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A QUANTITATIVE EXTENSION OF SZLENK'S THEOREM

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Abstract

We show that for a bounded subset A of the $L_1(\mu)$ space with finite measure μ, the measure of weak noncompactness of A based on the convex separation of sequences coincides with the measure of deviation from the Banach–Saks property expressed by the arithmetic separation of sequences. A similar result holds for a related quantity with the alternating signs Banach–Saks property. The results provide a geometric and quantitative extension of Szlenk's theorem saying that every weakly convergent sequence in the Lebesgue space L_1 has a subsequence whose arithmetic means are norm convergent.

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1. Introduction

In [17], Szlenk proved that the Lebesgue space $L_1([0,1])$ has the weak Banach–Saks (WBS) property, that is every weakly convergent sequence (x_n) in $L_1([0,1])$ has a subsequence (x'_n) whose arithmetic means $(x'_1+\cdots+x'_n)/n$ converge in norm. The WBS property of a Banach space is a weakened form of the Banach–Saks (BS) property, where Cesàro summability of bounded sequences is considered. The BS property of $L_p([0,1])$ with $1<p<\infty$ was proved in the classical paper [3].

Properties of this type can also be considered for sets. We say that a bounded subset D of a Banach space X has the BS property (or D is a BS set), if every sequence (x_n) in D contains a subsequence (x'_n) whose arithmetic means $(x'_1+\cdots+x'_n)/n$ converge in X. If the alternating sign means $(-x'_1+\cdots+(-1)^n x'_n)/n$ are norm convergent in X, we say that D has the alternating signs Banach–Saks (ABS) property (or D is an ABS set).

Nishiura and Waterman [16] proved that every Banach space with the BS property is reflexive. This result also has a counterpart for sets: every BS subset of a Banach space is relatively weakly compact (see [13]). On the other hand, the reflexive version of the Schreier space given by Baernstein [2] shows that, in general, these implications cannot be reversed.

In the above terminology, Szlenk's result can be read as follows: a bounded subset D in $L_1([0,1])$ is relatively weakly compact if and only if D is a BS set. In general, by Mazur's theorem, the weak limit of a sequence (x_n) in a Banach space can be approximated in norm by the convex combinations of the elements of (x_n). The BS property allows us to replace unknown convex combinations by arithmetic means which is especially important in applications where, for instance, the solutions of equations are the weak limits of sequences of approximate solutions. In this note we show that the rule described in Szlenk's theorem is deeper and—in a certain quantitative sense—concerns all bounded subsets of L_1. We prove that the deviations from weak compactness and from the BS and ABS properties, for a bounded subset in $L_1(\mu)$ with finite measure μ, coincide from a geometric viewpoint.

2. Quantitative results

We give geometric descriptions of weak compactness and the BS and ABS properties. Let X be a Banach space and $\mathsf{B}(X)$ its closed unit ball. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. The cardinality of $A \subset \mathbb{N}$ will be denoted by $|A|$. For a bounded set $D \subset X$, we consider its measure of weak noncompactness,

$$\gamma(D) = \sup\{\operatorname{csep}(x_n) \colon (x_n) \subset D\},$$

where

$$\operatorname{csep}(x_n) = \inf_m \operatorname{dist}(\operatorname{conv}\{x_n\}_{n=1}^m, \operatorname{conv}\{x_n\}_{n=m+1}^\infty).$$

The quantity csep is based on James' criterion of weak compactness (see [12]) and will be called the *convex separation* of a sequence. The measure γ was defined in [15] with the supremum over all sequences in $\operatorname{conv} D$. By the quantitative version of Krein's theorem proved in [11], combined with [15, Theorem 2.5], the supremum can be restricted to all sequences in D.

In [5], Beauzamy proved with the use of spreading models that a Banach space X does not have the BS property if and only if there exist $\delta > 0$ and a bounded sequence (x_n) in X such that for all subsequences (x'_n) of (x_n) and all integers $1 \le k \le m$,

$$\left\| \frac{1}{m} \left(\sum_{n=1}^{k} x'_n - \sum_{n=k+1}^{m} x'_n \right) \right\| \ge \delta.$$

We consider a quantity based on a modification of Beauzamy's condition, called the *arithmetic separation* of a sequence (see [13]), which measures distances between the arithmetic means over successive equipollent blocks of (x_n):

$$\operatorname{asep}(x_n) = \inf \left\| \frac{1}{m} \left(\sum_{n \in A} x_n - \sum_{n \in B} x_n \right) \right\|,$$

the infimum being taken over all $m \in \mathbb{N}$ and all finite subsets $A, B \subset \mathbb{N}$ with $|A| = |B| = m$ and $\max A < \min B$. Then the deviation of D from the BS property will be measured by

$$\varphi(D) = \sup\{\operatorname{asep}(x_n) \colon (x_n) \subset D\}.$$

The measure of deviation from the ABS property is given by

$$\widehat{\varphi}(D) = \sup\{\phi(x_n) : (x_n) \subset D\},$$

where

$$\phi(x_n) = \inf \Big\| |A|^{-1} \sum_{n\in A} \epsilon_n x_n \Big\|,$$

the infimum being taken over all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n), with $\epsilon_n = \pm 1$. Another approach to the quantification of the Banach–Saks properties can be found in the recent paper [7].

The measure γ satisfies the axiomatic definition of a measure of weak noncompactness given in [4]. In particular, $\gamma(D) = 0$ if and only if D is relatively weakly compact. Recall the basic properties of φ and $\widehat{\varphi}$ proved in [13].

Proposition 2.1. *Let D and E be bounded subsets of a Banach space X.*

(i) $\varphi(D) = 0$ *if and only if D is a BS set.*
(ii) $\widehat{\varphi}(D) = 0$ *if and only if D is an ABS set.*
(iii) $\gamma(D) \leq \varphi(D)$ *and* $2\widehat{\varphi}(D) \leq \varphi(D)$.
(iv) *If $D \subset E$, then* $\varphi(D) \leq \varphi(E)$ *and* $\widehat{\varphi}(D) \leq \widehat{\varphi}(E)$.
(v) $\varphi(tD) = |t|\varphi(D)$ *and* $\widehat{\varphi}(tD) = |t|\widehat{\varphi}(D)$ *for every scalar t.*
(vi) *If D, E are convex, then* $\varphi(D + E) \leq \varphi(D) + \varphi(E)$.
(vii) *If D, E are convex and symmetric, then* $\widehat{\varphi}(D + E) \leq \widehat{\varphi}(D) + \widehat{\varphi}(E)$.

The inequality $\gamma(D) \leq \varphi(D)$ is a quantitative extension for sets of the result of Nishiura and Waterman [16]. The unit ball of the Baernstein space defined in [2] shows that this inequality can be sharp.

The subadditivity of the measure φ for convex sets is implied by the fact that the variations of the arithmetic separation of a sequence can be reduced to an arbitrary small interval by passing to a certain sequence of arithmetic means over successive equipollent blocks. A similar stabilising result holds for the quantity ϕ related to the ABS property (see [13]). Both results are based on Ramsey's theorem and the spreading models of Brunel and Sucheston [8].

The following theorem provides a quantitative extension of Szlenk's result.

Theorem 2.2. *Let D be a nonempty bounded subset of $L_1(\mu)$ with finite measure μ. Then*

(i) $\varphi(D) = \gamma(D)$;
(ii) $2\widehat{\varphi}(D) = \gamma(D)$.

Proof. Let (Ω, Σ, μ) be a finite measure space. Put $L_1 = L_1(\mu)$. Clearly, for the measure φ it is enough to prove that $\varphi(D) \leq \gamma(D)$. By [6, Theorem II.3.5], a Banach space X does not have the ABS property if and only if X has a spreading model isomorphic to

l_1. In particular, for L_1 and, by [6, Theorem II.2.3], for each $\varepsilon > 0$, there exists (z_n) in $\mathsf{B}(L_1)$ such that

$$\Big\| \sum_{n \in A} \epsilon_n z_n \Big\| \geq |A|(1-\varepsilon)$$

for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n), with $\epsilon_n = \pm 1$. Thus

$$\varphi(\mathsf{B}(L_1)) = 2.$$

We use Grothendieck's characterisation of weak compactness and related to it the measure of weak noncompactness ω introduced by De Blasi [9] for a bounded subset D in a Banach space X:

$$\omega(D) = \inf\{t > 0 \colon D \subset K + t\mathsf{B}(X),\ K \text{ is weakly compact}\}.$$

Let $D \subset K + t\mathsf{B}(L_1)$ for some $t > 0$ and a weakly compact set $K \subset L_1$. By Proposition 2.1,

$$\begin{aligned}\varphi(D) &\leq \varphi(K + t\mathsf{B}(L_1))\\ &\leq \varphi(\operatorname{conv}(K) + t\mathsf{B}(L_1)) \leq \varphi(\operatorname{conv}(K)) + t\varphi(\mathsf{B}(L_1)).\end{aligned}$$

Since $\operatorname{conv}(K)$ is relatively weakly compact, by Szlenk's result [17] extended to L_1 with finite measure μ (see [10, page 85]), $\operatorname{conv}(K)$ is a BS set. Thus $\varphi(\operatorname{conv}(K)) = 0$. It follows that $\varphi(D) \leq 2t$ and consequently,

$$\varphi(D) \leq 2\omega(D).$$

Since $2\widehat{\varphi}(D) \leq \varphi(D)$, we have also

$$\widehat{\varphi}(D) \leq \omega(D).$$

The measure ω in L_1 can be expressed as the deviation from uniform integrability by the following formula given in [1]:

$$\omega(D) = \inf_{\delta > 0} \sup_{\mu(C) \leq \delta} \sup_{x \in D} \int_C |x(t)|\, d\mu(t).$$

To make our exposition more self-contained, we use a part of the proof of Theorem 2.5 of [14]. The case of relatively weakly compact D is solved by Szlenk's theorem. Therefore we can assume that $\omega(D) > 2\varepsilon > 0$. There is a sequence (x_n) in D and a sequence (E_n) of pairwise disjoint sets in Ω such that

$$\omega(D) - 2\varepsilon < \int_{E_n} |x_n(t)|\, d\mu(t)$$

for all n. Passing to a subsequence of (x_n) by Rosenthal's lemma (as in the proof of the Kadec–Pełczyński theorem in [10]), we can obtain almost disjoint elements x_n over (E_n) and assume that

$$\int_{E \setminus E_n} |x_n(t)|\, d\mu(t) < \varepsilon,$$

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where $E = \bigcup_{n\geq 1} E_n$. Let χ_E denote the characteristic function of the set E. Then for all scalars $\lambda_1, \ldots, \lambda_m$,

$$\begin{aligned}\Big\|\sum_{n=1}^{m}\lambda_n x_n\Big\| &\geq \Big\|\sum_{n=1}^{m}\lambda_n x_n\chi_E\Big\| \\ &\geq \sum_{n=1}^{m}\int_{E_n}|\lambda_n x_n(t)|\,d\mu(t) - \Big\|\sum_{n=1}^{m}\lambda_n x_n\chi_{E\setminus E_n}\Big\| \\ &\geq \sum_{n=1}^{m}\int_{E_n}|\lambda_n x_n(t)|\,d\mu(t) - \sum_{n=1}^{m}|\lambda_n|\int_{E\setminus E_n}|x_n(t)|\,d\mu(t) \\ &\geq (\omega(D)-2\varepsilon)\sum_{n=1}^{m}|\lambda_n| - \varepsilon\sum_{n=1}^{m}|\lambda_n| \\ &= (\omega(D)-3\varepsilon)\sum_{n=1}^{m}|\lambda_n|.\end{aligned}$$

For the coefficients of the convex separation of (x_n),

$$2(\omega(D)-3\varepsilon) \leq \mathrm{csep}(x_n),$$

and consequently,

$$\varphi(D) \leq 2\omega(D) \leq \gamma(D).$$

For the alternate signs means of (x_n),

$$\omega(D)-3\varepsilon \leq \phi(x_n).$$

It follows that $\widehat{\varphi}(D) = \omega(D)$ and finally,

$$2\widehat{\varphi}(D) = \gamma(D).$$

This completes the proof. □

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