Proceedings of the Edinburgh Mathematical Society (2023) **66**, 100–116
doi:10.1017/S0013091523000056

ON THE EXISTENCE OF BABY SKYRMIONS STABILIZED BY VECTOR MESONS

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(Received 26 January 2022)

Abstract In this paper, we prove the existence of topologically non-trivial solutions of the two-dimensional Adkins–Nappi model of nuclear physics; to this end, we minimize the energy functional by using the classical Skyrme ansatz, as well as a non-radially symmetric generalization of it. In both cases, we show that the minimization procedure preserves the topological degree of the minimization sequence.

Keywords: skyrmion; vector meson; minimization; Adkins-Nappi model; Skyrme ansatz

2020 Mathematics subject classification: Primary 58E50;
Secondary 81V05

1. Introduction

The Skyrme model for baryons and mesons is a non-linear field theory in 3+1 dimensions which admits topologically non-trivial solutions, called skyrmions (see [26, 27]). In this model, the lagrangian includes a quadratic term and a term of the fourth order in the derivatives (the Skyrme term) which is essential for the existence of a minimum amount of energy, since it enables us to evade Derrick's theorem [8]. A different model was developed later by Adkins and Nappi [2] to improve the fit to the experimental data. In this theory, vector mesons are also considered, and the Skyrme term is replaced with a term that describes the coupling of the meson field to the pions and that stabilizes the skyrmion.

Recently, many authors have studied both the Skyrme model and the Adkins and Nappi model in two spatial dimensions (see [1, 3, 6, 10, 11, 13–23, 25, 30, 31]) because the two-dimensional skyrmions (usually called baby skyrmions) are more tractable, from the numerical point of view, than their three-dimensional analogues. Moreover, baby skyrmions have applications to condensed matter physics (see for instance, [28]), so they have their own interest.

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https://doi.org/10.1017/S0013091523000056 Published online by Cambridge University Press

In two spatial dimensions, the static Skyrme field, namely the pions field, is a map $u\colon \mathbb{R}^2 \to S^2$ whose energy is

$$E(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{2}|\nabla \omega_u|^2 - \frac{M^2}{2}|\omega_u|^2 + \frac{g}{4\pi}\omega_u u \cdot \partial_1 u \times \partial_2 u + V(u) \right) \mathrm{d}x, \quad (1.1)$$

where the function $\omega_u\colon \mathbb{R}^2 \to \mathbb{R}$ is the solution, vanishing at infinity, of the equation

$$\Delta \omega - M^2 \omega + \frac{g}{4\pi} u \cdot \partial_1 u \times \partial_2 u = 0. \quad (1.2)$$

In Equations (1.1) and (1.2), M is the mass of the ω field, g is a coupling constant and $V(y) = V(y_1, y_2, y_3)$ is a suitable smooth non-negative potential defined on S^2, which vanishes at the North Pole $e_3 = (0, 0, 1)$ of S^2; in most cases, the potential $V(y)$ has the form $V(y) = m(1 - y_3)$, where m is the mass of the pion field (see [11]). From Equation (1.1), provided $\omega_u(x) \to 0$ appropriately for $|x| \to +\infty$, we get

$$\int_{\mathbb{R}^2} \left(|\nabla \omega_u|^2 + M^2 \omega_u^2\right) \mathrm{d}x = \int_{\mathbb{R}^2} \frac{g}{4\pi}\omega_u u \cdot \partial_1 u \times \partial_2 u \,\mathrm{d}x,$$

so that the functional $E(u)$ becomes

$$E(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla \omega_u|^2 + \frac{1}{2}M^2|\omega_u|^2 + V(u) \right) \mathrm{d}x.$$

Clearly $E(u) \geq 0$, and it has a trivial global minimum for $u(x) \equiv e_3$. On the other hand, the finite energy requirement implies $u(\infty) = e_3$, so that $\mathbb{R}^2$ can be compactified to S^2, and the map $u\colon \mathbb{R}^2 \to S^2$ can be identified with a map from S^2 to S^2, with a well-defined topological degree $Q(u) \in \mathbb{Z}$, where

$$Q(u) = \frac{1}{4\pi} \int_{\mathbb{R}^2} u \cdot \partial_1 u \times \partial_2 u \,\mathrm{d}x.$$

On each topological sector Q_k ($k \in \mathbb{Z}$), we have the well-known topological lower bound on the energy:

$$E(u) \geq \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x \geq 4\pi|k|,$$

so that $E_k \equiv \inf\{E(u) \mid u \in Q_k\} > 0$ for $k \neq 0$, and we can search for functions $u \in Q_k$ such that $E(u) = E_k$, namely for baby skyrmions with topological degree $Q(u) = k$.

In this paper, we limit ourselves to consider functions $u\colon \mathbb{R}^2 \to S^2$ of the form

$$u_f(x) = \left(\sin(f(x))\cos(k\varphi), \sin(f(x))\sin(k\varphi), \cos(f(x))\right), \quad (1.3)$$

where $k \in \mathbb{Z}$ and $f\colon \mathbb{R}^2 \to \mathbb{R}$. In this case, the functional $E(u)$ becomes

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$$E(f)=\int_{\mathbb{R}^2}\left(\frac{1}{2}\left(\frac{k^2\,\sin^2(f(x))}{|x|^2}+|\nabla f|^2\right)-\frac{1}{2}|\nabla\omega_f|^2-\frac{M^2}{2}|\omega_f|^2\right)\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^2}\left(-\frac{g}{4\pi}\omega_f(x)\frac{k\,\sin(f(x))}{|x|^2}x\cdot\nabla f(x)+V(u_f)\right)\,\mathrm{d}x,$$

and $f(x)$ and $\omega_f(x)$ are coupled by the equation

$$\Delta\,\omega_f-M^2\omega_f+\frac{g}{4\pi}\frac{k\,\sin(f(x))}{|x|^2}x\cdot\nabla f(x)=0. \tag{1.4}$$

We assume that the potential $V\colon S^2\to\mathbb{R}$ is a smooth non-negative function and that there exist $a_0,\,a_1>0$ and $t\in]0,1[$, such that

$$V(y)\le a_1|y-e_3|^2\quad\text{for every } y\in S^2; \tag{1.5}$$

$$V(y)\ge a_0|y-e_3|^2\quad\text{for every } y\in S^2 \text{ with } y_3>t; \tag{1.6}$$

we have

$$V(y_1,y_2,y_3)=V(-y_1,-y_2,y_3)\quad\text{for every } y=(y_1,y_2,y_3)\in S^2. \tag{1.7}$$

Set

$$Y=\left\{f\in W^{1,2}(\mathbb{R}^2,\mathbb{R})\,\middle|\,\int_{\mathbb{R}^2}\frac{\sin^2(f(x))}{|x|^2}\,\mathrm{d}x<+\infty\right\}$$

and denote by X the set of the functions $f\in Y$ such that there exists a weak solution ω_f of Equation (1.4) with $\omega_f\in W^{1,2}(\mathbb{R}^2,\mathbb{R})$. If $f\in X$, by multiplying Equation (1.4) by ω_f and integrating over $\mathbb{R}^2$, we get

$$\int_{\mathbb{R}^2}\left(|\nabla\omega_f|^2+M^2|\omega_f|^2\right)\,\mathrm{d}x=\int_{\mathbb{R}^2}\frac{g}{4\pi}\omega_f(x)\frac{k\,\sin(f(x))}{|x|^2}x\cdot\nabla f(x)\,\mathrm{d}x<+\infty,$$

so that X is the natural domain of definition of the functional $E(f)$, and we also have

$$E(f)=\int_{\mathbb{R}^2}\left(\frac{1}{2}\left(\frac{k^2\,\sin^2(f(x))}{|x|^2}+|\nabla f|^2\right)+\frac{1}{2}|\nabla\omega_f|^2+\frac{M^2}{2}|\omega_f|^2+V(u_f)\right)\,\mathrm{d}x.$$

Moreover, if $f\in Y$, we have $u_f-e_3\in W^{1,2}(\mathbb{R}^2,\mathbb{R}^3)$ because of $|u_f(x)-e_3|^2=2(1-\cos(f(x)))\le f(x)^2$, and

$$Q(u_f)=\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{k\,\sin(f(x))}{|x|^2}x\cdot\nabla f(x)\,\mathrm{d}x.$$

Clearly, if $k=0$, then $Q(u_f)=0$; for $k\neq0$, we have (see Lemma 2) $Q(u_f)=0$ or $Q(u_f)=-k$; let us suppose from now on $k\neq0$ and let $X_{-k}=\{f\in X\mid Q(u_f)=-k\}$.

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On X_{-k}, we have

$$E_{-k} = \inf\left\{E(f) \mid f \in X_{-k}\right\} \geq 4\pi|k| > 0.$$

We prove the following theorem.

Theorem 1. *Assume that V satisfy assumptions (1.5)–(1.7). Then, for every $k \in \mathbb{Z}$, $k \neq 0$, there exists $f \in X_{-k}$ such that $0 \leq f(x) \leq \pi$ and $E(f) = \inf\left\{E(f) \mid f \in X_{-k}\right\}$.*

Starting from the paper [2], the existence of skyrmions stabilized by vector mesons was studied by many authors, but almost exclusively through numerical simulations; the axially symmetric ansatz (the Skyrme ansatz)

$$u_f(x) = (\sin(f(r))\cos(k\varphi), \sin(f(r))\sin(k\varphi), \cos(f(r))), \tag{1.8}$$

where $r = |x|$ or the rational map ansatz (see [29]) was usually used in order to reduce the problem to ordinary differential equations.

Existence and regularity results for critical points of a functional with a non-local term are given in [7]; in this paper, no symmetry assumptions are made, but the field configurations are taken over a compact manifold (the two-dimensional torus).

In Theorem 1, we prove the existence of baby skyrmions, which minimize the energy functional on a class of field configurations wider than the axially symmetric ones usually considered because we do not assume $f(x) = f(|x|)$ in ansatz (1.3), namely that $f(x)$ is radially symmetric.

Notice that the assumptions (1.5)–(1.7) on the potential are satisfied, of course, by the potential $V(y) = m(1 - y_3)$ used in most papers and also, for instance, by 'double vacuum' potentials like $V(y) = m(1 - y_3^2)$ (see [11]).

The skyrmions found in Theorem 1 are not necessarily axially symmetric (as far as we know); however, the existence of baby skyrmions of the form (1.8) is proved in Theorem 2 (see § 3); a similar result in the three-dimensional case was obtained by [9]. Clearly, the baby skyrmions found in Theorem 2 minimize the energy only on the class of axially symmetric field configurations.

2. Proof of Theorem 1

To simplify the notation, in this section, we set $M = 1$ and $g = 4\pi$; moreover, since $Q(u_f) = k\,Q(v_f)$, where

$$v_f(x) = (\sin(f(x))\cos(\varphi), \sin(f(x))\sin(\varphi), \cos(f(x))),$$

we can also assume, without loss of generality, that $k = 1$, so that the functional $E(f)$ in the Introduction becomes

$$E(f) = \int_{\mathbb{R}^2} \left(\frac{1}{2}\left(\frac{\sin^2(f(x))}{|x|^2} + |\nabla f|^2\right) + \frac{1}{2}|\nabla\omega_f|^2 + \frac{1}{2}|\omega_f|^2 + V(u_f)\right) \mathrm{d}x.$$

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For every $f\colon \mathbb{R}^2 \to \mathbb{R}$, we set $\hat{f}(r,\theta) = f(r\cos\theta, r\sin\theta)$, where $r \geq 0$ and $\theta \in [0, 2\pi]$; we prove the following simple lemma.

Lemma 1. *Let $f \in Y$, then the function*

$$g(r) \equiv \int_0^{2\pi} \cos(\hat{f}(r,\theta))\,\mathrm{d}\theta$$

is well-defined and continuous on $[0, +\infty[$, and $\lim_{r\to+\infty} g(r) = 2\pi$.

Proof. Since

$$|g(b) - g(a)| \leq \frac{1}{2}\int_a^b \int_0^{2\pi} \left(\frac{\sin^2(\hat{f}(r,\theta))}{r} + \hat{f}_r(r,\theta)^2 r\right) \mathrm{d}\theta\,\mathrm{d}r$$
$$\leq \frac{1}{2}\int_{a<|x|<b} \left(\frac{\sin^2(f(x))}{|x|^2} + |\nabla f(x)|^2\right) \mathrm{d}x,$$

$g(r)$ is continuous because of the absolute continuity of the integral. Moreover, since for every $\varepsilon > 0$

$$\int_{|x|>R} \left(\frac{\sin^2(f(x))}{|x|^2} + |\nabla f(x)|^2\right) \mathrm{d}x < \varepsilon$$

provided R is large enough, the Cauchy condition at infinity is satisfied, so the limit $\lim_{r\to+\infty} g(r)$ exists, and clearly it is $\leq 2\pi$; then, since

$$\int_{|x|>1} f(x)^2\,\mathrm{d}x \geq \int_{|x|>1} (1 - \cos(f(x)))\,\mathrm{d}x = \int_1^{+\infty}\int_0^{2\pi} (1 - \cos(\hat{f}(r,\theta)))r\,\mathrm{d}\theta\,\mathrm{d}r$$
$$\geq \int_1^{+\infty}\int_0^{2\pi} (1 - \cos(\hat{f}(r,\theta)))\,\mathrm{d}\theta\,\mathrm{d}r = \int_1^{+\infty} (2\pi - g(r))\,\mathrm{d}r,$$

and $\int_{\mathbb{R}^2} f(x)^2\,\mathrm{d}x < \infty$, we must have $\lim_{r\to+\infty} g(r) = 2\pi$. □

Lemma 2. *Let $f \in Y$, then*

$$\ell \equiv \lim_{r\to 0}\int_0^{2\pi} \cos(\hat{f}(r,\theta))\,\mathrm{d}\theta = \pm 2\pi;$$

moreover, if we set $u_f(x) = (\sin(f(x))\cos(\varphi), \sin(f(x))\sin(\varphi), \cos(f(x)))$, then $Q(u_f) = 0$ if $\ell = 2\pi$ and $Q(u_f) = -1$ if $\ell = -2\pi$.

Proof. Fix $f \in Y$, and let $u_f(x)$ be as in the statement of the lemma. Let r, R be such that $0 < r < R$ and set $\Omega_{r,R} = \{x \in \mathbb{R}^2 \mid r < |x| < R\}$; from the summability of the

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function

$$\frac{\sin(f(x))}{|x|^2}x\cdot\nabla f(x)=\operatorname{div}\left(\frac{1-\cos(f(x))}{|x|^2}x\right),$$

we have

$$\begin{aligned}4\pi\,Q(u_f)&=\int_{\mathbb{R}^2}\frac{\sin(f(x))}{|x|^2}x\cdot\nabla f(x)\,\mathrm{d}x=\lim_{r\to 0}\lim_{R\to+\infty}\int_{\Omega_{r,R}}\operatorname{div}\left(\frac{1-\cos(f(x))}{|x|^2}x\right)\mathrm{d}x\\&=\lim_{r\to 0}\lim_{R\to+\infty}\left(-\int_0^{2\pi}\Big(1-\cos(\hat f(r,\theta))\Big)\,\mathrm{d}\theta+\int_0^{2\pi}\Big(1-\cos(\hat f(R,\theta))\Big)\,\mathrm{d}\theta\right).\end{aligned}$$

From the previous lemma, we have $\lim_{R\to+\infty}\int_0^{2\pi}\Big(1-\cos(\hat f(R,\theta))\Big)\,\mathrm{d}\theta=0$, so

$$Q(u_f)=-\frac{1}{4\pi}\lim_{r\to 0}\int_0^{2\pi}\Big(1-\cos(\hat f(r,\theta))\Big)\,\mathrm{d}\theta=-\frac{1}{4\pi}(2\pi-\ell).$$

Since $\ell\in[-2\pi,2\pi]$ and $Q(u_f)\in\mathbb{Z}$, we have $\ell=2\pi$ or $\ell=-2\pi$, so that $Q(u_f)=0$ or $Q(u_f)=-1$, and the lemma follows. □

In the following lemma, we show that a function $f\in X$ can be replaced with a function $g\in X$ such that $0\le g(x)\le\pi$, leaving the energy and the topological charge unchanged.

Lemma 3. *Under the assumptions (1.5)–(1.7), for every $f\in X$, there exists $g\in X$ such that $0\le g(x)\le\pi$ a.e. in $\mathbb{R}^2$, $E(g)=E(f)$ and $Q(u_g)=Q(u_f)$.*

Proof. Let $f\in X$ and set $g(x)=h(f(x))$, where $h(s)=\arccos(\cos(s))$. Clearly, $0\le g(x)\le\pi$ a.e. in $\mathbb{R}^2$, and $\sin^2(g(x))=\sin^2(f(x))$, $\cos(g(x))=\cos(f(x))$; moreover, $g\in L^2(\mathbb{R}^2,\mathbb{R})$, in fact, since $s^2\le\pi^2(1-\cos s)/2\le\pi^2 s^2/4$ for $s\in[0,\pi]$, we have

$$\int_{\mathbb{R}^2}g(x)^2\,\mathrm{d}x\le\frac{\pi^2}{2}\int_{\mathbb{R}^2}(1-\cos(g(x)))\,\mathrm{d}x=\\ \frac{\pi^2}{2}\int_{\mathbb{R}^2}(1-\cos(f(x)))\,\mathrm{d}x\le\frac{\pi^2}{4}\int_{\mathbb{R}^2}f(x)^2\,\mathrm{d}x<+\infty.$$

Since h is Lipschitz, g is weakly derivable, and the chain rule holds true, so that $\nabla g(x)=h'(f(x))\nabla f(x)$ if h is derivable at $f(x)$, and $\nabla g(x)=0$ if h is not derivable at $f(x)$ (see, for instance [12], Theorem 7.8). But $h(s)$ is not derivable at $s=m\pi$, $m\in\mathbb{Z}$, whereas $h'(s)=\pm1$ for $s\neq m\pi$; set $A_m=\{x\in\mathbb{R}^2\mid f(x)=m\pi\}$. Since $f(x)$ is constant on each A_m, we have $\nabla f(x)=0$ a.e. on $A\equiv\cup A_m$ (see [12], Lemma 7.7), so that

$$\int_{\mathbb{R}^2}|\nabla g|^2\,\mathrm{d}x=\int_{\mathbb{R}^2\setminus A}|\nabla g|^2\,\mathrm{d}x=\int_{\mathbb{R}^2\setminus A}|\nabla f|^2\,\mathrm{d}x=\int_{\mathbb{R}^2}|\nabla f|^2\,\mathrm{d}x.$$

Next we observe that $-\sin(g(x))x\cdot\nabla g(x)/|x|^2=-\sin(f(x))x\cdot\nabla f(x)/|x|^2$ a.e. on $\mathbb{R}^2$; in fact, both sides of the equation vanish on A, whereas for $x\in\mathbb{R}^2\setminus A$, we have

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$\nabla g(x) = \pm\nabla f(x)$ and, respectively, $\sin(g(x)) = \pm\sin(f(x))$, so the equality holds true. Then $\omega_f = \omega_g$ so that $g \in X$, and moreover $Q(u_g) = Q(u_f)$. Finally, by assumption (1.7), we have

$$\begin{aligned} V(u_g(x)) &= V(\sin(g(x))\cos(\varphi), \sin(g(x))\sin(\varphi), \cos(g(x))) \\ &= V(\pm\sin(f(x))\cos(\varphi), \pm\sin(f(x))\sin(\varphi), \cos(f(x))) = V(u_f(x)), \end{aligned}$$

so that $E(g) = E(f)$. □

Notice that we require $V(y) \geq a_0|y-e_3|^2$ only for $y_3 > t$, where $0 < t < 1$ (see assumption (1.6)), in order to include in Theorem 1 the 'multiple vacuum' case. Nevertheless, the energy functional $E(f)$ is coercive in the sense of Remark 2, as shown in the following two lemmas.

We denote by $C_c^\infty(\mathbb{R}^2, S^2)$ the set of smooth functions from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $|u(x)| = 1$ for every $x \in \mathbb{R}^2$, and $u - e_3$ has compact support; moreover, $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^2$.

Lemma 4. *There exist $c_1 > 0$ such that, for every $u = (u_1, u_2, u_3) \in C_c^\infty(\mathbb{R}^2, S^2)$, we have*

$$|A| \leq c_1 \left(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(u) \right) \mathrm{d}x \right)^2,$$

where $A = \{x \in \mathbb{R}^2 \mid u_3(x) < t\}$.

Proof. Set

$$c_1 = \frac{1}{4\pi} \frac{1}{(1-t)^3} \frac{9}{16} \max\left(\frac{1}{4a_0^2}, 1 \right),$$

where a_0 and $t \in]0,1[$ are as in assumption (1.6). Let $u = (u_1, u_2, u_3) \in C_c^\infty(\mathbb{R}^2, S^2)$, and set $A = \{x \in \mathbb{R}^2 \mid u_3(x) < t\}$ and $B = \{x \in \mathbb{R}^2 \mid u_3(x) > t\}$. Since $\nabla(1-u_3(x))^{\frac{3}{2}} = -\frac{3}{2}(1-u_3(x))^{\frac{1}{2}}\nabla u_3(x)$, we have

$$\begin{aligned} \int_B |\nabla(1-u_3(x))^{3/2}| \,\mathrm{d}x &\leq \frac{3}{2} \left(\int_B (1-u_3(x)) \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_B |\nabla u_3(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq \frac{3}{4} \left(\int_B (1-u_3(x)) \,\mathrm{d}x + \int_B |\nabla u_3(x)|^2 \,\mathrm{d}x \right). \end{aligned}$$

But $u_3(x) > t$, so that, by assumption (1.6), we have $V(u(x)) \geq a_0|u(x) - e_3|^2 = 2a_0(1-u_3(x))$ on B (notice that $|u(x)| = 1$ implies $|u(x) - e_3|^2 = 2(1-u_3(x))$), so that

$$\int_B (1-u_3(x)) \,\mathrm{d}x \leq \frac{1}{2a_0} \int_B V(u(x)) \,\mathrm{d}x \leq \frac{1}{2a_0} \int_{\mathbb{R}^2} V(u(x)) \,\mathrm{d}x,$$

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and then

$$\int_B |\nabla(1-u_3(x))^{3/2}|\,\mathrm{d}x \le \frac{3}{4}\left(\frac{1}{2a_0}\int_{\mathbb{R}^2} V(u)\,\mathrm{d}x + \int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right).$$

On the other hand, since on B we have $0 \le 1-u_3(x) \le 1-t$, from the co-area formula, it follows that

$$\int_0^{(1-t)^{3/2}} \mathcal{H}^1(\Gamma_y)\,\mathrm{d}y = \int_B |\nabla(1-u_3(x))^{3/2}|\,\mathrm{d}x \le \frac{3}{4}\left(\frac{1}{2a_0}\int_{\mathbb{R}^2} V(u)\,\mathrm{d}x + \int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right),$$

where $\Gamma_y = \{x \in \mathbb{R}^2 \mid (1-u_3(x))^{3/2} = y\}$. Then, there exists $y_0 \in [0,(1-t)^{3/2}]$ such that

$$\mathcal{H}^1(\Gamma_{y_0}) \le \frac{1}{(1-t)^{3/2}}\frac{3}{4}\left(\frac{1}{2a_0}\int_{\mathbb{R}^2} V(u)\,\mathrm{d}x + \int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right).$$

Let $C = \{x \in \mathbb{R}^2 \mid u_3(x) < 1-(y_0)^{2/3}\}$; clearly, $x \in A$ implies $y_0 \le (1-t)^{3/2} < (1-u_3(x))^{3/2}$, so that $A \subset C$. Moreover, C is an open and bounded subset of $\mathbb{R}^2$, with $\partial C = \Gamma_{y_0}$; therefore, by the isoperimetric inequality,

$$|A| \le |C| \le \frac{1}{4\pi}\mathcal{H}^1(\Gamma_{y_0})^2 \le \frac{1}{4\pi}\frac{1}{(1-t)^3}\frac{9}{16}\left(\frac{1}{2a_0}\int_{\mathbb{R}^2} V(u)\,\mathrm{d}x + \int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^2,$$

and the lemma is proved. □

Remark 1. Let us consider $f \in Y$ and let u_f as in (1.3), then $u_f - e_3 \in W^{1,2}(\mathbb{R}^2,\mathbb{R}^3)$, as observed in the Introduction. By virtue of the well-known density theorem of [24, Part 4] (see also [5]), there exists a sequence $(v_n)_n \subset C_c^\infty(\mathbb{R}^2,S^2)$, such that $v_n - e_3 \to u_f - e_3$ in $W^{1,2}(\mathbb{R}^2,\mathbb{R}^3)$. Clearly, we also have, eventually passing to a subsequence, $\int_{\mathbb{R}^2} V(v_n)\,\mathrm{d}x \to \int_{\mathbb{R}^2} V(u_f)\,\mathrm{d}x$. In fact, since $v_n - e_3 \to u_f - e_3$ in $L^2(\mathbb{R}^2,\mathbb{R}^3)$, we have (modulo subsequences) $v_n - e_3 \to u_f - e_3$ a.e. on $\mathbb{R}^2$, and moreover, there exist $h \in L^2(\mathbb{R}^2,\mathbb{R}^3)$ such that $|v_n(x)-e_3| \le h(x)$ a.e. on $\mathbb{R}^2$ (see for instance [4], Theorem 4.9). Then, $V(v_n(x)) \le a_1|v_n(x)-e_3|^2 \le a_1 h(x)^2$ because of assumption (1.5), and we can apply the dominated convergence theorem.

Lemma 5. *For every $f \in Y$, we have*

$$\int_{\mathbb{R}^2}|u_f - e_3|^2\,\mathrm{d}x \le \frac{1}{a_0}\int_{\mathbb{R}^2} V(u_f)\,\mathrm{d}x + 4c_1\left(\int_{\mathbb{R}^2}\left(|\nabla u_f|^2 + V(u_f)\right)dx\right)^2, \tag{2.1}$$

where $c_1 > 0$ is the constant in the previous Lemma, and u_f is as in formula (1.3).

Proof. Because of the previous Remark, it is enough to demonstrate that for every $v \in C_c^\infty(\mathbb{R}^2,S^2)$, inequality (2.1) holds true. In fact, fix $v \in C_c^\infty(\mathbb{R}^2,S^2)$ and observe that

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$$\int_{\mathbb{R}^2} |v - e_3|^2 \, \mathrm{d}x = \int_{v_3(x) \geq t} |v - e_3|^2 \, \mathrm{d}x + \int_{v_3(x) < t} |v - e_3|^2 \, \mathrm{d}x;$$

since $v_3(x) \geq t$ implies $a_0 |v(x) - e_3|^2 \leq V(v(x))$, for the first integral, we have

$$\int_{v_3(x) \geq t} |v - e_3|^2 \, \mathrm{d}x \leq \frac{1}{a_0} \int_{v_3(x) \geq t} V(v) \, \mathrm{d}x \leq \frac{1}{a_0} \int_{\mathbb{R}^2} V(v) \, \mathrm{d}x.$$

Moreover, if we set $A = \{x \in \mathbb{R}^2 \mid v_3(x) < t\}$, since $|v(x) - e_3| \leq 2$, from the previous lemma, we get

$$\int_{v_3(x) < t} |v - e_3|^2 \, \mathrm{d}x \leq 4|A| \leq 4c_1 \left(\int_{\mathbb{R}^2} \left(|\nabla v|^2 + V(v) \right) \, \mathrm{d}x \right)^2,$$

so inequality (2.1) is proved. □

Remark 2. If $f \in X$ and $0 \leq f(x) \leq \pi$ a.e. on $\mathbb{R}^2$, since $s^2 \leq \pi^2(1 - \cos s)/2$ for $s \in [0, \pi]$, inequality (2.1) implies

$$\begin{aligned} \int_{\mathbb{R}^2} |f(x)|^2 \, \mathrm{d}x &\leq \frac{\pi^2}{2} \int_{\mathbb{R}^2} \left(1 - \cos(f(x))\right) \, \mathrm{d}x = \frac{\pi^2}{4} \int_{\mathbb{R}^2} |u_f - e_3|^2 \, \mathrm{d}x \\ &\leq \frac{\pi^2}{4} \left(\frac{1}{a_0} \int_{\mathbb{R}^2} V(u_f) \, \mathrm{d}x + 4c_1 \left(\int_{\mathbb{R}^2} \left(|\nabla u_f|^2 + V(u_f) \right) \, \mathrm{d}x \right)^2 \right). \end{aligned}$$

In particular, if $(f_n)_n \subset X$ is a sequence such that $0 \leq f_n(x) \leq \pi$ and $(E(f_n))_n$ is bounded, then $(f_n)_n$ is bounded in $W^{1,2}(\mathbb{R}^2, \mathbb{R})$.

We consider now a family of functions obtained by truncation and rescaling from a function like $\log|\log x|$ as described in the following lemma. We denote by $C_c^\infty(\mathbb{R}^2, \mathbb{R})$ the set of smooth functions from $\mathbb{R}^2$ to $\mathbb{R}$ with compact support, by $\|\cdot\|_{W^{1,2}}$ the norm on the Sobolev space $W^{1,2}(\mathbb{R}^2, \mathbb{R})$ and by $\|\cdot\|_\infty$ the norm on $L^\infty(\mathbb{R}^2, \mathbb{R})$.

Lemma 6. *There exists $C_* > 0$ and a family of functions $(\psi_{n,\lambda})_{n,\lambda} \subset C_c^\infty(\mathbb{R}^2, \mathbb{R})$, with $n \in \mathbb{N}$ and $\lambda \in [1, +\infty[$, such that for every $n \in \mathbb{N}$ and every $\lambda \geq 1$,*

$$\|\psi_{n,\lambda}\|_{W^{1,2}} < C_*; \quad \|\psi_{n,\lambda}\|_\infty \leq n + 1; \quad \psi_{n,\lambda}(0) = n; \quad \mathrm{supp}(\psi_{n,\lambda}) \subset B_{\frac{2}{\lambda}}(0).$$

Proof. Let $g(x) = \log(1 - \log|x|)$ if $|x| \leq 1$, $g(x) = 0$ if $|x| > 1$ and set $g_n(x) = \min(g(x), n)$; clearly, g, $g_n \in W^{1,2}(\mathbb{R}^2, \mathbb{R})$ and $\|g_n\|_{W^{1,2}} < \|g\|_{W^{1,2}}$, $\|g_n\|_\infty = n$; moreover, $\mathrm{supp}(g_n) = B_1(0)$.

For every $n \in \mathbb{N}$, there exists a mollification $\psi_n \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$ of g_n such that $\|\psi_n\|_{W^{1,2}} < \|g\|_{W^{1,2}}$, $\|\psi_n\|_\infty \leq n + 1$, $\psi_n(x) = n$ in a neighborhood of zero and $\mathrm{supp}(\psi_n) \subset B_2(0)$.

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Finally, for every $n \in \mathbb{N}$ and every $\lambda \geq 1$, let $\psi_{n,\lambda}(x) = \psi_n(\lambda x)$; clearly, setting $C_* = \|g\|_{W^{1,2}}$ and observing that $\lambda \to \|\psi_{n,\lambda}\|_{W^{1,2}}$ is decreasing, we see immediately that the family $(\psi_{n,\lambda})_{n,\lambda} \subset C_c^\infty(\mathbb{R}^2,\mathbb{R})$ of functions verify the lemma. $\square$

Remark 3. Let $f \in Y$, then there exists $\lim_{r\to 0} \int_0^{2\pi} \cos(\hat{f}(r,\theta))\,\mathrm{d}\theta = \pm 2\pi$ (see Lemma 2). Now, let $\varphi \in C_c^\infty(\mathbb{R}^2,\mathbb{R})$; setting $\Omega_{r,R} = \{x \in \mathbb{R}^2 \mid r < |x| < R\}$, we have, for R large enough,

$$\begin{aligned}\int_{\Omega_{r,R}} &\sin(f(x))\frac{x}{|x|^2}\cdot\nabla f(x)\varphi(x)\,\mathrm{d}x\\ &= \int_{\Omega_{r,R}} \frac{\cos(f(x))}{|x|^2} x\cdot\nabla\varphi(x)\,\mathrm{d}x - \int_{\Omega_{r,R}} \operatorname{div}\left(\frac{\cos(f(x))}{|x|^2}\varphi(x)x\right)\mathrm{d}x\\ &= \int_{\Omega_{r,R}} \frac{\cos(f(x))}{|x|^2} x\cdot\nabla\varphi(x)\,\mathrm{d}x + \int_0^{2\pi} \hat{\varphi}(r,\theta)\,\cos(\hat{f}(r,\theta))\,\mathrm{d}\theta,\end{aligned}$$

so that we obtain, for $r\to 0$ and $R \to +\infty$,

$$\begin{aligned}\int_{\mathbb{R}^2} &\sin(f(x))\frac{x}{|x|^2}\cdot\nabla f(x)\varphi(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^2} \frac{x}{|x|^2}\cdot\nabla\varphi(x)\cos(f(x))\,\mathrm{d}x + \varphi(0)\lim_{r\to 0}\int_0^{2\pi}\cos(\hat{f}(r,\theta))\,\mathrm{d}\theta.\end{aligned}$$

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let $(f_n)_n \subset X_{-1}$ be a minimizing sequence for $E(f)$, so that $E(f_n) \to E_{-1} = \inf\{E(f) \mid f \in X_{-1}\}$. Recall that $Q(u_{f_n}) = -1$ and that

$$\lim_{r\to 0}\int_0^{2\pi}\cos(\hat{f}_n(r,\theta))\,\mathrm{d}\theta = -2\pi$$

(see Lemma 2). Because of Lemma 3, we can assume that $0 \leq f_n(x) \leq \pi$ a.e. on $\mathbb{R}^2$. Since $(E(f_n))_n$ is bounded, $(f_n)_n$ is bounded in $W^{1,2}(\mathbb{R}^2,\mathbb{R})$ (see Remark 2), so $f_n \to f \in W^{1,2}(\mathbb{R}^2,\mathbb{R})$ weakly (up subsequences). We can also suppose that $f_n \to f$ a.e. on $\mathbb{R}^2$. Clearly, by the Fatou lemma

$$\int_{\mathbb{R}^2}\frac{\sin^2(f(x))}{|x|^2}\,\mathrm{d}x \leq \liminf_{n\to+\infty}\int_{\mathbb{R}^2}\frac{\sin^2(f_n(x))}{|x|^2}\,\mathrm{d}x < +\infty, \tag{2.2}$$

so that $f \in Y$. We will prove that $f \in X_{-1}$ and that $E(f) \leq \liminf_{n\to+\infty} E(f_n) = E_{-1}$, and therefore $E(f) = E_{-1}$, namely E_{-1} is attained on X_{-1}.

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For every $n \in \mathbb{N}$, let $\omega_n \in W^{1,2}(\mathbb{R}^2, \mathbb{R})$ be the weak solution of the equation

$$\Delta\omega_n - \omega_n = -\frac{1}{|x|^2}\sin(f_n(x))x \cdot \nabla f_n(x).$$

Multiplying by $\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$ and integrating, we have (see Remark 3)

$$\int_{\mathbb{R}^2} \nabla\omega_n \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^2} \omega_n \varphi \,\mathrm{d}x = \int_{\mathbb{R}^2} \frac{x}{|x|^2} \cdot \nabla\varphi(x)\, \cos(f_n(x)) \,\mathrm{d}x - 2\pi\varphi(0). \tag{2.3}$$

Since $(E(f_n))_n$ is bounded, $(\omega_n)_n$ is bounded in $W^{1,2}(\mathbb{R}^2, \mathbb{R})$ so that $\omega_n \to \omega \in W^{1,2}(\mathbb{R}^2, \mathbb{R})$ weakly (up subsequences). Moreover,

$$\int_{\mathbb{R}^2} \frac{x}{|x|^2} \cdot \nabla\varphi(x)\, \cos(f_n(x)) \,\mathrm{d}x \to \int_{\mathbb{R}^2} \frac{x}{|x|^2} \cdot \nabla\varphi(x)\, \cos(f(x)) \,\mathrm{d}x$$

by the dominated convergence theorem, so that passing to the limit in Equation (2.3), we get

$$\int_{\mathbb{R}^2} \nabla\omega \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^2} \omega \varphi \,\mathrm{d}x = \int_{\mathbb{R}^2} \frac{x}{|x|^2} \cdot \nabla\varphi(x)\, \cos(f(x)) \,\mathrm{d}x - 2\pi\varphi(0). \tag{2.4}$$

On the other hand, from Remark 3, setting $\lim_{r\to 0} \int_0^{2\pi} \cos(\hat{f}(r,\theta)) \,\mathrm{d}\theta = \ell$, we also have

$$\int_{\mathbb{R}^2} \sin(f(x)) \frac{x}{|x|^2} \cdot \nabla f(x) \varphi(x) \,\mathrm{d}x = \int_{\mathbb{R}^2} \frac{x}{|x|^2} \cdot \nabla\varphi(x) \cos(f(x)) \,\mathrm{d}x + \ell\varphi(0),$$

so that we can recast Equation (2.4) in the form

$$\int_{\mathbb{R}^2} \nabla\omega \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^2} \omega \varphi \,\mathrm{d}x = \int_{\mathbb{R}^2} \sin(f(x)) \frac{x}{|x|^2} \cdot \nabla f(x) \varphi(x) \,\mathrm{d}x - (2\pi + \ell)\varphi(0). \tag{2.5}$$

We claim now that $2\pi + \ell = 0$, namely $\ell = -2\pi$. In fact, let us consider the family $(\psi_{n,\lambda})_{n,\lambda} \subset C_c^\infty(\mathbb{R}^2, \mathbb{R})$ as in Lemma 6; from Equation (2.5), we get

$$\int_{\mathbb{R}^2} \nabla\omega \nabla\psi_{n,\lambda} \,\mathrm{d}x + \int_{\mathbb{R}^2} \omega \psi_{n,\lambda} \,\mathrm{d}x = \int_{\mathbb{R}^2} \sin(f(x)) \frac{x}{|x|^2} \cdot \nabla f(x) \psi_{n,\lambda}(x) \,\mathrm{d}x - (2\pi + \ell)n,$$

so that

$$(2\pi + \ell)n \le \|\omega\|_{W^{1,2}} \|\psi_{n,\lambda}\|_{W^{1,2}} + \left| \int_{\mathbb{R}^2} \sin(f(x)) \frac{x}{|x|^2} \cdot \nabla f(x) \psi_{n,\lambda}(x) \,\mathrm{d}x \right|,$$

and then

$$(2\pi + \ell)n \le C_* \|\omega\|_{W^{1,2}} + (n+1) \int_{B_{2/\lambda}(0)} \left| \sin(f(x)) \frac{x}{|x|^2} \cdot \nabla f(x) \right| \,\mathrm{d}x. \tag{2.6}$$

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If $2\pi + \ell \neq 0$, there exists $n_0 \in \mathbb{N}$ such that $(2\pi + \ell)n_0 > C_* \|\omega\|_{W^{1,2}} + 1$. Then, we can choose $\lambda_0 \geq 1$ large enough that $\int_{B_{2/\lambda_0}(0)} |\sin(f(x)) \frac{x}{|x|^2} \cdot \nabla f(x)| \, \mathrm{d}x < 1/(n_0 + 1)$, so that from inequality (2.6), we get $C_* \|\omega\|_{W^{1,2}} + 1 < C_* \|\omega\|_{W^{1,2}} + 1$, which is impossible, and the claim is proved. So, Equation (2.5) becomes

$$\int_{\mathbb{R}^2} \nabla\omega \nabla\varphi \, \mathrm{d}x + \int_{\mathbb{R}^2} \omega\varphi \, \mathrm{d}x = \int_{\mathbb{R}^2} \sin(f(x)) \frac{x}{|x|^2} \cdot \nabla f(x) \varphi(x) \, \mathrm{d}x,$$

and since φ was arbitrary, this show that $\omega \in W^{1,2}(\mathbb{R}^2, \mathbb{R})$ is a weak solution of the equation $\Delta\omega - \omega = -\sin(f(x)) x \cdot \nabla f(x)/|x|^2$, so that $f \in X$; moreover, since $\ell = -2\pi$, we have $f \in X_{-1}$, namely u_f is topologically nontrivial.

Finally, we have, by inequality (2.2), the weak lower semicontinuity of the norm and the fact that $\int_{\mathbb{R}^2} V(u_f) \, \mathrm{d}x \leq \liminf_{n \to +\infty} \int_{\mathbb{R}^2} V(u_{f_n}) \, \mathrm{d}x$ (by the Fatou lemma), that $E(f) \leq \liminf_{n \to +\infty} E(f_n)$, so that $E(f) = E_{-1}$, and Theorem 1 is proved. □

3. The axially symmetric case

In this section, we study the functional (1.1) by using the Skyrme ansatz, namely $E(u)$ is restricted to maps u_f of the form (1.8).

Let us denote by X^r the set of the functions $f \colon]0, +\infty[\to \mathbb{R}$ absolutely continuous on every compact subinterval of $]0, +\infty[$, such that

$$\int_0^{+\infty} \left(\frac{\sin^2(f(r))}{r} + f'(r)^2 r + f(r)^2 r \right) \mathrm{d}r < +\infty.$$

It is easy to verify that $f \in X^r$ implies that there exist the limits of $f(r)$ for $r \to 0$ and for $r \to +\infty$, and $f(0) = m\pi$ $(m \in \mathbb{Z})$, $f(+\infty) = 0$, so that we can assume f continuous on $[0, +\infty[$. Moreover,

$$Q(u_f) = \frac{1}{2} \int_0^{+\infty} k \sin(f(r)) f'(r) \, \mathrm{d}r = -\frac{k}{2} (1 - \cos(m\pi))$$

is equal to zero or $-k$. Now, suppose that $k \neq 0$ and set $X^r_{-k} = \{f \in X^r \mid Q(u_f) = -k\}$. Then, we have the following theorem, which is analogous to Theorem 1 (we set, for simplicity, $E(u_f) = E(f)$).

Theorem 2. *Assume that V satisfies assumptions (1.5)–(1.7). Then, for every $k \in \mathbb{Z}$, $k \neq 0$, there exists $f \in X^r_{-k}$ such that $0 \leq f(r) \leq \pi$, $f(0) = \pi$, $f(+\infty) = 0$ and $E(f) = \inf\{E(f) \mid f \in X^r_{-k}\}$.*

As in the previous section, we assume for simplicity, and without loss of generality, that $M = 1$, $g = 4\pi$ and $k = 1$. The coupling equation (1.4) becomes

$$r\,\omega_f''(r) + \omega_f'(r) - r\,\omega_f(r) = -\sin(f(r)) f'(r). \tag{3.1}$$

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For every $f \in X^r$, Equation (3.1) has a unique solution ω_f such that

$$\int_0^{+\infty} \left(\omega_f(r)^2 + \omega_f'(r)^2\right) r \, \mathrm{d}r < +\infty$$

(see Remark 4); so by multiplying Equation (3.1) by ω_f and integrating, we can write the functional (1.1) in the form

$$E(f) = 2\pi \int_0^{+\infty} \left(\frac{1}{2} \left(\frac{\sin^2(f(r))}{r} + f'(r)^2 r \right) + \frac{1}{2} \left(\omega_f'(r)^2 + \omega_f(r)^2\right) r + V(u_f(r)) r \right) \mathrm{d}r.$$

Notice that if $f \in X^r$, then $\omega_f(r)$ can be expressed in terms of the Green's function

$$G(r,s) = \begin{cases} I_0(s)K_0(r) & s \le r \\ I_0(r)K_0(s) & s > r \end{cases}$$

as $\omega_f(r) = \int_0^{+\infty} \sin(f(s)) f'(s) G(r,s) \, \mathrm{d}s$, where $I_0(r)$ and $K_0(r)$ are the modified Bessel functions of the first and second kind, respectively. Clearly, $\omega_f \in C^1\left(]0,+\infty[\right)$; moreover, if we set

$$H(r,s) = \begin{cases} (I_0(s)-1)K_0(r) & s \le r \\ I_0(r)K_0(s) & s > r \end{cases}, \quad H_r(r,s) = \begin{cases} -(I_0(s)-1)K_1(r) & s \le r \\ I_1(r)K_0(s) & s > r \end{cases}$$

where $I_1(r) = I_0'(r)$ and $K_1(r) = -K_0'(r)$, we have

$$\omega_f(r) = -K_0(r)(\cos(f(r)) - \cos(f(0))) + \int_0^{+\infty} \sin(f(s)) f'(s) H(r,s) \, \mathrm{d}s, \tag{3.2}$$

$$\omega_f'(r) = K_1(r)(\cos(f(r)) - \cos(f(0))) + \int_0^{+\infty} \sin(f(s)) f'(s) H_r(r,s) \, \mathrm{d}s. \tag{3.3}$$

We have now the following lemma.

Lemma 7. *If $f \in X^r$, then*

$$\int_0^{+\infty} (\omega_f(r)^2 + \omega_f'(r)^2) r \, \mathrm{d}r < +\infty$$

and moreover $\omega_f \in C\left([0,+\infty[\right)$.

Proof. We note that for $r \to 0$, we have $I_0(r) - 1 \simeq r^2$ and $K_0(r) \simeq |\log r|$; for $r \to +\infty$, we have $I_0(r) - 1 \simeq \mathrm{e}^r/\sqrt{r}$, $K_0(r) \simeq 1/(\mathrm{e}^r\sqrt{r})$; then, for $r \to 0$ and $r \to +\infty$, we also have, respectively,

$$\int_0^r \frac{(I_0(s)-1)^2}{s} \, \mathrm{d}s \simeq r^4, \qquad \int_r^{+\infty} \frac{K_0(s)^2}{s} \, \mathrm{d}s \simeq |\log r|^3;$$

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$$\int_0^r \frac{(I_0(s)-1)^2}{s}\,\mathrm{d}s \simeq \frac{e^{2r}}{r^2}, \qquad \int_r^{+\infty} \frac{K_0(s)^2}{s}\,\mathrm{d}s \simeq \frac{1}{e^{2r}r^2}.$$

Since

$$\int_0^{+\infty} \frac{H(r,s)^2}{s}\,\mathrm{d}s = K_0(r)^2 \int_0^r \frac{(I_0(s)-1)^2}{s}\,\mathrm{d}s + I_0(r)^2 \int_r^{+\infty} \frac{K_0(s)^2}{s}\,\mathrm{d}s,$$
$$\int_0^{+\infty} \frac{H_r(r,s)^2}{s}\,\mathrm{d}s = K_1(r)^2 \int_0^r \frac{(I_0(s)-1)^2}{s}\,\mathrm{d}s + I_1(r)^2 \int_r^{+\infty} \frac{K_0(s)^2}{s}\,\mathrm{d}s,$$

we get (recalling also that for $r \to 0$, we have $I_1(r) \simeq r$ and $K_1(r) \simeq 1/r$, and for $r \to +\infty$, we have $I_1(r) \simeq \mathrm{e}^r/\sqrt{r}$ and $K_1(r) \simeq 1/(\mathrm{e}^r\sqrt{r})$)

$$\int_0^{+\infty} \frac{H(r,s)^2}{s}\,\mathrm{d}s \simeq \begin{cases} \log^2(r)r^4 + |\log r|^3 & r \to 0 \\ \frac{1}{r^3} & r \to +\infty, \end{cases}$$

$$\int_0^{+\infty} \frac{H_r(r,s)^2}{s}\,\mathrm{d}s \simeq \begin{cases} r^2 + r^2|\log r|^3 & r \to 0 \\ \frac{1}{r^3} & r \to +\infty. \end{cases}$$

Then, from formula (3.2), by using the Holder inequality

$$\left|\int_0^{+\infty} \sin(f(s))f'(s)H(r,s)\,\mathrm{d}s\right| \le \left(\int_0^{+\infty} f'(s)^2 s\,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^{+\infty} \frac{H(r,s)^2}{s}\,\mathrm{d}s\right)^{\frac{1}{2}},$$

we have

$$|\omega_f(r)| \lesssim \begin{cases} |\log r| + |\log r|r^2 + |\log r|^{3/2} & r \to 0 \\ \frac{1}{e^r\sqrt{r}} + \frac{1}{r^{3/2}} & r \to +\infty \end{cases}$$

so that $\int_0^{+\infty} \omega_f(r)^2 r\,\mathrm{d}r < +\infty$.

We observe now that $f(0) = m\pi$, with $m \in \mathbb{Z}$, and since $|\cos t - \cos(m\pi)| \le \sin^2 t$ for $t \simeq m\pi$, we have also $|\cos(f(r)) - \cos(f(0))| \le \sin^2(f(r))$ for $r \simeq 0$. Then, from formula (3.3), by using again the Holder inequality, we get

$$|\omega_f'(r)| \lesssim \begin{cases} \frac{\sin^2(f(r))}{r} + r^2 + r^2|\log r|^3 & r \to 0 \\ \frac{1}{\mathrm{e}^r\sqrt{r}} + \frac{1}{r^{3/2}} & r \to +\infty, \end{cases}$$

and since $\int_0^{+\infty} \frac{\sin^2(f(r))}{r}\,\mathrm{d}r < +\infty$ because of $f \in X^r$, we also get $\int_0^{+\infty} \omega_f'(r)^2 r\,\mathrm{d}r < +\infty$. Finally, from $\omega_f' \in L^1(]0,+\infty[)$, we have $\omega_f \in C\big([0,+\infty[\big)$, and the lemma is proved. □

Remark 4. The solutions of Equation (3.1) are $\omega(r) = A\,K_0(r) + B\,I_0(r) + \omega_f(r)$, with A, $B \in \mathbb{R}$, so that $\int_0^{+\infty}(\omega(r)^2 + \omega'(r)^2)r\,\mathrm{d}r < +\infty$ implies $A = B = 0$, namely $\omega = \omega_f$.

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Remark 5. If $f \in X^r$, then $\bar{f} \in X$, where $\bar{f}(x) = f(|x|)$, and X is defined in § 1, so that, by Lemma 3, it is easy to see that there exists $g \in X^r$ such that $0 \le g(r) \le \pi$ on $[0, +\infty[$, $E(g) = E(f)$ and $Q(u_g) = Q(u_f)$. Moreover, as in Remark 2, if $(f_n)_n \subset X^r$ is a sequence such that $0 \le f_n(r) \le \pi$ and $(E(f_n))_n$ is bounded, then the sequence $\left(\int_0^{+\infty}(f_n'(r)^2 + f_n(r)^2) r \, \mathrm{d}r\right)_n$ is bounded.

Now we can prove Theorem 2.

Proof of Theorem 2. Let $(f_n)_n \subset X^r_{-1}$ be a minimizing sequence for $E(f)$, namely $E(f_n) \to E_{-1} = \inf\{E(f) \mid f \in X^r_{-1}\}$, then $E(f_n) \le C$ for some $C > 0$. Because of the previous Remark, we can also assume that $0 \le f_n(r) \le \pi$ and

$$\int_0^{+\infty} \left(\frac{\sin^2(f_n(r))}{r} + \left(f_n'(r)^2 + f_n(r)^2\right) r\right) \mathrm{d}r \le C.$$

Clearly, $f_n(+\infty) = 0$ and $f_n(0) = \pi$ for every $n \in \mathbb{N}$. Since $(f_n)_n$ is bounded on $W^{1,2}([a,b])$ for every $[a,b] \subset]0, +\infty[$, a standard diagonal subsequence argument (see for instance [3]) shows that there exists $f \in W^{1,2}_{\mathrm{loc}}(]0, +\infty[)$ such that we have (up subsequences) for every $[a,b] \subset]0, +\infty[$: $f_n \to f$ weakly in $W^{1,2}([a,b])$ and $f_n \to f$ uniformly in $[a,b]$. Clearly, f is absolutely continuous on every compact subinterval of $]0, +\infty[$, and, by using in the usual way the Fatou lemma and the weak lower semicontinuity of the norm, we have

$$\int_a^b \left(\frac{\sin^2(f(r))}{r} + \left(f'(r)^2 + f(r)^2\right) r\right) \mathrm{d}r \le \liminf_{n \to +\infty} \int_a^b \left(\frac{\sin^2(f_n(r))}{r} + \left(f_n'(r)^2 + f_n(r)^2\right) r\right) \mathrm{d}r \le C,$$

so that, since a and b are arbitrary, we have $f \in X^r$, and we can consider the function ω_f that we write in the form (see (3.2))

$$\omega_f(r) = -K_0(r)(\cos(f(r)) - \cos(f(0))) + \int_0^{+\infty} \sin(f(s)) f'(s) H(r,s) \, \mathrm{d}s.$$

Of course, $0 \le f(r) \le \pi$, $f(+\infty) = 0$ and $f(0) = 0$ or $f(0) = \pi$. We will soon see that, in fact, $f(0) = \pi$, and therefore $f \in X^r_{-1}$. To this end, let us consider the sequence $(\omega_{f_n})_n \subset W^{1,2}(]0, +\infty[, \mathbb{R})$; since $E(f_n) \le C$, we also have

$$\int_0^{+\infty} \left(\omega_{f_n}'(r)^2 + \omega_{f_n}(r)^2\right) r \, \mathrm{d}r \le 2C,$$

and, arguing as for $(f_n)_n$, we get $\omega \in W^{1,2}_{\mathrm{loc}}(]0, +\infty[)$ such that, up subsequences, for every $[a,b] \subset]0, +\infty[$: $\omega_{f_n} \to \omega$ weakly in $W^{1,2}([a,b])$, $\omega_{f_n} \to \omega$ uniformly in $[a,b]$, and

$$\int_0^{+\infty} \left(\omega'(r)^2 + \omega(r)^2\right) r \, \mathrm{d}r \le 2C. \tag{3.4}$$

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By using again formula (3.2) and recalling that $f_n(0)=\pi$, we can write, for every $r>0$,

$$\omega_{f_n}(r) = -K_0(r)(\cos(f_n(r))+1) + \int_0^{+\infty} \sin(f_n(s)) f_n'(s)\sqrt{s}\,\frac{H(r,s)}{\sqrt{s}}\,\mathrm{d}s. \tag{3.5}$$

Notice that $\sin(f_n(s))f_n'(s)\sqrt{s} \to \sin(f(s))f'(s)\sqrt{s}$ weakly in $L^2(]0,+\infty[)$. In fact, the sequence $(\sin(f_n(s))f_n'(s)\sqrt{s})_n$ is bounded in $L^2(]0,+\infty[)$, and, on every subinterval $[a,b] \subset]0,+\infty[$, we have $\sin(f_n(s)) \to \sin(f(s))$ uniformly in $[a,b]$, and $f_n'(s)\sqrt{s} \to f'(s)\sqrt{s}$ weakly in $L^2([a,b])$, so that we get the claim.

Then, passing to the limit in Equation (3.5) and recalling that $\omega_{f_n}(r) \to \omega(r)$ for every $r>0$, we have

$$\omega(r) = -K_0(r)(\cos(f(r))+1) + \int_0^{+\infty} \sin(f(s))f'(s)H(r,s)\,\mathrm{d}s.$$

But then we have $\omega(r) = \omega_f(r) - K_0(r)(\cos(f(0))+1)$ and so $\omega'(r) = \omega_f'(r) + K_1(r)(\cos(f(0))+1)$; by multiplying for $\sqrt{r}$, we get $K_1(r)\sqrt{r}(\cos(f(0))+1) = \left(\omega'(r) - \omega_f'(r)\right)\sqrt{r}$.

Since $\omega'(r)\sqrt{r} \in L^2(]0,+\infty[)$ because of inequality (3.4) and $\omega_f'(r)\sqrt{r} \in L^2(]0,+\infty[)$ for the Lemma 7, we must have $K_1(r)\sqrt{r}(\cos(f(0))+1) \in L^2(]0,+\infty[)$.

But $K_1(r)\sqrt{r} \simeq 1/\sqrt{r}$ as $r\to 0$, so we get $\cos(f(0)) = -1$, namely $f(0)=\pi$; therefore, $f \in X_{-1}^r$, $\omega = \omega_f$ and we have $E(f) \le \liminf_{n\to+\infty} E(f_n) = E_{-1}$ so that $E(f) = E_{-1}$, and the theorem is proved. □

Competing interests. The author declare none.

References

(1) C. Adam, T. Romańczukiewicz, J. Sánchez-Guillén and A. Wereszczyński, Investigation of restricted baby Skyrme models, *Phys. Rev. D* **81**(8) (2010), 085007–085017.

(2) G. S. Adkins and C. R. Nappi, Stabilization of chiral solitons via vector mesons, *Phys. Lett. B* **137**(3) (1984), 251–256.

(3) K. Arthur, G. Roche, D. H. Tchrakian and Y. Yang, Skyrme models with self-dual limits: $d=2,3$, *J. Math. Phys.* **37**(6) (1996), 2569–2584.

(4) H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext. (Springer, New York, 2011).

(5) H. Brezis and J. M. Coron, Large solutions for harmonic maps in two dimensions, *Comm. Math. Phys.* **92** (1983), 203–215.

(6) M. de Innocentis and R. S. Ward, Skyrmions on the 2-sphere, *Nonlinearity* **14**(3) (2001), 663–671.

(7) S. Demoulini and D. M. A. Stuart, Existence and regularity for generalised harmonic maps associated to a nonlocal polyconvex energy of Skyrme type, *Calc. Var. Partial Differential Equations* **30**(4) (2007), 523–546.

(8) G. H. Derrick, Comments on nonlinear wave equations as models for elementary particles, *J. Math. Phys.* **5**(9) (1964), 1252–1254.

https://doi.org/10.1017/S0013091523000056 Published online by Cambridge University Press

(9) J. Dolbeault, Existence de solutions symétriques pour un modèle de champs de mésons: le modèle d'Adkins et Nappi, *Comm. Partial Differential Equations* **15**(12) (1990), 1743–1786.

(10) P. Eslami, M. Sarbishaei and W. Zakrzewski, Baby Skyrme models for a class of potentials, *Nonlinearity* **13**(5) (2000), 1867–1881.

(11) D. Foster and P. Sutcliffe, Baby skyrmions stabilized by vector mesons, *Phys. Rev. D* **79** (2009), 125026.

(12) D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order* (Springer, Berlin, 2001). Reprint of the 1998 ed.

(13) C. Greco, Existence criteria for baby skyrmions for a wide range of potentials, *J. Math. Anal. Appl.* **487**(2) (2020), 124039.

(14) I. Hen and M. Karliner, Rotational symmetry breaking in baby Skyrme models, *Nonlinearity* **21**(3) (2008), 399–408.

(15) H. Hu and K. Hu, Existence of solutions for a baby-Skyrme model, *J. Appl. Math.* **2014** (2014), 1–4.

(16) J. Jäykkä, M. Speight and P. Sutcliffe, Broken baby skyrmions, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **468** (2012), 1085–1104.

(17) P. Jennings and T. Winyard, Broken planar skyrmions – statics and dynamics, *J. High Energy Phys.* **2014** (2014), 122.

(18) A. Kudryavtsev, B. Piette and W. Zakrzewski, Mesons, baryons and waves in the baby skyrmion model, *Eur. Phys. J. C* **1** (1998), 333–341.

(19) A. Kudryavtsev, B. M. A. G. Piette and W. J. Zakrzewski, Skyrmions and domain walls in $(2 + 1)$ dimensions, *Nonlinearity* **11**(4) (1998), 783–795.

(20) J. Li and X. Zhu, Existence of 2D skyrmions, *Math. Z.* **268**(1) (2011), 305–315.

(21) F. Lin and Y. Yang, Existence of two-dimensional skyrmions via the concentration-compactness method, *Comm. Pure Appl. Math.* **57**(10) (2004), 1332–1351.

(22) F. Lin and Y. Yang, Energy splitting, substantial inequality, and minimization for the Faddeev and Skyrme models, *Comm. Math. Phys.* **269**(1) (2007), 137–152.

(23) B. M. A. G. Piette, B. J. Schroers and W. J. Zakrzewski, Multisolitons in a two-dimensional Skyrme model, *Z. Phys. C* **65**(1) (1995), 165–174.

(24) R. Schoen and K. Uhlenbeck, Boundary regularity and the Dirichlet problem for harmonic maps, *J. Differential Geom.* **18**(2) (1983), 253–268.

(25) N. N. Scoccola and D. R. Bes, Two dimensional skyrmions on the sphere, *J. High Energy Phys.* **1998**(9) (1998), 012.

(26) T. H. R. Skyrme, A non-linear field theory, *Proc. Roy. Soc. London Ser. A* **260** (1961), 127–138.

(27) T. H. R. Skyrme, A unified field theory of mesons and baryons, *Nuclear Phys.* **31** (1962), 556–569.

(28) S. L. Sondhi, A. Karlhede, S. A. Kivelson and E. H. Rezayi, Skyrmions and the crossover from the integer to fractional quantum Hall effect at small Zeeman energies, *Phys. Rev. B* **47**(24) (1993), 16419–16426.

(29) P. Sutcliffe, Multi-skyrmions with vector mesons, *Phys. Rev. D* **79**(8) (2009), 085014.

(30) R. S. Ward, Planar skyrmions at high and low density, *Nonlinearity* **17**(3) (2004), 1033–1040.

(31) T. Weidig, The baby Skyrme models and their multi-skyrmions, *Nonlinearity* **12**(6) (1999), 1489–1503.

https://doi.org/10.1017/S0013091523000056 Published online by Cambridge University Press