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On p-Adic Properties of Central L-Values of Quadratic Twists of an Elliptic Curve

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Abstract. We study p-indivisibility of the central values $L(1, E_d)$ of quadratic twists E_d of a semi-stable elliptic curve E of conductor N. A consideration of the conjecture of Birch and Swinnerton-Dyer shows that the set of quadratic discriminants d splits naturally into several families $\mathcal{F}_S$, indexed by subsets S of the primes dividing N. Let $\delta_S = \gcd_{d\in\mathcal{F}_S} L(1, E_d)^{\mathrm{alg}}$, where $L(1, E_d)^{\mathrm{alg}}$ denotes the algebraic part of the central L-value, $L(1, E_d)$. Our main theorem relates the p-adic valuations of δ_S as S varies. As a consequence we present an application to a refined version of a question of Kolyvagin. Finally we explain an intriguing (albeit speculative) relation between Waldspurger packets on $\widetilde{\mathrm{SL}_2}$ and congruences of modular forms of integral and half-integral weight. In this context, we formulate a conjecture on congruences of half-integral weight forms and explain its relevance to the problem of p-indivisibility of L-values of quadratic twists.

1 Introduction

Let E be a semi-stable elliptic curve of (square-free) conductor N. Let d be a quadratic fundamental discriminant and consider twists E_d of E by d. The central L-values $L(1, E_d)$ have received much attention in the literature. It is known, for instance, that infinitely many of these are nonvanishing, even if we restrict attention to quadratic fields satisfying specific splitting conditions at a finite set of (finite) primes. Much less is known about the mod p properties of (the algebraic part of) these L-values. For instance, for a fixed prime p, it is not known if there is even a single quadratic discriminant d such that $L(1, E_d)^{\mathrm{alg}}$ is non-vanishing mod p.

The starting point for all work on these L-values so far is a result of Waldspurger that identifies them as squares of coefficients of modular forms of half-integral weight. One then exploits the simple idea that if a modular form has at least one nonvanishing coefficient, it must have infinitely many such; and in the case of half-integral weights, that one can even find, under suitable conditions, infinitely many square classes of nonvanishing coefficients. This idea can be made to work mod p along with suitable additional input, and in this way one may also obtain estimates on the density of d such that $L(1, E_d)^{\mathrm{alg}}$ is nonvanishing mod p, provided that one knows the existence of a single d with this property. The reader is referred to [5, 7] for quantitative results in this direction.

Our approach to this problem is different in two respects. First, instead of directly using Waldspurger's result (which is really about ratios of Fourier coefficients), we build on Waldspurger's work by studying in detail the p-integrality of the theta correspondence for the dual pair $(\widetilde{\mathrm{SL}_2}, \mathrm{PB}^\times)$, where B is an indefinite quaternion algebra over $\mathbb{Q}$. (The reader is referred to the forthcoming articles [9, 10] on these

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issues.) Secondly, we exploit crucially the structure of the Waldspurger packet on $\widetilde{\mathrm{SL}}_2$ corresponding to the automorphic representation π of PGL_2 associated with E, as is explained below.

The arithmetic of the values $L(1, E_d)$ is predicted by the Birch and Swinnerton-Dyer (BSD) conjecture. A consideration of the factors appearing in the BSD conjecture shows that the set of quadratic discriminants d splits up naturally into several families $\mathcal{F}_S$, indexed by subsets S of the primes dividing N. The reader is referred to the theorem stated below for the definition of $\mathcal{F}_S$; for the moment we will just say that a given quadratic discriminant d lies in $\mathcal{F}_S$ if it satisfies a prescribed set of splitting conditions at the primes dividing the conductor N. The key observation of this article is that these families $\mathcal{F}_S$ correspond in a very precise way to the different representations in the Waldspurger packet. Let us denote, for the moment, the representation of $\widetilde{\mathrm{SL}}_2$ associated with S by the symbol $\tilde{\pi}_S$. Then the set of L-values $\{L(1, E_d), d \in \mathcal{F}_S\}$ occurs as the set of squares of Fourier coefficients of a modular form h_S in $\tilde{\pi}_S$. Having made this observation, one can apply the results of [9, 10] to compare the p-adic valuations of the forms h_S as S varies. In particular we get the following result (Theorem 2.2).

Theorem *Assume that N is odd and $p \nmid \tilde{N} := \prod_{q|N} q(q^2 - 1)$. Let c_q be the order of the component group of the Neron model of E at q and $w_q = \pm 1$ the sign of the Atkin–Lehner involution at q. Let Σ be the set of primes dividing N and for S any subset of Σ set*

$$\mathcal{F}_S = \left\{ d \;\middle|\; \left(\frac{d}{q}\right) = -w_q \text{ for } q \in S, \left(\frac{d}{q}\right) \neq -w_q \text{ for } q \in \Sigma \setminus S \right\}$$

and $\mathcal{F}'_S = \{d \in \mathcal{F}_S \mid d \equiv 1 \mod 4\}$. Define $c_S := \prod_{q \in S} c_q$, $\delta_S := \gcd_{d \in \mathcal{F}_S} L(1, E_d)^{\mathrm{alg}}$, and $\delta'_S = \gcd_{d \in \mathcal{F}'_S} L(1, E_d)^{\mathrm{alg}}$. Then for any $S_0 \subseteq S$ such that $|S_0| \equiv |S| \mod 2$,

$$\mathrm{v}_p(c_{S-S_0}) - \mathrm{v}_p(\delta'_{S_0}) \leq \mathrm{v}_p(\delta_S) \leq \mathrm{v}_p(c_{S-S_0}) + \mathrm{v}_p(\delta'_{S_0}). \tag{1.1}$$

We remark that the two inequalities in the theorem correspond to two instances of integrality in the theory of the theta correspondence. The inequality on the left corresponds to the theta lift $\mathrm{PB}^\times \mapsto \widetilde{\mathrm{SL}}_2$, while that on the right (which is conjecturally an equality) comes from studying the lift in the opposite direction.

As explained in the text (see Corollaries 2.3, 2.4), the theorem has applications to a refined version of a question of Kolyvagin on p-indivisibility of $|\text{Ш}_d|$ (where Ш_d denotes the Tate–Shafarevich group of the twist E_d) via the exact version of BSD for elliptic curves of rank 0. It is expected that ongoing work of Skinner and Urban combined with previous results of Kato [3] will establish this exact version for primes p of good ordinary reduction in the near future. Also, the author certainly expects that the terms δ'_{S_0} in (1.1) can be replaced by δ_{S_0}. The only obstacle to doing so is that one needs to rework some of the computations in [9, 10] with less restrictive assumptions at the prime 2.

The curious reader may have noticed the condition $|S_0| \equiv |S| \mod 2$ in the theorem and wondered if it could be removed. The main obstacle here is that one does not yet have results analogous to those of [9, 10] for the dual pair $(\widetilde{\mathrm{SL}}_2, \mathrm{PB}^\times)$ with B

a *definite* quaternion algebra over $\mathbb{Q}$. This is in a sense surprising, since one would expect the definite case to be simpler than the indefinite case. The precise obstacle to working out the definite case is explained in Section 5.1. We isolate from this discussion a certain conjecture (Conjecture 5.1) about Petersson norms of half-integral weight forms. The conjecture is explained in Section 5.2, as is the following consequence.

Proposition *Suppose that Conjecture* 5.1 *is true, that E, p are as above, and p is not a congruence prime for E,* i.e., *p does not divide the degree of the modular parametrization $\varphi\colon X_0(N) \to E$. Then for each S, there exists $d \in \mathcal{F}_S$ such that $p \nmid L(1, E_d)^{alg}$.*

Finally, the last section also contains a rather speculative discussion of consequences of Conjecture 5.1 for congruences of modular forms of half-integral weight and their relation to congruences of modular forms of integral weight, as well as some heuristic justification for why the conjecture might be true.

2 A Refined Version of a Question of Kolyvagin

As in the introduction, let E be a semi-stable elliptic curve of odd (square-free) conductor N and let Ш_d denote the Tate–Shafarevich group of the quadratic twist E_d, where d is a fundamental quadratic discriminant. The following question was apparently first posed by Kolyvagin.

Question A: Let p be a prime number. Is there a quadratic discriminant d such that $p \nmid \text{Ш}_d$?

In this section, we pose a refined version of this question and reformulate it in terms of L-values. To motivate the refined version, we begin by considering the form of the BSD conjecture for E_d. This conjecture predicts that if $L(1, E_d) \neq 0$, then

$$\frac{L(1, E_d)}{\Omega_{E_d}} = \frac{\prod_{q \mid Nd} c_q(E_d) \cdot |\text{Ш}_d|}{|T_d|^2},$$

where Ω_{E_d} is a period of the Neron differential ω on E_d, $\Omega_{E_d} = \int_{E_d(\mathbb{R})} |\omega|$, $c_q(E_d) = E_d(\mathbb{Q}_q)/E_d^0(\mathbb{Q}_q)$, and T_d is the order of the torsion subgroup of E_d.

Let c_q denote the order of the component group of the Neron model of E over $\overline{\mathbb{F}}_q$. Also let $p > 3$ be a prime and say $a \sim b$ if a and b differ by a p-adic unit. Let w_q (resp. $w_q(E_d)$) be the sign of the Atkin–Lehner involution of E (resp. E_d) at q. We begin with the following easy lemma.

Lemma 2.1 *Let q be a prime dividing Nd. Then $c_q(E_d) \sim 1$ unless $\left(\frac{d}{q}\right) = -w_q$, in which case $c_q(E_d) \sim c_q$.*

Proof By [12, Corollary 15.2.1], $c_q(E_d) \sim 1$ unless E_d has split multiplicative reduction at q, *i.e.*, unless $q \nmid d$ and $w_q(E_d) = -1$. But if $q \nmid d$, it is easy to see that $w_q(E_d) = \left(\frac{d}{q}\right) w_q$. Thus $c_q(E_d) \sim 1$ unless $\left(\frac{d}{q}\right) = -w_q$. In this case $c_q(E_d) = -\mathrm{v}_q(j(E_d)) = -\mathrm{v}_q(j(E)) = c_q$. ■

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Now assume further that p is not an Eisenstein prime for E, *i.e.*, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ representation on the p-torsion $E[p]$ is irreducible. The same is true then for $E_d[p]$, and hence the torsion subgroup T_d has order prime to p. In this case, we see that

$$\frac{L(1,E_d)}{\Omega_{E_d}} \sim \Big(\prod_{\substack{q\mid N\\ (d/q)=-w_q}} c_q\Big)\cdot \text{Ш}_d. \tag{2.1}$$

Let Σ be the set of primes dividing N and S any subset of Σ. Let $\mathcal{F}_S$ be the collection of quadratic discriminants defined by

$$\mathcal{F}_S = \Big\{ d \;\Big|\; \Big(\frac{d}{q}\Big) = -w_q \text{ for } q \in S, \Big(\frac{d}{q}\Big) \neq -w_q \text{ for } q \in \Sigma \setminus S \Big\}$$

We can now pose the following refined version of Question A.

Question B(S): Is there $d \in \mathcal{F}_S$ such that $L(1,E_d) \neq 0$ and $p \nmid |\text{Ш}_d|$?

Let $\delta_S = \gcd_{d\in\mathcal{F}_S} \frac{L(1,E_d)}{\Omega_{E_d}}$ and $c_S = \prod_{q\in S} c_q$. Assuming BSD, we see from (2.1) that Question B(S) is equivalent to the following.

Question C(S): Is $\delta_S \sim c_S$?

The following is a slightly stronger version that we need for technical reasons.

Question C′(S): Is $\delta'_S \sim c_S$?

For the rest of this article we make the following assumption on p.

Assumption I: $p \nmid \tilde{N} = \prod_{q\mid N} q(q^2-1)$.

This assumption implies among other things that $p > 3$ and that p is not Eisenstein for E. (See [15] for the computation of the order of the cuspidal divisor class group for $X_0(N)$. It is known that Eisenstein primes for E must divide the order of this group.)

Theorem 2.2 *Suppose $S_0 \subseteq S$ and $|S| \equiv |S_0| \mod 2$. Then*

$$\mathrm{v}_p\Big(\frac{c_S}{c_{S_0}}\Big) - \mathrm{v}_p(\delta'_{S_0}) \le \mathrm{v}_p(\delta_S) \le \mathrm{v}_p\Big(\frac{c_S}{c_{S_0}}\Big) + \mathrm{v}_p(\delta'_{S_0})$$

or, equivalently, $-\mathrm{v}_p(\delta'_{S_0}) - \mathrm{v}_p(c_{S_0}) \le \mathrm{v}_p(\delta_S) - \mathrm{v}_p(c_S) \le \mathrm{v}_p(\delta'_{S_0}) - \mathrm{v}_p(c_{S_0})$.

Corollary 2.3 *If $C'(\varnothing)$ is true, then $C(S)$ is true for all S with $|S|$ even.*

Corollary 2.4 *Assume the exact form of BSD for curves of rank 0. If there exists $d \in \mathcal{F}'_{S_0}$ such that $L(1,E_d) \neq 0$ and $p \nmid \text{Ш}_d$, then for all sets S with $S_0 \subset S$ and $|S| \equiv |S_0| \mod 2$, there exists $d \in \mathcal{F}_S$ such that $L(1,E_d) \neq 0$ and $p \nmid \text{Ш}_d$.*

The proofs of the theorem and the corollaries will be given in Section 4. As remarked before, if p is a prime of good ordinary reduction for E, the exact form of the BSD up to p-adic units should follow from ongoing work of Skinner and Urban on the main conjecture for $L(1,E_d)$, combined with earlier work of Kato [3].

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3 The Theta Correspondence and Waldspurger Packets

The main idea behind the proofs is that the L-values $L(1, E_d), d \in \mathcal{F}_S$ are related to the Fourier coefficients of a holomorphic modular form of weight $3/2$. While such a modular form is not determined uniquely, the corresponding automorphic representation of $\widetilde{\mathrm{SL}_2}$ is determined by S. Let us denote the representation associated with S by $\tilde{\pi}_S$. A key observation is that, roughly speaking, as S varies, $\tilde{\pi}_S$ varies over all representations in a certain Waldspurger packet on $\widetilde{\mathrm{SL}_2}$. (This is not exactly correct: for the more precise relationship, see the last section of this article.) The automorphic representations in this packet may be constructed either as theta lifts from PGL_2 or as theta lifts from $\mathrm{PB}^\times$ for B a suitable quaternion algebra. The main theorem then results from a comparison of the integrality properties of these various theta correspondences as B varies. With this motivation, let us review more precisely some results on the Waldspurger packet and Fourier coefficients of modular forms of half-integral weight.

3.1 Local Theta Correspondence

In this section, let B be a non-split quaternion algebra over $\mathbb{Q}_v$. Let π (resp. π') denote an infinite dimensional irreducible admissible unitary representation of (the Hecke algebra of) $\mathrm{PGL}_2(\mathbb{Q}_v)$ (resp. finite dimensional irreducible representation of $\mathrm{PB}^\times(\mathbb{Q}_v)$), and ψ a nontrivial additive character of $\mathbb{Q}_v$. Attached to (π, ψ) (resp. (π', ψ)) is an irreducible representation $\theta(\pi, \psi)$ (resp. $\theta(\pi', \psi)$) of (the Hecke algebra of) $\widetilde{\mathrm{SL}_2}(\mathbb{Q}_v)$. A fundamental observation of Waldspurger is that for a fixed choice of additive character, these local theta lifts do not commute with the Jacquet–Langlands correspondence ($\mathrm{PGL}_2 \leftrightarrow \mathrm{PB}^\times$). For any $\nu \in \mathbb{Q}_v^\times$, let ψ^ν be the additive character defined by $\psi^\nu(x) = \psi(\nu x)$ and χ_ν the quadratic character associated with the quadratic extension $\mathbb{Q}_v(\sqrt{\nu})/\mathbb{Q}_v$.

Theorem 3.1 **(Waldspurger [19, Proposition 17, Theorem 2])** (i) *Suppose π is a principal series representation. Then $\theta(\pi \otimes \chi_\nu, \psi^\nu) = \theta(\pi, \psi)$.*

(ii) *Suppose that π is either special or supercuspidal if $v < \infty$ or that π is a discrete series if $v = \infty$. Let $\pi' := JL(\pi)$. There exists a nontrivial partition $\mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2 = \mathbb{Q}_v(\pi) \sqcup \mathbb{Q}_v(\pi')$ such that*

$$\begin{aligned}
\nu \in \mathbb{Q}_v(\pi) &\iff \theta(\pi \otimes \chi_\nu, \psi^\nu) = \theta(\pi, \psi) \iff \theta(\pi' \otimes \chi_\nu, \psi^\nu) = \theta(\pi', \psi)\\
&\iff \varepsilon(\pi \otimes \chi_\nu, 1/2) = \chi_\nu(-1)\varepsilon(\pi, 1/2),\\
\nu \in \mathbb{Q}_v(\pi') &\iff \theta(\pi \otimes \chi_\nu, \psi^\nu) = \theta(\pi', \psi) \iff \theta(\pi' \otimes \chi_\nu, \psi^\nu) = \theta(\pi, \psi)\\
&\iff \varepsilon(\pi \otimes \chi_\nu, 1/2) = -\chi_\nu(-1)\varepsilon(\pi, 1/2).
\end{aligned}$$

In particular, $\theta(\pi, \psi) \neq \theta(\pi', \psi)$.

The local Waldspurger packet attached to (π, ψ) is the set of isomorphism classes of $\theta(\pi \otimes \chi_\nu, \psi^\nu)$ as ν varies over $\mathbb{Q}_v^\times$. We see from Theorem 3.1 that this packet has cardinality either 1 or 2 according as we are in case (i) or (ii) of the theorem.

3.2 Global Theta Correspondence

Now suppose π is an automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ and π' an automorphic representation of $\mathrm{PB}_{\mathbb{A}}^{\times}$ for B a quaternion algebra (split or not) over $\mathbb{Q}$. Let ψ be a nontrivial character of $\mathbb{Q} \setminus \mathbb{A}_{\mathbb{Q}}$, and for $\nu \in \mathbb{Q}^{\times}$, let ψ^{ν} be given by $\psi^{\nu}(x) = \psi(\nu x)$. Let $V := \{x \in B, \mathrm{tr}(x) = 0\}$, where tr is the reduced trace and $\mathcal{S}(V(\mathbb{A}))$ is the Schwartz space of $V(\mathbb{A})$.

Let $\widetilde{\mathrm{S}}_{\mathbb{A}}$ be the metaplectic twofold cover of $\mathrm{SL}_2(\mathbb{A})$, $\mathrm{S}_{\mathbb{Q}}$ the natural image of $\mathrm{SL}_2(\mathbb{Q})$ in $\widetilde{\mathrm{S}}_{\mathbb{A}}$ and let $\tilde{\mathcal{A}}_{00}$ denote the space of genuine cuspidal automorphic forms on $\mathrm{SL}_2(\mathbb{Q}) \setminus \widetilde{\mathrm{S}}_{\mathbb{A}}$ that are orthogonal to the span of one variable theta series. For $f \in \pi', \varphi \in \mathcal{S}(V(\mathbb{A})), \sigma \in \widetilde{\mathrm{S}}_{\mathbb{A}}$ one defines, with a suitable choice of Haar measure,

$$t(\psi, \varphi, \sigma, f) = \int_{PB_{\mathbb{Q}}^{\times} \setminus PB_{\mathbb{A}}^{\times}} \sum_{x \in V} r_{\psi}(\sigma, g)\varphi(x) f(g)\, dg,$$

where r_{ψ} is the Weil representation.

Theorem 3.2 **(Waldspurger, [19, Proposition 20, Proposition 22])** *Let $\Theta(\pi', \psi)$ be the span of $t(\psi, \varphi, \cdot, f)$ as φ, f vary over elements of $\mathcal{S}(V(\mathbb{A})), \pi'$, respectively. Then $\Theta(\pi', \psi)$ is an irreducible automorphic representation of $\widetilde{\mathrm{S}}_{\mathbb{A}}$ contained in $\tilde{\mathcal{A}}_{00}$. It is non-zero precisely when $L(\frac{1}{2}, \pi') \neq 0$ and in that case, factorizes as*

$$\Theta(\pi', \psi) = \bigotimes_{\mathrm{v}} \theta(\pi'_{\mathrm{v}}, \psi_{\mathrm{v}}),$$

where $\theta(\pi'_{\mathrm{v}}, \psi_{\mathrm{v}})$ denote the local theta lifts from the previous section.

The global Waldspurger packet associated with (π, ψ) is defined to be the set of isomorphism classes of $\Theta(\pi \otimes \chi_{\nu}, \psi^{\nu})$ as ν varies over $\mathbb{Q}^{\times}$. If t is the number of primes at which π_{v} is a discrete series, the packet has size 2^{t-1}. (The local packet at each such prime has size 2, but globally, to get an automorphic representation one needs an additional condition on the product of the central characters, which cuts the number of possibilities by a factor of 2.) If B is a quaternion algebra such that π admits a Jacquet–Langlands transfer π' to B, then $\Theta(\pi', \psi)$ lies in the packet associated with π. However, $\Theta(\pi', \psi) \neq \Theta(\pi, \psi)$ if $B \neq M_2(\mathbb{Q})$. In fact, keeping the additive character ψ fixed and varying B, one gets all representations in the packet, this assignment giving a bijection between quaternion algebras that admit a Jacquet–Langlands-transfer of π and automorphic representations in the packet.

There is also a theta lift in the other direction. Namely, for an automorphic representation $\tilde{\pi}$ on $\tilde{S}_{\mathbb{A}}$ contained in $\tilde{\mathcal{A}}_{00}$, and for $h \in \tilde{\pi}$, one may define

$$T(\psi, \varphi, g, h) = \int_{\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})} \sum_{x \in V} r_{\psi}(\sigma, g)\varphi(x) h(\sigma)\, d\sigma.$$

The vanishing or nonvanishing of this lift is more subtle. One must first deal with the case of $B = M_2(\mathbb{Q})$.

Theorem 3.3 **(Waldspurger, [17])** *Let $B = M_2(\mathbb{Q})$ and let $\Theta(\tilde{\pi}, \psi)$ denote the span of $T(\psi, \varphi, g, h)$ as φ, h vary over $\mathcal{S}(V(\mathbb{A}))$, $\tilde{\pi}$, respectively.*

(i) *$\Theta(\tilde{\pi},\psi)$ is an irreducible automorphic representation of* $\mathrm{PB}_{\mathbb{A}}^{\times}=\mathrm{PGL}_2(\mathbb{A})$.
(ii) *There always exists $\nu\in\mathbb{Q}^{\times}$ such that $\Theta(\tilde{\pi},\psi^{\nu})$ is non-zero, and for any such ν, the representation $\Theta(\tilde{\pi},\psi^{\nu})\otimes\chi_{\nu}$ is independent of the choice of ν.*

One sets $S_{\psi}(\tilde{\pi})=\Theta(\tilde{\pi},\psi^{\nu})\otimes\chi_{\nu}$ for any choice of ν such that $\Theta(\tilde{\pi},\psi^{\nu})\neq 0$. It turns out that the Waldspurger packet associated with a representation π on PGL_2 (and a character ψ) may also be described as the pre-image of π under the assignment $\tilde{\pi}\rightsquigarrow S_{\psi}(\tilde{\pi})$.

With this preparation, one can state the condition on nonvanishing for arbitrary B.

Theorem 3.4 **(Waldspurger, [19, Proposition 21, p. 292])** *Let B be a quaternion algebra over $\mathbb{Q}$, and let $\Theta_B(\tilde{\pi},\psi)$ denote the span of $T(\psi,\varphi,g,h)$ as φ,h vary over $\mathcal{S}(V(\mathbb{A}))$, $\tilde{\pi}$, respectively. Then $\Theta_B(\tilde{\pi},\psi)$ is an irreducible automorphic representation of $\mathrm{PB}_{\mathbb{A}}^{\times}$. It is nonzero if and only if the following two conditions are satisfied:*
(i) *$L(\frac{1}{2},\pi)\neq 0$ for $\pi:=S_{\psi}(\tilde{\pi})$.*
(ii) *For all places v, $\tilde{\pi}_{\mathrm{v}}$ admits a ψ_{v}-Whittaker model exactly when B is split at* v.

3.3 The Shimura Correspondence

Henceforth we restrict ourselves to holomorphic forms and explain the translation from the classical situation. Let k be a positive integer, M a positive integer divisible by 4, and $\boldsymbol{\chi}$ a character of conductor dividing M. Denote by $S'_{k+(1/2)}(M,\boldsymbol{\chi})$ the space of modular forms of weight $k+\frac{1}{2}$ on $\Gamma_0(M)$ with central character $\boldsymbol{\chi}$ and by $S_{k+(1/2)}(M,\boldsymbol{\chi})$ the subspace consisting of forms that are orthogonal to the span of all one-variable theta series. Let $f_0\in S_{2k}^{\mathrm{new}}(\boldsymbol{\chi}^2)$ be a newform of weight $2k$ and character $\boldsymbol{\chi}^2$ and level $N(f_0)$. Set

$$S_{k+\frac{1}{2}}(M,\boldsymbol{\chi},f_0)=\{h\in S_{k+\frac{1}{2}}(M,\boldsymbol{\chi})\mid T_{q^2}h=a_q(f_0)h\ \forall q\nmid M\},$$

where $a_q(f_0)$ denotes the q-th Hecke eigenvalue of f_0. In his famous article [13], Shimura showed that if $h\in S_{k+(1/2)}(M,\boldsymbol{\chi})$ is an eigenvector of almost all Hecke operators T_{q^2}, then $h\in S_{k+(1/2)}(M,\boldsymbol{\chi},f_0)$ for some (uniquely determined) newform f_0. The assignment $h\rightsquigarrow f_0$ is called the Shimura correspondence.

Let $\chi_0=\boldsymbol{\chi}\cdot\chi_{-1}^{k}$ and denote by $\tilde{A}'_{k+(1/2)}(M,\chi_0)$ the space of automorphic forms on $\mathrm{S}_{\mathbb{Q}}\backslash\widetilde{\mathrm{S}}_{\mathbb{A}}$ with character χ_0 and level M [18, p. 386]. There is a natural bijection $S'_{k+(1/2)}(M,\boldsymbol{\chi})\simeq\tilde{A}'_{k+(1/2)}(M,\chi_0)$ via which we may think of h as an automorphic form on $\mathrm{S}_{\mathbb{Q}}\backslash\widetilde{\mathrm{S}}_{\mathbb{A}}$. Henceforth, let ψ denote the usual additive character on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$.

Proposition 3.5 **(Waldspurger, [18, Proposition 4])** *Suppose that $h\rightsquigarrow f_0$ under the Shimura correspondence. Let $\tilde{\pi}$ denote the automorphic representation of $\mathrm{S}_{\mathbb{Q}}\backslash\widetilde{\mathrm{S}}_{\mathbb{A}}$ associated with h and π_0 the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated with f_0. Then $S_{\psi}(\tilde{\pi})=\pi_0\otimes\chi_0^{-1}$.*

4 Proofs of the Main Results

For simplicity, let us assume that N is odd. Let $S_0\subseteq\Sigma$ and let $\nu\in\mathcal{F}'_{S_0}$ be such that $\mathrm{v}_p(\frac{L(1,E_\nu)}{\Omega_{E_\nu}})$ is minimal, *i.e.*, such that $\frac{L(1,E_\nu)}{\Omega_{E_\nu}}\sim\delta'_{S_0}$. Let $S\subseteq\Sigma$ be another subset containing S_0 such that $|S| \equiv |S_0| \mod 2$, and let $\tau = 0$ or 1 be such that $(|S_0| + 1) \mod 2 = \tau$. Also denote by π the automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ corresponding to E.

Lemma 4.1 (i) $\varepsilon(\pi \otimes \chi_\nu, \frac{1}{2}) = (-1)^\tau \operatorname{sign}(\nu)$.
(ii) $\operatorname{sign}(\nu) = (-1)^\tau$.

Proof For part (i) see the proof of [9, Proposition 3.3]. As for part (ii), since we have assumed that $L(1, E_\nu)^{\mathrm{alg}}$ has minimal p-adic valuation among $\nu \in \mathcal{F}'_{S_0}$, certainly $L(1, E_\nu) \neq 0$. Hence $\varepsilon(\pi \otimes \chi_\nu, \frac{1}{2}) \neq 0$ and $\operatorname{sign}(\nu) = (-1)^\tau$. ■

Let $\boldsymbol{\chi}$ be a character of conductor dividing $4N$ with $\boldsymbol{\chi}(-1) = 1$. Set

$$\chi = \boldsymbol{\chi} \cdot \left(\frac{-1}{\cdot}\right)^{1+\tau} \quad \text{and} \quad \chi_0 = \boldsymbol{\chi} \cdot \left(\frac{-1}{\cdot}\right).$$

Let us assume that $\boldsymbol{\chi}$ has been chosen to satisfy the following conditions:

- $\boldsymbol{\chi}$ is unramified at the primes in $\Sigma \setminus S$.
- $\chi_q(-1) = \chi_{0,q}(-1) = -1$ for $q \in S$. In particular, χ is ramified at all such primes.

Since $\boldsymbol{\chi}(-1) = 1$, we must have $\boldsymbol{\chi}_2(-1) = (-1)^{|S|} = (-1)^{1+\tau}$. Hence $\chi_2(-1) = 1$ and χ is unramified at 2.

Let B be the indefinite quaternion algebra ramified exactly at the primes in $S \setminus S_0$ and V the subspace of B consisting of elements with reduced trace 0. Denote by π' the automorphic representations of $\mathrm{PB}_{\mathbb{A}}^\times$ which is the Jacquet–Langlands transfer of π. Let f (resp. f_χ, resp. g, resp. g_χ) be newforms in π, (resp. $\pi \otimes \chi$, resp. π', resp. $\pi' \otimes \chi$), normalized to be p-adic units with respect to the integral structure provided by the integral models of the relevant Shimura curves. Let $s = g_\chi \otimes ((\chi^{-1}\chi_\nu) \circ \mathrm{Nm}) \in \pi' \otimes \chi_\nu$ and consider the theta lift $h' := \theta(s, \varphi, \psi^{1/|\nu|})$ to $\widetilde{\mathrm{SL}_2}$ where $\varphi \in \mathcal{S}_\psi(V(\mathbb{A}))$. Then for a suitable choice of φ one has the following five results, proved in [9, 10]:

(1) $h' \in S_{3/2}(\Gamma_0(M), \chi))$ where $M = \operatorname{lcm}(4, NN_\chi)$. (See [9, Proposition 3.4].)

(2) Let ξ_0 be a fundamental discriminant, $\xi = (-1)^\tau \xi_0$. Then $a_{|\xi_0|}(h') = 0$ unless $\operatorname{sign}(\xi_0) = \operatorname{sign}(\nu)$, $\xi_0 \equiv 0$ or $1 \mod 4$ and $\xi_0 \in \mathcal{F}_S$. In that case (see [10, Main Theorem])

$$|a_\xi(h')|^2 \sim \pi^{-2}\sqrt{|\nu\xi|} \cdot L(\frac{1}{2}, \pi \otimes \chi_\nu) \cdot L(\frac{1}{2}, \pi \otimes \chi_{\xi_0}) \cdot \frac{\langle g, g\rangle}{\langle f, f\rangle}. \tag{4.1}$$

Note that [10, Main Theorem] involves the ratio $\langle g_\chi, g_\chi\rangle / \langle f_\chi, f_\chi\rangle$, but one can check that this differs from $\langle g, g\rangle / \langle f, f\rangle$ by a p-adic unit because of our assumption that $p \nmid \tilde{N}$. (Indeed, by a mild generalization of the results of [8], one has, for $p \nmid \tilde{N}$,

$$\frac{\langle f_\chi, f_\chi\rangle}{\langle g_\chi, g_\chi\rangle} \sim \gcd_{K,\eta'} L(2, f_{\chi,K} \otimes \eta')^{\mathrm{alg}} \sim \gcd_{K,\eta} L(2, f_K \otimes \eta)^{\mathrm{alg}} \sim \frac{\langle f, f\rangle}{\langle g, g\rangle},$$

where K runs over imaginary quadratic fields that are *Heegner* for f, and η (resp. η') runs over Hecke characters of K of weight 2 that are unramified (resp. of the form $\eta \cdot \chi^{-1} \circ \mathrm{Nm}_{K/\mathbb{Q}}$ for η unramified).

(3) The relation (4.1) implies that $h' \in S^+_{3/2}(\Gamma_0(M), \chi)$, the *Kohnen subspace* (see [9, Proposition 2.3]). This subspace is one-dimensional and generated by a modular form h whose Fourier coefficients are algebraic integers. By Waldspurger's theorem, $a_\xi(h)^2$ (and not just its absolute value) is proportional to $\sqrt{|\xi|}L(\frac{1}{2}, \pi \otimes \chi_{\xi_0})$. Since the algebraic parts of these L-values $\in \mathbb{Q}$, we can pick h such that at least one of its coefficients is a p-unit, *i.e.,* a unit at all primes above p. With such a choice of h, we have $h' \sim \alpha \cdot u_\delta(g) \cdot h$, where α is p-integral and u_δ (with $\delta = \pm$) is a suitable fundamental period associated with g, well defined up to a p-adic unit. In our setting, $\delta = \pm$ according as $|S|$ is even or odd. (See [9, Corollary 4.7].)

(4) Let $g' = \theta^t(h, \varphi, \psi^{1/|\nu|})$ be the theta lift of h to $\mathrm{PB}^\times_{\mathbb{A}}$. Then $g' = \beta s$ for a p-adic integer β. (See [9, Corollary 5.3].)

(5) We have the following fundamental formula: (see [9, Theorem 6.1])

$$\pi^{-1}|\nu|^{1/2}L\Big(\frac{1}{2}, \pi \otimes \chi_\nu\Big) \sim \alpha\beta u_\delta(g) \tag{4.2}$$

Note now that by Faltings' isogeny theorem and our assumption that p is not Eisenstein, $u_\delta(g) \sim u_\delta(f) \sim \pi^{-1}|d|^{1/2}\Omega_{E_d}$. Furthermore, the discussion in [8, §2.2.1] (see also [1] for a discussion of the Manin constant), along with the main results of [11, 16], show that

$$\frac{\langle f, f\rangle}{\langle g, g\rangle} \sim \frac{c_S}{c_{S_0}}.$$

Theorem 2.2 can now be deduced easily using the facts mentioned above. First, using that $\overline{u_\delta(g)} \sim u_\delta(g) \sim u_\delta(f)$, we see from (4.1) that

$$\alpha\bar{\alpha}|a_\xi(h)|^2\frac{\langle f, f\rangle}{\langle g, g\rangle} \sim \frac{\sqrt{|\nu|}L(\frac{1}{2}, \pi \otimes \chi_\nu)}{\pi u_\delta(f)} \cdot \frac{\sqrt{|\xi_0|}L(\frac{1}{2}, \pi \otimes \chi_{\xi_0})}{\pi u_\delta(f)} \tag{4.3}$$

$$\Rightarrow \mathrm{v}_p\Big(\frac{\langle f, f\rangle}{\langle g, g\rangle}\Big) \le \mathrm{v}_p\Big(\frac{\sqrt{|\nu|}L(\frac{1}{2}, \pi \otimes \chi_\nu)}{\pi u_\delta(f)}\Big) + \mathrm{v}_p\Big(\frac{\sqrt{|\xi_0|}L(\frac{1}{2}, \pi \otimes \chi_{\xi_0})}{\pi u_\delta(f)}\Big) \quad \forall \xi_0 \in \mathcal{F}_S \tag{4.4}$$

$$\Rightarrow \mathrm{v}_p\Big(\frac{\langle f, f\rangle}{\langle g, g\rangle}\Big) \le \mathrm{v}_p(\delta'_{S_0}) + \mathrm{v}_p(\delta_S) \tag{4.5}$$

$$\Rightarrow \mathrm{v}_p\Big(\frac{c_S}{c_{S_0}}\Big) \le \mathrm{v}_p(\delta'_{S_0}) + \mathrm{v}_p(\delta_S). \tag{4.6}$$

Now, picking ξ such that $a_\xi(h)$ is a p-unit, we see from (4.3) that

$$\mathrm{v}_p(\delta_S) = \mathrm{v}_p\Big(\frac{\sqrt{|\xi_0|}L(\frac{1}{2}, \pi \otimes \chi_{\xi_0})}{\pi u_\delta(f)}\Big)$$

for such a ξ, and hence

$$\mathrm{v}_p(\alpha\bar{\alpha}) = \mathrm{v}_p(\delta'_{S_0}) + \mathrm{v}_p(\delta_S) - \mathrm{v}_p\Big(\frac{c_S}{c_{S_0}}\Big).$$

Next, from (4.2), we find that

$$\Big(\frac{|\nu|^{1/2}L(\frac{1}{2},\pi\otimes\chi_\nu)}{\pi u_\delta(f)}\Big)^2 \sim \alpha\bar{\alpha}\beta\bar{\beta}.$$

Hence, using that β is a p-integer, we have

$$2\mathrm{v}_p(\delta'_{S_0}) = \mathrm{v}_p(\alpha\bar{\alpha}) + \mathrm{v}_p(\beta\bar{\beta}) \geq \mathrm{v}_p(\alpha\bar{\alpha}),$$

whence from (4.7)

$$\mathrm{v}_p\left(\frac{c_S}{c_{S_0}}\right) \geq \mathrm{v}_p(\delta_S) - \mathrm{v}_p(\delta'_{S_0}). \tag{4.9}$$

Finally, combining (4.6) and (4.9) we get

$$\mathrm{v}_p\left(\frac{c_S}{c_{S_0}}\right) - \mathrm{v}_p(\delta'_{S_0}) \leq \mathrm{v}_p(\delta_S) \leq \mathrm{v}_p\left(\frac{c_S}{c_{S_0}}\right) + \mathrm{v}_p(\delta'_{S_0}),$$

which is the statement of Theorem 2.2. Note that the inequality on the left uses the integrality of α while that on the right uses the integrality of β, the latter being substantially harder than the former.

Proofs of Corollaries 2.3 and 2.4 Apply the theorem with $S_0 = \Phi$. Then $c_{S_0} = 1$. Furthermore, if $C'(\varnothing)$ is true, then $\delta'_{S_0} \sim 1$. Thus $\mathrm{v}_p(\delta_S) = \mathrm{v}_p(c_S)$, so that $C(S)$ is true. This proves Corollary 2.3.

Now, if there exists $d \in \mathcal{F}'_{S_0}$ such that $L(1, E_d) \neq 0$ and $p \nmid Ш_d$, then $\delta'_{S_0} \sim c_{S_0}$. By the theorem, $\mathrm{v}_p(\delta_S) \leq \mathrm{v}_p(c_S/c_{S_0}) + \mathrm{v}_p(\delta'_{S_0}) = \mathrm{v}_p(c_S)$. On the other hand, assuming BSD, $\mathrm{v}_p(\delta_S) \geq \mathrm{v}_p(c_S)$ so that $\mathrm{v}_p(\delta_S) = \mathrm{v}_p(c_S)$. Thus $C(S)$ is true and there exists $d \in \mathcal{F}_S$ with $L(1, E_d) \neq 0$ and $L(1, E_d)^{\mathrm{alg}} \sim c_S$. Assuming the exact form of BSD for E_d, we get that $|Ш_d| \sim 1$, which proves Corollary 2.4. ■

5 A Conjecture on Congruences of Half-Integral Weight Forms and Some Consequences

5.1 Motivation: Theta Lifts to Definite Quaternion Algebras

Let B be a definite quaternion algebra over $\mathbb{Q}$ such that f admits a Jacquet–Langlands transfer to B. Thus we may write $N = N^+N^-$, where $N^- = \mathrm{disc}(B)$ is a product of an odd number of primes, and we set $g = \mathrm{JL}(f)$. Since f has weight 2, we may think of g as a function on $B^\times \setminus B^\times_{\mathbb{A}}/UB^\times_\infty$ for a suitable open compact $U \subset B^\times_{\mathbb{A}_f}$. Further, we may normalize g so that all its values lie in $\mathbb{Z}$ and at least one of its values is a p-unit.

Since this section is mainly of a motivational nature, we shall be slightly less precise in what follows. Let us fix some character ψ_0 and a Schwartz function φ and consider the theta lift $H' := \theta(g)$ to $\tilde{S}_{\mathbb{A}}$. We may pick ψ_0 and φ such that H' is a holomorphic eigenform of weight 3/2. Suppose that $H' = \alpha H$ for some p-adically normalized form H. Now consider the theta lift in the reverse direction. By suitably adjusting φ, one can arrange that $g' := \theta^t(H) = \beta' g$ for some scalar β'. Let us assume that $\beta' \in \mathbb{R}$, to simplify the discussion. By see-saw duality $\langle \alpha H, H\rangle = \langle g, \beta' g\rangle$. Now the Fourier coefficients of H' are linear combinations with (algebraic) integer coefficients of integrals of g on certain tori in $B^\times$, hence they can be shown to be $\mathbb{Q}$-rational and even p-integral. Thus $\alpha \in \mathbb{Q}$ and is a p-integer. Since $\langle g, g\rangle \in \mathbb{Q}$, we see that $\langle H, H\rangle/\beta' \in \mathbb{Q}$. Since the transcendental part of $\langle H, H\rangle$ is $\pi^{-1}u_-(f)$ by a theorem of Shimura [14], we find that $\beta'/\pi^{-1}u_-(f)$ is algebraic. Let us then write $\beta' = \beta\pi^{-1}u_-(f)$. A general principle in such a situation is that one expects β to be a p-integer. Indeed, this is more or less the statement needed to prove the analogs of the results in Sections 2 and 4 if $|S_0| \not\equiv |S| \mod 2$. Now one would hope to be able to show that β is a p-integer using the following integrality criterion, which follows quite easily from the equidistribution theorem [6, Theorem 10].

Integrality criterion: Let $K^\times/\mathbb{Q}^\times \hookrightarrow PB^\times$ be a torus corresponding to a *Heegner* embedding of an imaginary quadratic field K and η' a finite order character of $K_{\mathbb{A}}^\times$. For any multiple $\tilde{g}$ of g, define

$$L_{\eta'}(\tilde{g}) = \int_{K^\times\backslash K_{\mathbb{A}}^\times} \tilde{g}(x)\cdot\eta'(x)d^\times x$$

for a Haar measure on $d^\times x$ on $K_{\mathbb{A}}^\times$ normalized such that $\mathrm{vol}(U_v) = 1$ for finite v and $\mathrm{vol}((\mathbb{R}^\times)^+ \backslash K_\infty^\times) = 1$. Then $\tilde{g}$ is p-integral if and only if for sufficiently many K with $p \nmid h_K$, $L_{\eta'}(\tilde{g})$ is a p-integer for all η that are unramified.

To compute $L_{\eta'}(g')$ in our case, let η be a finite order character of $K_{\mathbb{A}}^\times$ such that $\eta(\eta^\rho)^{-1} = \eta'$ and such that $\eta'|_{\mathbb{Q}_{\mathbb{A}}^\times} = 1$. Also let $\mu = \eta|_{\mathbb{Q}_{\mathbb{A}}^\times}$. Then by an application of see-saw duality, one gets roughly $L_{\eta'}(g') = \langle H, \theta(\eta)\theta(\mu)\rangle$, where

$$\theta_\eta = \sum_{\mathfrak{a}} \eta(\mathfrak{a})e^{2\pi i N\mathfrak{a} z} \quad \text{and} \quad \theta_\mu = \sum_n \mu(n)e^{2\pi i n^2 z}$$

are modular forms of weight 1 and $1/2$, respectively. Thus, showing that β is integral is equivalent to showing that

$$\pi\frac{\langle H, \theta(\eta)\theta(\mu)\rangle}{u_-(f)}$$

is a p-integer. If one is optimistic, one would even hope that the expression

$$\pi\frac{\langle H, G\rangle}{u_-(f)}$$

is p-integral as long as G is p-integral and one has suitable control over the level of G. This motivates the conjecture stated in the next section.

5.2 A Conjecture on Petersson Inner Products of Half-Integral Weight Forms and Congruences of Modular Forms

Conjecture 5.1 Suppose $h \in S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_0)$ is a newform and $h' \in S_{k+\frac{1}{2}}(M, \boldsymbol{\chi})$. Let p be a prime such that $p > 2k$, $p \nmid M$, and p is not Eisenstein for f_0. Assume that h, h' are $\overline{\mathbb{Q}}$-rational and p-integral (*i.e.,* integral at all primes above p), and further that h is a p-unit. Then the ratio

$$\pi i \mathfrak{g}(\chi) \frac{\langle h, h' \rangle}{u_-(f_0)}$$

is a p-integer.

Remarks 5.2 (1) $\mathfrak{g}(\chi)$ is the Gauss sum attached to χ. It does not play a role in p-integrality issues since we have assumed that $p \nmid N$ and $c_\chi \mid N$. There are various definitions of newforms in the literature for modular forms of half-integral weight. We have in mind Kohnen's definition [4].

(2) By a theorem of Shimura, the above ratio is algebraic and lies in the field generated by the Fourier coefficients of h, h'. Unfortunately, the author does not know how to translate Shimura's proof of rationality to the p-adic setting.

(3) A standard principle shows that Conjecture 5.1 is equivalent to saying that the ratio $\delta_h := \mathfrak{g}(\chi)\pi\langle h, h\rangle / u_-(f_0)$ is a p-integer and is divisible by all congruences satisfied by the form h. It is known that the Petersson inner product for integral weight forms has the following property: $\pi\langle f, f\rangle = \delta_f u_+(f) u_-(f)$, where δ_f is a p-integer that "counts" congruences satisfied by f. The conjecture above thus suggests that congruences of half-integral weight forms may be measured similarly by the ratio of the Petersson inner product to a canonical period.

(4) As explained in the previous section, the truth of Conjecture 5.1 should imply the main results of this article even in the case $|S_0| \not\equiv |S| \mod 2$.

Let us now explain the relation of Conjecture 5.1 to the problem of p-indivisibility of quadratic twists of E. Recall that f is the GL_2 form corresponding to the elliptic curve E.

Proposition 5.3 *Suppose that Conjecture* 5.1 *holds and that $p \nmid \delta_f$. Then for each S, there exists $d \in \mathcal{F}_S$ such that $L(1, E_d)^{\mathrm{alg}}$ is a p-unit.*

Proof By [2, Theorem 10.1], one has

$$\frac{|a_\xi(h)|^2}{\langle h, h\rangle} \sim \frac{|\xi|^{1/2} L(\frac{1}{2}, f \otimes \chi_{\xi_0})}{\pi\langle f, f\rangle}$$

for $\xi_0 \in \mathcal{F}_S$. Thus

$$|a_\xi(h)|^2 \frac{\pi\langle f, f\rangle}{u_+(f)u_-(f)} \sim \frac{|\xi|^{1/2} L(\frac{1}{2}, f \otimes \chi_{\xi_0})}{\pi u_\delta(f)} \cdot \frac{\pi\langle h, h\rangle}{u_{-\delta}(f)}.$$

In the notation of Proposition 3.5,

$$S_\psi(\tilde{\pi}) = \pi_0 \otimes \chi_0^{-1} = (\pi \otimes \chi) \otimes \chi_0^{-1} = \pi \otimes \left(\frac{-1}{\cdot}\right)^{\tau}.$$

Recall that f_0 is the newform in $\pi \otimes \chi$, and $\chi_\infty(-1) = (-1)^{1+\tau}$, $u_-(f_0) \sim u_{-\delta}(f)$. If we now pick ξ such that $a_\xi(h)$ is a unit at all primes above p, we get $\delta_f \sim \delta_S \cdot \delta_h$, *i.e.*, $\mathrm{v}_p(\delta_S) + \mathrm{v}_p(\delta_h) = \mathrm{v}_p(\delta_f)$. Now the truth of the conjecture (for the special case $h' = h$) would imply that $\mathrm{v}_p(\delta_h) \geq 0$. If also $p \nmid \delta_f$, then $\mathrm{v}_p(\delta_S) = 0$, as required. ■

The proof above tells us more, namely it gives us the equality

$$\mathrm{v}_p(\delta_S) + \mathrm{v}_p(\delta_h) = \mathrm{v}_p(\delta_f),$$

which, of course, holds independently of the conjecture.

5.3 Arithmetic of the Waldspurger Packet: Some Speculations

The author asks for the reader's indulgence in the speculation that follows. Recall that question $C(S)$ asks if $\delta_S \sim c_S := \prod_{q \in S} c_q$. If this were indeed true, we would have

$$\delta_h \sim \frac{\delta_f}{\prod_{q \in S} c_q}.$$

Here is how one might make sense of this last equation: δ_f counts all congruences satisfied by the form f. The term c_q is known to count *level-lowering* congruences at q, *i.e.*, congruences between f and forms of level dividing N that are old at q. Thus $\delta_f / \prod_{q \in S} c_q$ counts all congruences satisfied by f except that it leaves out level-lowering congruences at S. Now, if δ_h were to count congruences satisfied by h, as the conjecture suggests, we would have the following consequence: the form h basically enjoys the same congruences satisfied by its Shimura lift f except perhaps for certain level-lowering congruences, and which level-lowering congruences are left out depends exactly on the automorphic representation in the Waldspurger packet that h belongs to.

Note that if t is the number of primes dividing N, there are 2^t choices of subsets S and roughly speaking, these correspond exactly to the 2^t representations in the Waldspurger packet associated with f. More precisely, since $S_\psi(\tilde{\pi}) = \pi \otimes \left(\frac{-1}{\cdot}\right)^{\tau}$, the automorphic representation associated to h lies in the packet $W_{f,\tau}$ associated with $f \otimes \left(\frac{-1}{\cdot}\right)^{\tau}$. As $\tau = 0$ or 1, by varying S, one gets half the representations in $W_{f,0}$ and half the representations of $W_{f,1}$. This is to be expected since the other half in each case consists of representations that are antiholomorphic discrete series at infinity, while we have chosen to work with holomorphic forms h. However if $h \in \tilde{\pi} \in W_{f,0}$, we have $\bar{h} \in \bar{\tilde{\pi}} \in W_{f,1}$ and conversely. Thus if we allow ourselves both holomorphic and antiholomorphic forms h, we can always restrict ourselves to a fixed packet.

We end by giving some heuristic evidence for the picture described above. Clearly if $h \equiv h_0 \mod p$ for some other eigenform h_0, their Shimura lifts must be congruent modulo p. Thus the size of the congruence module for h should be bounded above by the size of the congruence module for f. Conversely, suppose $f \equiv F \mod p$ for some other newform F which is a Shimura lift of a half-integral form H. In that case, we have a congruence of L-values $L(\frac{1}{2}, \pi_f \otimes \chi_{\xi_0})^{\mathrm{alg}} \equiv L(\frac{1}{2}, \pi_F \otimes \chi_{\xi_0})^{\mathrm{alg}}$. Now the Fourier coefficients of h and H are basically square-roots of these L-values, and one may imagine that there is a certain coherence in taking the square-roots that preserves congruences, so that congruences between h and other newforms of level N are always preserved on the half-integral side. On the other hand, if F is an oldform, say coming from a newform of level N/q, the corresponding half-integral weight form will have more nonvanishing Fourier coefficients in general, since there is no condition now on the quadratic symbol $\left(\frac{\xi_0}{q}\right)$. If one wants the congruence to be preserved, these extra coefficients should be congruent to $0 \mod p$. One can check in simple cases from the description of the Fourier coefficients in [18] (and using the fact that F admits a *level raising* congruence at q) that this can only happen if the extra coefficients a_ξ are such that ξ_0 satisfies a certain congruence condition at q, or, what is the same thing, that h corresponds to a particular representation in the local Waldspurger packet at q.

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