v_{2211} = j^6 + j^2 + j + 1, \quad v_{2212} = -w(\frac{j^4}{2} + 2j^3 + 3j^2 + 2 + \frac{7j}{2}) + \frac{j^6+1}{2},\\
&v_{2221} = w(j^5 + \frac{3j^4}{2} + 2j^3 + 3j^2 + \frac{7j}{2} + 3) - \frac{j^6+1}{2},\\
&v_{2222} = \frac{1}{2}\big(w^2(-4j^5 + 10j^3 + 14j^2 + 13j + 6) - w(j^3 + 2j^2 + 2j + 1)\big).
\end{aligned}
\tag{S4}
$$

Solution 5

$$
\begin{aligned}
&g = j, \quad p = j^2, \quad w = c_{22}(-j^6 + j^5),\\
&v_{1212} = 1, \quad v_{2112} = -1, \quad v_{2121} = -j^2,\\
&v_{2211} = j^2, \quad v_{2212} = (j^4 - j^3 + j^2)c_{22},\\
&v_{2221} = c_{22}(2j^5 + 2j^3 + j + 1),\\
&v_{2222} = c_{22}^2(j^6 - 2j^5 - j^3 - j^2 - 2).
\end{aligned}
\tag{S5}
$$

The number j occurring in Solutions 4 and 5 satisfies $j^6 + j^5 + j^4 + j^3 + j^2 + j + 1 = 0$.

We check Hilbert series of them by using Diamond Lemma [**5**] to calculate the Gröbner bases. Before that, we show a lemma to help us compare the Hilbert series with other series in low degrees. In the following, we fix an arbitrary monomial ordering on X^*. For any $u, v \in X^*$, we say v is a *factor* of u, if there exist $w, w' \in X^*$ such that $u = wvw'$ denoted by $v|u$. Let any nonzero polynomial $f \in k\langle X\rangle$, the *leading monomial* $\mathrm{LM}(f)$ of f is the largest monomial in f. Let $\mathcal{G}$ be the reduced monic Gröbner basis of I, and $\mathcal{G} = \bigcup_i \mathcal{G}_i$ where $\mathcal{G}_i = \{f \in \mathcal{G} \mid \deg f \leq i\}$. Then, the set

$$
\mathrm{NW}(\mathcal{G}) = \{u \in X^* \mid \mathrm{LM}(g) \nmid u \text{ for any } g \in \mathcal{G}\} = \bigcup_i \mathrm{NW}(\mathcal{G})_i
$$

is a k-basis of A, where $\mathrm{NW}(\mathcal{G})_i$ consists of the elements of degree i in $\mathrm{NW}(\mathcal{G})$. Hence, $\dim_k A_m = \#(\mathrm{NW}(\mathcal{G})_m)$. Notice that $\mathrm{NW}(\mathcal{G})_i = \{u \in X^* \mid \mathrm{LM}(g) \nmid u \text{ for any } g \in \mathcal{G}_i \text{ and } \deg u = i\}$.

LEMMA 4.2. *Let $A = k\langle X\rangle/I$ be a connected graded algebra, $\mathcal{G}$ is a Gröbner basis of I, and let $A' = k\langle X\rangle/(\mathrm{LM}(\mathcal{G}_m))$. Then, $H_{A'}(t) - H_A(t) = \sum_{i>m} a_i t^i$ with $a_i \geq 0$.*

Proof. For any $i \geq 0$, $\mathrm{LM}(\mathcal{G}_m)_i \subset \mathrm{LM}(\mathcal{G})_i$ since $\mathrm{LM}(\mathcal{G}_m) \subset \mathrm{LM}(\mathcal{G})$. Then, $\mathrm{NW}(\mathrm{LM}(\mathcal{G}))_i \subset \mathrm{NW}(\mathrm{LM}(\mathcal{G}_m))_i$ and $\dim_k A'_i \geq \dim_k A_i$. $H_{A'}(t) - H_A(t) = \sum_{i\geq 0} a_i t^i$ with $a_i \geq 0$.

Moreover, for $i \leq m$,

$$
\begin{aligned}
\mathrm{NW}(\mathrm{LM}(\mathcal{G}_m))_i &= \{u \in X^* \mid \mathrm{LM}(g) \nmid u \text{ for any } g \in \mathrm{LM}(\mathcal{G}_m) \text{ and } \deg u = i\}\\
&= \{u \in X^* \mid \mathrm{LM}(g) \nmid u \text{ for any } g \in \mathcal{G}_m \text{ and } \deg u = i\}\\
&= \{u \in X^* \mid \mathrm{LM}(g) \nmid u \text{ for any } g \in \mathcal{G}_i \text{ and } \deg u = i\}\\
&= \mathrm{NW}(\mathcal{G})_i.
\end{aligned}
$$

Hence, $\dim_k A_i = \dim_k A'_i$ if $i \leq m$, which implies $H_{A'}(t) - H_A(t) = \sum_{i>m} a_i t^i$ with $a_i \geq 0$. □

We choose a monomial ordering $\prec_{gr-lex}$ on the free monoid $\{x_1, x_2\}^*$ as follows: For any $u = x_{i_1}x_{i_2}\cdots x_{i_s}$, $v = x_{j_1}x_{j_2}\cdots x_{j_t} \in X^*$, to say $u \prec_{gr-lex} v$ we mean either

(a) $s < t$, or

(b) $s = t$ and there exists p such that $x_{i_l} = x_{j_l}$ for $l < p$ and $i_p > j_p$.

Keep in mind that Hilbert series of type (12221) is

$$H_A(t) = 1 + 2t + 4t^2 + 7t^3 + 11t^4 + 16t^5 + 23t^6 + 31t^7 + \cdots . \tag{HS}$$

The algebra corresponding to (S1) is $U(g, h) = k\langle x_1, x_2\rangle/(f_{11}, f_{12})$, where

$$f_{11} = x_1x_2^2 - 2x_2x_1x_2 + x_2^2x_1,$$
$$f_{12} = x_1x_2x_1x_2 - x_2x_1^2x_2 - x_2x_1x_2x_1 + x_2^2x_1^2 + gx_2^2x_1x_2 - gx_2^3x_1 + hx_2^4,$$

with $g, h \in k$.

$V(w, l) = k\langle x_1, x_2\rangle/(f_{21}, f_{22})$ is the algebra corresponding to (S5), where

$$\begin{aligned} f_{21} =& x_1x_2^2 - (1 + j^2)x_2x_1x_2 + j^2x_2^2x_1 + w(-j^6 + j^5)x_2^3,\\ f_{22} =& x_1x_2x_1x_2 - x_2x_1^2x_2 - j^2x_2x_1x_2x_1 + j^2x_2^2x_1^2 + l(j^4 - j^3 + j^2)x_2^2x_1x_2\\ &+ l(2j^5 + 2j^3 + j + 1)x_2^3x_1 + l^2(j^6 - 2j^5 - j^3 - j^2 - 2)x_2^4, \end{aligned}$$

with $j^6 + j^5 + j^4 + j^3 + j^2 + j + 1 = 0$ and $w, l \in k$.

Lemma 4.3. *$U(g, h)$, $V(w, l)$ are not AS-regular.*

Proof. By Diamond Lemma, we know that $\{f_{11}, f_{12}\}$ and $\{f_{21}, f_{22}\}$ are Gröbner bases of (f_{11}, f_{12}) and (f_{21}, f_{22}), respectively. Then, their leading monomials of Gröbner bases are the same, that is, $\mathrm{LM}(\mathcal{G}) = \{x_1x_2^2, x_1x_2x_1x_2\}$. Let $\mathrm{MON} := k\langle x_1, x_2\rangle/(\mathrm{LM}(\mathcal{G}))$. Hence,

$$H_{U(g,h)}(t) = H_{V(w,l)}(t) = H_{\mathrm{MON}}(t) = 1 + 2t + 4t^2 + 7t^3 + 11t^4 + 17t^5 + \cdots ,$$

which is different from (HS). □

The algebras corresponding to solutions (S2)–(S4) are listed below.

(S2) $O(w) = k\langle x_1, x_2\rangle/(f_{31}, f_{32})$, where

$$\begin{aligned} f_{31} =& x_1x_2^2 - x_2^2x_1 + wx_2^3,\\ f_{32} =& x_1x_2x_1x_2 + x_2x_1^2x_2 + x_2x_1x_2x_1 - 3x_2^2x_1^2 + (1 - \frac{w}{2})x_2^2x_1x_2\\ &+ (\frac{7w}{2} - 1)x_2^3x_1 + (-\frac{3w^2}{2} + \frac{w}{2})x_2^4, \end{aligned}$$

with $w \in k$.

(S3) $P(a) = k\langle x_1, x_2\rangle/(f_{41}, f_{42})$, where

$$f_{41} = x_1x_2^2 - x_2^2x_1 + \frac{2}{7}x_2^3,$$

$$f_{42} = x_1x_2x_1x_2 + x_2x_1^2x_2 + x_2x_1x_2x_1 - 3x_2^2x_1^2 + \frac{6}{7}x_2^2x_1x_2 + ax_2^4,$$

with $a \in k$.

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(S4) $Q(d) = k\langle x_1, x_2\rangle/(f_{51}, f_{52})$, where

$$\begin{aligned} f_{51} =& x_1x_2^2 - (1-j^3)x_2x_1x_2 + j^3x_2^2x_1 + dx_2^3, \\ f_{52} =& x_1x_2x_1x_2 + jx_2x_1^2x_2 - (j^6+j^2+2j+2)x_2x_1x_2x_1 + (j^6+j^2+j+1)x_2^2x_1^2 \\ &+ (\frac{j^6+1}{2} - d(\frac{j^4}{2} + 2j^3 + 3j^2 + 2 + \frac{7j}{2}))x_2^2x_1x_2 \\ &+ \big(d(j^5 + \frac{3j^4}{2} + 2j^3 + 3j^2 + \frac{7j}{2} + 3) - \frac{j^6+1}{2}\big)x_2^3x_1 \\ &+ \frac{1}{2}\Big(d^2(-4j^5 + 10j^3 + 14j^2 + 13j + 6) - d(j^3 + 2j^2 + 2j + 1)\Big)x_2^4, \end{aligned}$$

with $j^6 + j^5 + j^4 + j^3 + j^2 + j + 1 = 0$ and $d \in k$.

LEMMA 4.4. *$O(w)$, $P(a)$, $Q(d)$ are not AS-regular.*

Proof. Only consider $\mathcal{G}_7$. Then, we obtain that the leading monomials in $\mathcal{G}_7$ of them are the same, that is,

$$\mathrm{LM}(\mathcal{G}_7) = \{x_1x_2^2, x_1x_2x_1x_2, x_2^2x_1^2x_2, x_2^2x_1^3x_2, x_2^2x_1x_2x_1^2x_2, x_2^2x_1^4x_2\}.$$

Let $\mathrm{MON}_7 := k\langle x_1, x_2\rangle/(\mathcal{G}_7)$.

Suppose they are AS-regular, then they have the same Hilbert series (HS) denoted $H(t)$. Then, $H_{\mathrm{MON}_7}(t) - H(t) = \sum_{i\geq 8} a_it^i$ with $a_i \geq 0$ by Lemma 4.2.

However, $H_{\mathrm{MON}_7}(t) = 1 + 2t + 4t^2 + 7t^3 + 11t^4 + 16t^5 + 23t^6 + 32t^7 + \cdots$ and

$$H_{\mathrm{MON}_7}(t) - H(t) = t^7 + \Sigma_{i\geq 8}a_it^i.$$

It is a contradiction. □

4.2. Case 2: AS-regular. Now we turn to Case 2. From (SI(5a)), we find that $v_{1112} = -\frac{M}{2}$. Assume

$$v_{1112} = 1.$$

The same method is used as in Case 1. Using (3.0.1)–(3.0.16) again, represent $\{v_{ijkh}\}$ by $\{w, c_{11}, c_{21}, c_{31}, c_{12}, c_{22}, c_{32}, v_{2221}, v_{2222}\}$. Find expressions for $\{u_{isjkh}\}$ from SI(5a). We also omit those explicit formulas. Then, input the formulas into SI(6a) which produces 2^7 equations involving the variables $\{w, c_{11}, c_{21}, c_{31}, c_{12}, c_{22}, c_{32}, v_{2221}, v_{2222}\}$ and solve them. All those steps are computed by Maple. There exits only one solution, and taking it back to $\{v_{ijkh}\}$ we obtain:

Solution 6

$$\begin{aligned} &g = 1, \quad p = 1, \quad w = 0, \\ &v_{1112} = 1, \quad v_{1121} = -3, \quad v_{1211} = 3, \\ &v_{1212} = -c_{21} + 1, \quad v_{2111} = -1, \quad v_{2112} = c_{21}, \\ &v_{2121} = c_{21} - 3, \quad v_{2211} = -c_{21} + 2, \quad v_{2212} = -v_{2221}, \\ &v_{2221} = v_{2221}, \quad v_{2222} = v_{2222}. \end{aligned} \tag{S6}$$

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The corresponding algebra is $\mathcal{J} = k\langle x_1, x_2\rangle/(f_1, f_2)$ where

$$\begin{aligned} f_1 =& x_1x_2^2 - 2x_2x_1x_2 + x_2^2x_1, \\ f_2 =& x_1^3x_2 - 3x_1^2x_2x_1 + 3x_1x_2x_1^2 - x_2x_1^3 + (1-u)x_1x_2x_1x_2 + ux_2x_1^2x_2 \\ &+ (u-3)x_2x_1x_2x_1 + (2-u)x_2^2x_1^2 - vx_2^2x_1x_2 + vx_2^3x_1 + wx_2^4, \end{aligned}$$

and $u, v, w \in k$.

Then, we define a $\mathbb{Z}^2$-grading on $k\langle x_1, x_2\rangle$ with $\deg^2 x_1 = (1, 0)$, $\deg^2 x_2 = (0, 1)$. We choose $\prec_{gr-lex}$ on $\{x_1, x_2\}^*$ as the monomial ordering defined in Section 4.1. The admissible ordering $\prec_{\mathbb{Z}^2}$ is defined as in Section 2. Let $I = (f_1, f_2)$ and $\mathcal{G}$ be the Gröbner basis of I respect to $\prec_{\mathbb{Z}^2}$. Then,

$$G^2(\mathcal{J}) \cong k\langle x_1, x_2\rangle/(\mathrm{LH}(I)) \cong k\langle x_1, x_2\rangle/(\mathrm{LH}(\mathcal{G})).$$

Applying the Diamond Lemma, the Gröbner basis $\mathcal{G}$ is $\{f_1, f_2, f_3\}$ where

$$\begin{aligned} f_3 =& x_1^2x_2x_1x_2 - 3x_1x_2x_1^2x_2 + 2x_1x_2x_1x_2x_1 + 3x_2x_1^2x_2x_1 - 5x_2x_1x_2x_1^2 \\ &+ (2u-2)x_2x_1x_2x_1x_2 + 2x_2^2x_1^3 - 2ux_2^2x_1^2x_2 + (6-2u)x_2^2x_1x_2x_1 \\ &+ (2u-4)x_2^3x_1^2 + 2vx_2^3x_1x_2 - 2vx_2^4x_1 - 2wx_2^5. \end{aligned}$$

So, $\mathrm{LH}(\mathcal{G}) = \{\mathrm{LH}(f_1), \mathrm{LH}(f_2), \mathrm{LH}(f_3)\}$, where

$$\begin{aligned} \mathrm{LH}(f_1) &= x_1x_2^2 - 2x_2x_1x_2 + x_2^2x_1, \\ \mathrm{LH}(f_2) &= x_1^3x_2 - 3x_1^2x_2x_1 + 3x_1x_2x_1^2 - x_2x_1^3, \\ \mathrm{LH}(f_3) &= x_1^2x_2x_1x_2 - 3x_1x_2x_1^2x_2 + 2x_1x_2x_1x_2x_1 + 3x_2x_1^2x_2x_1 - 5x_2x_1x_2x_1^2 + 2x_2^2x_1^3. \end{aligned}$$

However, $\mathrm{LH}(f_3) = \mathrm{LH}(f_2)x_2 - x_2\,\mathrm{LH}(f_2) + \mathrm{LH}(f_1)x_1^2 - x_1^2\,\mathrm{LH}(f_1) - x_1\,\mathrm{LH}(f_1)x_1$. Therefore,

$$G^2(\mathcal{J}) = k\langle x_1, x_2\rangle/(x_1x_2^2 - 2x_2x_1x_2 + x_2^2x_1, x_1^3x_2 - 3x_1^2x_2x_1 + 3x_1x_2x_1^2 - x_2x_1^3).$$

This is just $D(-2, -1)$ in [**9**] which is AS-regular. By Theorem 0.1, we have

PROPOSITION 4.5. *The algebra $\mathcal{J}$ is an AS-regular algebra of global dimension* 4.

5. Properties of the algebras. In this section, we show some properties of $\mathcal{J}$ about ring-theoretic, homology and geometry.

The algebra $D(-2, -1)$ has been proved to be noetherian, strongly noetherian and Auslander regular in [**9**]. By Corollary 2.15, we immediately obtain the following:

THEOREM 5.1. *The algebra $\mathcal{J}$ is strongly noetherian and Auslander regular.*

Besides, we still want to know whether $\mathcal{J}$ is Cohen–Macaulay. An Ore extension is constructed below.

THEOREM 5.2. *The algebra $\mathcal{J}$ is Cohen–Macaulay.*

Proof. We claim $\mathcal{J}$ is an Ore extension of an algebra which is Cohen–Macaulay. Hence, $\mathcal{J}$ is Cohen–Macaulay by [**13**, Lemma 1.3].

Take a graded polynomial algebra $B = k[x_2, z_1, z_2]$ with $\deg x_2 = 1$, $\deg z_1 = 2$, and $\deg z_2 = 3$. This is Cohen–Macaulay since it is an iterated Ore extension.

Let x_1 be a new variable with degree 1 and $C = B[x_1; \sigma, \delta]$ where σ is the identity and

$$\delta(x_2) = z_1, \quad \delta(z_1) = z_2, \quad \delta(z_2) = (u-1)z_1^2 - x_2z_2 + vx_2^2z_1 - wx_2^4, \quad u, v, w \in k.$$

We rewrite the relations between x_1 and x_2, z_1 as

$$x_1x_2 = x_2x_1 + z_1, \quad x_1z_1 = z_1x_1 + z_2.$$

Then, z_1, z_2 can be generated by x_1, x_2 as

$$z_1 = x_1x_2 - x_2x_1, \quad z_2 = x_1z_1 - z_1x_1.$$

Hence, C is generated by x_1, x_2. The other four relations of C are listed below

$$\begin{aligned}
&x_2z_1 - z_1x_2,\\
&x_1z_2 - z_2x_1 + (1-u)z_1^2 + x_2z_2 - vx_2^2z_1 + wx_2^4,\\
&x_2z_2 - z_2x_2,\\
&z_1z_2 - z_2z_1.
\end{aligned}$$

Replacing z_1, z_2, the first relation is equivalent to the relation f_1 of $\mathcal{J}$. After being reduced by f_1 with respect to $\prec_{\mathbb{Z}^2}$, the second is equivalent to f_2 of $\mathcal{J}$. And the last two relations can be derived from f_1, f_2, f_3. Hence, $\mathcal{J} \cong C$. □

REMARK 5.3. The proof also shows that $\mathcal{J}$ is AS-regular of dimension 4, strongly noetherian and Auslander regular. However, finding an Ore extension is a tedious task, the method used in last section is more effective.

THEOREM 5.4. *The automorphism group of $\mathcal{J}$ is isomorphic to the group G, where*

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a \in k\backslash\{0\}, b \in k \right\}.$$

Proof. Let σ is an arbitrary automorphism of $\mathcal{J}$. Suppose that

$$\sigma(x_1) = a_1x_1 + a_2x_2, \quad \sigma(x_2) = b_1x_1 + b_2x_2,$$

and the matrix $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ is nonsingular, that is, $a_1b_2 - a_2b_1 \neq 0$. Then,

$$\begin{aligned}
\sigma(f_1) = {} & b_1(a_2b_1 - a_1b_2)(x_1^2x_2 - 2x_1x_2x_1 + x_2x_1^2)\\
& + b_2(a_1b_2 - a_2b_1)(x_1x_2^2 - 2x_2x_1x_2 + x_2^2x_1).
\end{aligned}$$

It is zero in $\mathcal{J}$, so it must be a scalar multiple of f_1. Hence,

$$b_1 = 0.$$

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To see the other relation f_2,

$$\begin{aligned}
\sigma(f_2) &= a_1^3b_2(x_1^3x_2 - 3x_1^2x_2x_1 + 3x_1x_2x_1^2 - x_2x_1^3) + a_1^2b_2(4a_2 + (1-u)b_2)x_1x_2x_1x_2 \\
&\quad + a_1^2b_2((u-3)b_2 - 4a_2)x_2x_1x_2x_1 + a_1^2b_2(2a_2 + (2-u)b_2)x_2^2x_1^2 \\
&\quad - 2a_1^2a_2b_2x_1^2x_2^2 + ua_1^2b_2^2x_2x_1^2x_2 + a_1a_2b_2(a_2 + (1-u)b_2)x_1x_2^3 \\
&\quad + a_1a_2b_2((2u-3)b_2 - 3a_2)x_2x_1x_2^2 + a_1b_2((3-u)a_2b_2 + 3a_2^2 - vb_2^2)x_2^2x_1x_2 \\
&\quad + a_1b_2(vb_2^2 - a_2^2 - a_2b_2)x_2^3x_1 + wb_2^4x_2^4 \\
&= (1-u)a_1^2b_2(b_2 - a_1)x_1x_2x_1x_2 + (u-3)a_1^2b_2(b_2 - a_1)x_2x_1x_2x_1 \\
&\quad + (2-u)a_1^2b_2(b_2 - a_1)x_2^2x_1^2 + ua_1^2b_2(b_2 - a_1)x_2x_1^2x_2 \\
&\quad + va_1b_2(a_1^2 - b_2^2)x_2^2x_1x_2 + va_1b_2(b_2^2 - a_1^2)x_2^3x_1 + wb_2(b_2^3 - a_1^3)x_2^4 \\
&= 0.
\end{aligned}$$

Because the leading monomial $x_1x_2x_1x_2$ of right hand has no factors in $\mathrm{LM}(f_1)$, $\mathrm{LM}(f_2)$, it must be zero. While $a_1, b_2 \neq 0$, we obtain

$$a_1 = b_2.$$

Therefore, $\mathrm{Aut}(\mathcal{J}) \cong G$. $\square$

At last, we calculate the point modules of $\mathcal{J}$. Before that recall the definition.

DEFINITION 5.5 *[3]*. Let A be a connected graded algebra, a graded A-module M is called a *point module* if it satisfies the following conditions:

(a) M is generated in degree zero,
(b) $M_0 = k$,
(c) $\dim_k M_i = 1$, for all $i \geq 0$.

THEOREM 5.6. *The algebra $\mathcal{J}$ has two classes of point modules up to isomorphism.*

Proof. Let $M = \mathcal{J}e_0$ be a point module of $\mathcal{J}$. As vector space, $M = \bigoplus_{i=0}^{\infty} ke_i$ where $\deg e_i = i$. The $\mathcal{J}$-module structure on M can be described by generators as

$$x_1e_i = p_{i+1}e_{i+1}, \quad x_2e_i = q_{i+1}e_{i+1}, \text{ for any } i \geq 0,$$

where $p_i, q_i \in k$. For every $i > 0$, p_i, q_i cannot be zero simultaneously. Denote $\alpha_i = (p_i, q_i)$, then M determines a unique sequence of points $\{\alpha_i\}_{i=1}^{\infty}$ in $\mathbb{P}^1$.

Because of the $\mathcal{J}$-module structure on M, we have the equations

$$\begin{aligned}
&p_{i+3}q_{i+2}q_{i+1} - 2q_{i+3}p_{i+2}q_{i+1} + q_{i+3}q_{i+2}p_{i+1} = 0, \\
&p_{i+4}p_{i+3}p_{i+2}q_{i+1} - 3p_{i+4}p_{i+3}q_{i+2}p_{i+1} + 3p_{i+4}q_{i+3}p_{i+2}p_{i+1} + (1-u)p_{i+4}q_{i+3}p_{i+2}q_{i+1} \\
&\quad - q_{i+4}p_{i+3}p_{i+2}p_{i+1} + uq_{i+4}p_{i+3}p_{i+2}q_{i+1} + (u-3)q_{i+4}p_{i+3}q_{i+2}p_{i+1} \\
&\quad + (2-u)q_{i+4}q_{i+3}p_{i+2}p_{i+1} - vq_{i+4}q_{i+3}p_{i+2}q_{i+1} + vq_{i+4}q_{i+3}q_{i+2}p_{i+1} \\
&\quad + wq_{i+4}q_{i+3}q_{i+2}q_{i+1} = 0,
\end{aligned}$$

for any $i \geq 0$.

Notice that, the solutions of equations above are sequences of points $\{\alpha_i\}_{i=1}^{\infty}$ in $\mathbb{P}^1$. And those sequences of points also determines point modules of $\mathcal{J}$.

Suppose $S = \{(a_1, b_1), (a_2, b_2), \ldots\}$ is a sequence of points related to $\mathcal{J}$, it must be a solution of the equations. Now we consider the sequence $S_1 = \{(a_2, b_2), (a_3, b_3), \ldots\}$, it

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is also a solution of the equations. It always holds for $S_i = \{(a_{i+1}, b_{i+1}), (a_{i+2}, b_{i+2}), \ldots\}$ for any $i > 0$. In this sense, the minimum period length of those equations is 4. Hence, we solve the equation as follows:

$$\begin{aligned}
&p_3q_2q_1 - 2q_3p_2q_1 + q_3q_2p_1 = 0,\\
&p_4q_3q_2 - 2q_4p_3q_2 + q_4q_3p_2 = 0,\\
&p_4p_3p_2q_1 - 3p_4p_3q_2p_1 + 3p_4q_3p_2p_1 + (1-u)p_4q_3p_2q_1 - q_4p_3p_2p_1 + uq_4p_3p_2q_1\\
&\quad + (u-3)q_4p_3q_2p_1 + (2-u)q_4q_3p_2p_1 - vq_4q_3p_2q_1 + vq_4q_3q_2p_1 + wq_4q_3q_2q_1 = 0.
\end{aligned} \tag{EP}$$

If $p_i = 0$ (respectively, $q_i = 0$), then we assume $q_i = 1$ (respectively, $p_i = 1$) by some appropriate change of basis. If both q_i and p_i are nonzero, we assume $q_i = 1$. Then, the solutions of equations (EP) are listed below

$$\begin{cases} p_1 = 1,\ p_2 = 1,\ p_3 = 1,\ p_4 = 1,\\ q_1 = 0,\ q_2 = 0,\ q_3 = 0,\ q_4 = 0. \end{cases} \tag{P1}$$

$$\begin{cases} p_1 = p_1,\ p_2 = 1,\ p_3 = 1,\ p_4 = p_1 - u,\\ q_1 = 1,\quad q_2 = 0,\ q_3 = 0,\ q_4 = 1. \end{cases} \tag{P2}$$

$$\begin{cases} p_1 = p_1,\ p_2 = p_1 + d,\ p_3 = p_1 + 2d,\ p_4 = p_1 + 3d,\\ q_1 = 1,\quad q_2 = 1,\qquad q_3 = 1,\qquad q_4 = 1, \end{cases} \tag{P3}$$

where $d \in k$ satisfies $6d^3 + (3-u)d^2 - vd + w = 0$.

Assemble them under the rule that each S_i is a solution for any $i > 0$. Therefore, there exist two classes of point modules:

(a) (P1)(P1)(P1)(P1)(P1)(P1)$\cdot\cdots\cdots$,

(b) (P3)(P3)(P3)(P3)(P3)(P3)$\cdot\cdots\cdots$.

where (a) and (b) are sequenced by (P1) and (P3), respectively. □

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