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SOME STRONG FORMS OF THE LOCAL DUALITY OF ULTRAPRODUCTS

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Abstract. We obtain two refinements of the so called local duality of ultrapowers, that is, the ultrapower version of the well-known principle of local reflexivity.

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1. Introduction. A fundamental result in the modern theory of Banach spaces is the principle of local reflexivity due to J. Lindenstrauss and H. P. Rosenthal [**4**, Theorem 3.1]. This principle admits a version in the context of the theory of ultraproducts, the so-called theorem of the local duality of ultraproducts, which relates the finite dimensional geometry of the dual of an ultraproduct of Banach spaces and the ultraproduct of their duals [**2**, Theorem 7.3]. This theorem has become one of the most important model-theoretic tools in Banach space theory. Several authors have improved the principle of local reflexivity in the line of obtaining operators with additional properties. Two, (perhaps, the best known) of these stronger versions are [**1**] and [**3**].

Theorem 1.1 (Barton-Yu). *Let X be a Banach space, M a finite dimensional subspace of X^{**}, F a reflexive subspace of X^* and $\varepsilon > 0$. Then, there is an operator T from M into X such that*

(i) $(1-\varepsilon)\|m\| \leq \|Tm\| \leq (1+\varepsilon)\|m\|$, *for all* $m \in M$.

(ii) $Tm = m$, for all $m \in M \cap X$.

(iii) $\langle Tm, f\rangle = \langle m, f\rangle$, for all $m \in M$ and $f \in F$.

Theorem 1.2 (Johnson-Rosenthal-Zippin). *Let X be a Banach space, M and F finite dimensional subspaces of $X^{**}$and X^*, respectively, and $\varepsilon > 0$. Moreover, take a projection P from X^{**} onto M. Then, there is an operator T from M into X which satisfies conditions (i), (ii) and (iii) of the above theorem and there is a projection P_0 from X onto $T(M)$ such that*

$$\|P_0\| \leq (1+\varepsilon)\|P\|.$$

Therefore the following natural question appears. Is it possible to establish versions of these theorems for the local duality of ultraproducts? In this paper, we give a positive answer.

Our terminology and notations are standard and we refer the reader to the excellent monographs of Heinrich [**2**] and Sims [**5**] about the theory of ultraproducts.

2. Results. We begin by proving the corresponding version for ultraproducts of the result of Johnson, Rosenthal and Zippin.

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THEOREM 2.1. *Let $\mathcal{U}$ be an ultrafilter on a set $\mathcal{I}$, $(X_i)_{i\in\mathcal{I}}$ a family of Banach spaces, M and F finite dimensional subspaces of $(X_i)^*_{\mathcal{U}}$ and $(X_i)_{\mathcal{U}}$, respectively, and $\varepsilon > 0$. Take a projection P from $(X_i)^*_{\mathcal{U}}$ onto M . Then, there is an operator T from M into $(X_i^*)_{\mathcal{U}}$ and a projection P_0 from $(X_i^*)_{\mathcal{U}}$onto $T(M)$ such that*

(i) $(1-\varepsilon)\|m\| \le \|Tm\| \le (1+\varepsilon)\|m\|$, *for all* $m \in M$,
(ii) $Tm = m$, *for all* $m \in M \cap (X_i^*)_{\mathcal{U}}$,
(iii) $\langle Tm, f\rangle = \langle m, f\rangle$, *for all* $m \in M$ *and* $f \in F$,
(iv) $\|P_0\| \le \|P\|(1+\varepsilon)$.

Proof. Take $\delta > 0$ satisfying $(1+\delta)^2 < 1+\varepsilon$ and $\delta < \varepsilon$. Applying the principle of local reflexivity to $P^*M^* \subset (X_i)^{**}_{\mathcal{U}}$ and $M \subset (X_i)^*_{\mathcal{U}}$, we obtain an operator $U : P^*M^* \longrightarrow (X_i)_{\mathcal{U}}$ such that

$$(1-\delta)\|P^*m^*\| \le \|UP^*m^*\| \le (1+\delta)\|P^*m^*\|, \text{ for all } m^* \in M^*,$$

$$\langle UP^*m^*, m\rangle = \langle P^*m^*, m\rangle, \text{ for all } m^* \in M^* \text{ and } m \in M.$$

Define $S := UP^*$and denote by $\widetilde{F}$ the finite dimensional subspace of $(X_i)_{\mathcal{U}}$ spanned by F and $S(M^*)$. Now, applying the local duality of ultraproducts to $M \subset (X_i)^*_{\mathcal{U}}$ and $\widetilde{F} \subset (X_i)_{\mathcal{U}}$, we find an operator $T : M \to (X_i^*)_{\mathcal{U}}$ such that
(i) $(1-\delta)\|m\| \le \|Tm\| \le (1+\delta)\|m\|$, for all $m \in M$.
(ii) $Tm = m$ for all $m \in M \cap (X_i^*)_{\mathcal{U}}$.
(iii) $\langle Tm, f\rangle = \langle m, f\rangle$, for all $m \in M$ and $f \in \widetilde{F}$.

In particular, we note that

$$(1-\varepsilon)\|m\| \le \|Tm\| \le (1+\varepsilon)\|m\|, \text{for all } m \in M, \text{ and}$$

$$\langle Tm, f\rangle = \langle m, f\rangle \text{ for all } m \in M \text{ and } f \in F.$$

Obviously, we only have to find the projection P_0 onto $T(M)$ and show that it satisfies condition (iv). Consider the canonical embedding of $(X_i^*)_{\mathcal{U}}$ into $(X_i)^*_{\mathcal{U}}$ and take P_0 as the restriction of the operator TS^* to $(X_i^*)_{\mathcal{U}}$. Clearly

$$P_0 : (X_i^*)_{\mathcal{U}} \to T(M)$$

and

$$\|P_0\| \le \|TS^*\| \le \|T\|\|U\|\|P\| \le \|P\|(1+\varepsilon).$$

Moreover, if $m \in M$ and $m^* \in M^*$, we have

$$\langle S^*Tm, m^*\rangle = \langle Tm, Sm^*\rangle = \langle m, Sm^*\rangle = \langle m, UP^*m^*\rangle$$

$$= \langle m, P^*m^*\rangle = \langle Pm, m^*\rangle = \langle m, m^*\rangle.$$

Therefore, $P_0(Tm) = TS^*(Tm) = Tm$. □

Now, we present the variant of the local duality of ultraproducts related to the theorem of Barton and Yu. We use the following lemma, which can be considered the version for ultraproducts of [**6**, Theorem 4].

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LEMMA 2.2. *Let $\mathcal{U}$ be an ultrafilter on a set $\mathcal{I}$, $(X_i)_{i\in\mathcal{I}}$ a family of Banach spaces, F a reflexive subspace of $(X_i)_{\mathcal{U}}$, $h \in (X_i)^*_{\mathcal{U}}$, and $\varepsilon > 0$. Then there is $(x_i^*)_{\mathcal{U}} \in (X_i^*)_{\mathcal{U}}$ such that*

(i) $\|(x_i^*)_{\mathcal{U}}\| \leq (1+\varepsilon)\|h\|$,
(ii) $\langle h, f\rangle = \langle f, (x_i^*)_{\mathcal{U}}\rangle$, *for all* $f \in F$.

Proof. Consider the canonical embedding of $(X_i^*)_{\mathcal{U}}$ into $(X_i)^*_{\mathcal{U}}$ and define

$$u : (X_i^*)_{\mathcal{U}} \to F^*, \quad u((x_i^*)_{\mathcal{U}}) := f^*,$$

where f^* is the restriction of $(x_i^*)_{\mathcal{U}}$ to F. Likewise, denote

$$\mathcal{N}(u) := \{x \in X : u(x) = 0\}.$$

If $\mathcal{J}_1$ denotes the canonical embedding of $(X_i)_{\mathcal{U}}$ into $(X_i^*)^*_{\mathcal{U}}$ and i denotes the canonical embedding of F into $(X_i)_{\mathcal{U}}$, then it is clear that $u^* = \mathcal{J}_1 \circ i$. Therefore, u^* is an isometry and the induced map

$$\widetilde{u} : (X_i^*)_{\mathcal{U}}/\mathcal{N}(u) \longrightarrow F^*, \quad x + \mathcal{N}(u) \longmapsto u(x)$$

is a surjective isometry. Then, given a non-zero $h \in (X_i)^*_{\mathcal{U}}$ and denoting its restriction to F as $h|_F$, there is $\widetilde{x} + \mathcal{N}(u) \in (X_i^*)_{\mathcal{U}}/\mathcal{N}(u)$ such that

$$\widetilde{u}(\widetilde{x} + \mathcal{N}(u)) = h|_F, \quad \|\widetilde{x} + \mathcal{N}(u)\| = \|h|_F\|.$$

Therefore, there is $m \in \mathcal{N}(u)$ such that

$$\|\widetilde{x} + m\| - \varepsilon\|h\| \leq \|\widetilde{x} + \mathcal{N}(u)\|.$$

Let us check that $\widetilde{x} + m \in (X_i^*)_{\mathcal{U}}$ is the desired element. On the one hand,

$$\|\widetilde{x} + m\| \leq \|\widetilde{x} + \mathcal{N}(u)\| + \varepsilon\|h\| \leq (1+\varepsilon)\|h\|.$$

On the other hand, if $f \in F$,

$$\langle h, f\rangle = \langle \widetilde{u}(\widetilde{x} + \mathcal{N}(u)), f\rangle = \langle u(\widetilde{x}), f\rangle = \langle u(\widetilde{x} + m), f\rangle = \langle \widetilde{x} + m, f\rangle.$$

□

The starting point of the next proof is the same as [**5**, p. 84]; that is, the isometric identifications

$$(X \otimes_\pi Y)^* \equiv L(X, Y^*),$$
$$M \otimes_\pi (X_i)_{\mathcal{U}} \equiv (M \otimes_\pi X_i)_{\mathcal{U}} \ (M \text{ finite-dimensional}),$$

where $\otimes_\pi$ denotes the projective tensor product. After that, our arguments are different and, in a certain sense, more delicate.

THEOREM 2.3. *Let $\mathcal{U}$ be an ultrafilter on a set $\mathcal{I}$, $(X_i)_{i\in\mathcal{I}}$ a family of Banach spaces, M a finite dimensional subspace of $(X_i)^*_{\mathcal{U}}$, F a reflexive subspace of $(X_i)_{\mathcal{U}}$ and $\varepsilon > 0$.*

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Assume that $F=(F_i)_{\mathcal{U}}$, *where* F_i *is a subspace of* X_i, *for each* $i\in I$. *Then there is an operator* $T:M\to (X_i^*)_{\mathcal{U}}$ *such that*

(i) $(1-\varepsilon)\|m\|\le \|Tm\|\le \|m\|$ *for all* $m\in M$.

(ii) $Tm=m$ *for all* $m\in M\cap (X_i^*)_{\mathcal{U}}$.

(iii) $\langle Tm,f\rangle=\langle m,f\rangle$ *for all* $m\in M$ *and* $f\in F$.

Proof. Since M is finite dimensional, we may enlarge F if necessary so that, for all $m\in M$, we have

$$(*)\qquad (1-\varepsilon)\|m\|\le \sup\left\{\left|\langle m,f\rangle\right| : \left\|f\right\|=1, f\in F\right\}.$$

Let $I:M\to (X_i)^*_{\mathcal{U}}$ be the inclusion map. Then $I\in L(M,(X_i)^*_{\mathcal{U}})$ and we can identify I with a functional $I'\in ((M\otimes_\pi X_i)_{\mathcal{U}})^*$. Since M is a finite dimensional subspace, $M\otimes_\pi F$ is a reflexive subspace of $M\otimes_\pi (X_i)_{\mathcal{U}}$ and therefore $(M\otimes_\pi F_i)_{\mathcal{U}}$ is a reflexive subspace of $(M\otimes_\pi X_i)_{\mathcal{U}}$. By the above lemma, there is $S\in ((M\otimes_\pi X_i)^*)_{\mathcal{U}}$ such that

$$\|S\|\le \left\|I'\right\|(1+\varepsilon)=1+\varepsilon,$$

$$S(m\otimes_\pi f)=I'(m\otimes_\pi f),\ \text{for all } m\in M \text{ and } f\in F.$$

Indeed, the lemma says that $S=(S_i)_{\mathcal{U}}$ with $S_i\in (M\otimes_\pi X_i)^*$, $i\in\mathcal{I}$.

Let $M_1=M\cap (X_i^*)_{\mathcal{U}}$ and take M_2 such that $M=M_1\oplus M_2$. We assume that $M_1\neq\{0\}$. Let $\{h_1,\ldots,h_r\}$ be a basis of M_1 with $h_j=(h_i^j)_{\mathcal{U}}$ $(j=1,\ldots r)$ and choose, for each $j=1,\ldots,r$, a representative $(h_i^j)_{i\in\mathcal{I}}\in l_\infty(\mathcal{I},X_i^*)$.

Then, we can define the map

$$\widehat{S}_i:(M_1\otimes_\pi X_i)\oplus(M_2\otimes_\pi F_i)\to\mathbb{K},$$

$$\widehat{S}_i((m_1\otimes_\pi x_i)\oplus(m_2\otimes_\pi f_i)):=\langle m_i^1,x_i\rangle+S_i(m_2\otimes_\pi f_i),$$

where $m_i^1=\alpha_1h_i^1+\cdots+\alpha_rh_i^r$, if $m_1=\alpha_1h_1+\cdots+\alpha_rh_r$. Since the representatives are fixed from the beginning and M_1 is finite dimensional, we see that the map is well defined and linear. Moreover,

$$\left\|\widehat{S}_i\right\|\le 1+\left\|(S_i)_{i\in\mathcal{I}}\right\|_\infty,\ \text{for all } i\in\mathcal{I}.$$

However, this bound is not sufficient for our purpose and we are going to show that, in fact, $\left\|(\widehat{S}_i)_{\mathcal{U}}\right\|\le 1$.

Let $(m_1\otimes_\pi x_i+m_2\otimes_\pi f_i)_{\mathcal{U}}$ be an element of the unit ball of the space $(M_1\otimes_\pi X_i\oplus M_2\otimes_\pi F_i)_{\mathcal{U}}$. Then

$$\left|(\widehat{S}_i)_{\mathcal{U}}(m_1\otimes_\pi x_i+m_2\otimes_\pi f_i)_{\mathcal{U}}\right|=\lim{}_{\mathcal{U}}\left|\widehat{S}_i(m_1\otimes_\pi x_i+m_2\otimes_\pi f_i)\right|$$

$$=\lim{}_{\mathcal{U}}\left|\langle m_i^1,x_i\rangle+S_i(m_2\otimes_\pi f_i)\right|=\left|\langle m_1,(x_i)_{\mathcal{U}}\rangle+S(m_2\otimes_\pi (f_i)_{\mathcal{U}})\right|$$

$$=\left|\langle m_1,(x_i)_{\mathcal{U}}\rangle+I'(m_2\otimes_\pi (f_i)_{\mathcal{U}})\right|=\left|I'(m_1\otimes_\pi (x_i)_{\mathcal{U}}+m_2\otimes_\pi (f_i)_{\mathcal{U}})\right|\le 1.$$

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Now, we note that $(M_1 \otimes_\pi X_i) \oplus (M_2 \otimes_\pi F_i)$ is a subspace of $M \otimes_\pi X_i$, for all $i \in \mathcal{I}$, and so we can extend $\widehat{S_i}$ to $M \otimes_\pi X_i$ by the theorem of Hahn-Banach. Let $\widehat{T_i}$ be this extension. Of course, we may identify each $\widehat{T_i}$ with an operator $T_i \in L(M, X_i^*)$ such that $\|T_i\| = \left\|\widehat{T_i}\right\| = \left\|\widehat{S_i}\right\|$.

Define $T := (T_i)_{\mathcal{U}} \in (L(M, X_i^*))_{\mathcal{U}}$. Bearing in mind the isometric identification

$$L(M, (X_i^*)_{\mathcal{U}}) \equiv (L(M, X_i^*))_{\mathcal{U}},$$

we may also identify T as an operator from M to $(X_i^*)_{\mathcal{U}}$. Let us show that T is the desired operator.

First, if $m \in M \cap (X_i^*)_{\mathcal{U}}$ and $(x_i)_{\mathcal{U}} \in (X_i)_{\mathcal{U}}$,

$$\langle Tm, (x_i)_{\mathcal{U}} \rangle = \lim{}_{\mathcal{U}} \langle T_i m, x_i \rangle = \lim{}_{\mathcal{U}} \widehat{S_i}(m \otimes_\pi x_i) = \langle m, (x_i)_{\mathcal{U}} \rangle.$$

Second, if $m = m_1 + m_2 \in M_1 \oplus M_2 = M$ and $f = (f_i)_{\mathcal{U}} \in F$, then

$$\langle Tm, f \rangle = \lim{}_{\mathcal{U}} \langle T_i m, f_i \rangle = \lim{}_{\mathcal{U}} \widehat{S_i}(m \otimes_\pi f_i) = \lim{}_{\mathcal{U}} (\langle m_i^1, f_i \rangle + S_i(m_2 \otimes_\pi f_i))$$

$$= \langle m_1, (f_i)_{\mathcal{U}} \rangle + S(m_2 \otimes_\pi (f_i)_{\mathcal{U}}) = \langle m_1, (f_i)_{\mathcal{U}} \rangle + I'(m_2 \otimes_\pi (f_i)_{\mathcal{U}}) = \langle m, f \rangle.$$

Finally,

$$\|T\| = \lim{}_{\mathcal{U}} \|T_i\| = \lim{}_{\mathcal{U}} \left\|\widehat{S_i}\right\| = \left\|(\widehat{S_i})_{\mathcal{U}}\right\| \leq 1,$$

and, by $(*)$, we deduce that

$$\|Tm\| \geq \sup\left\{ \left| \langle Tm, f \rangle \right| : \|f\| = 1, f \in F \right\} = \sup\left\{ \left| \langle m, f \rangle \right| : \|f\| = 1, f \in F \right\}$$

$$\geq (1 - \varepsilon)\|m\|.$$

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