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ON TRACES OF BOCHNER REPRESENTABLE OPERATORS ON THE SPACE OF BOUNDED MEASURABLE FUNCTIONS

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Abstract Let Σ be a σ-algebra of subsets of a set Ω and $B(\Sigma)$ be the Banach space of all bounded Σ-measurable scalar functions on Ω. Let $\tau(B(\Sigma), ca(\Sigma))$ denote the natural Mackey topology on $B(\Sigma)$. It is shown that a linear operator T from $B(\Sigma)$ to a Banach space E is Bochner representable if and only if T is a nuclear operator between the locally convex space $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ and the Banach space E. We derive a formula for the trace of a Bochner representable operator $T : B(\mathcal{B}o) \to B(\mathcal{B}o)$ generated by a function $f \in L^1(\mathcal{B}o, C(\Omega))$, where Ω is a compact Hausdorff space.

Keywords: Bochner representable operators; kernel operators; nuclear operators; spaces of bounded measurable functions; trace of operator; vector measures

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1. Introduction and preliminaries

Let Σ be a σ-algebra of subsets of a set Ω and $B(\Sigma)$ be the Banach space of all bounded Σ-measurable scalar functions on Ω, equipped with the uniform norm $\|\cdot\|_\infty$. We assume that the field of scalars is either the set of real numbers or the set of complex numbers.

Let $ba(\Sigma)$ denote the Banach space of all bounded additive scalar-valued measures λ on Σ, equipped the total variation norm $\|\lambda\| := |\lambda|(\Omega)$. The Banach dual $B(\Sigma)'$ of $B(\Sigma)$ can be identified with $ba(\Sigma)$ throughout the mapping

$$\Phi : ba(\Sigma) \ni \lambda \mapsto \Phi_\lambda \in B(\Sigma)',$$

where $\Phi_\lambda(u) := \int_\Omega u(\omega)\, d\lambda$ for $u \in B(\Sigma)$ and $\|\Phi_\lambda\| = \|\lambda\|$. Let $ca(\Sigma)$ denote the closed subspace of $ba(\Sigma)$ consisting of all countably additive members of $ba(\Sigma)$.

From now on we assume that $(E, \|\cdot\|_E)$ is a Banach space and $(E', \|\cdot\|_{E'})$ denotes its dual. Assume that $m : \Sigma \to E$ is a finitely additive measure. By $|m|(A)$ (resp. $\|m\|(A)$)

we denote the variation (resp. semivariation) of m on A (see [7, Definition 4, p. 2]). Then $\|m\|(A) \leq |m|(A)$ for $A \in \Sigma$.

If $T : B(\Sigma) \to E$ is a bounded linear operator, let

$$m_T(A) = T(\mathbb{1}_A) \quad \text{for} \quad A \in \Sigma.$$

Then, $T(u) = \int_\Omega u(\omega) dm_T$ and $\|T\| = \|m_T\|(\Omega)$ (see [7, Theorem 13, p. 6]).

Different classes of linear operators $T : B(\Sigma) \to E$ (weakly compact, absolutely summing, nuclear, integral, σ-smooth) have been studied in numerous papers (see [5], [6], [7], [11], [18], [17]).

For $\mu \in ca(\Sigma)^+$, let $L^1(\mu, E)$ denote the Banach space of μ-equivalence classes of all E-valued Bochner μ-integrable functions f on Ω, equipped with norm $\|f\|_1 := \int_\Omega \|f(\omega)\|_E \, d\mu$.

Following [26] we can consider a class of linear operators on $B(\Sigma)$.

Definition 1.1. *We say that a linear operator $T : B(\Sigma) \to E$ is Bochner representable if there exist a measure $\mu \in ca(\Sigma)^+$ and a function $f \in L^1(\mu, E)$ so that*

$$T(u) = \int_\Omega u(\omega) \, f(\omega) \, d\mu, \quad \textit{for all} \quad u \in B(\Sigma).$$

The concept of nuclear operators between Banach spaces in due to Grothendieck [12], [13] (see also [28, p. 279], [21, Chap. 3], [22], [7, Chap. 6], [9, Chap. 5], [25], [23]).

Recall (see [28, p. 279], [25]) that a linear operator $T : B(\Sigma) \to E$ between Banach spaces $B(\Sigma)$ and E is said to be *nuclear* if there exist a bounded sequence (λ_n) in $ba(\Sigma)$, a bounded sequence (e_n) in E and a sequence $(\alpha_n) \in \ell^1$ so that

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \Phi_{\lambda_n}(u) \, e_n, \quad \text{for all } u \in B(\Sigma). \tag{1.1}$$

Then the *nuclear norm* of T is defined by

$$\|T\|_{nuc} := \inf \left\{ \sum_{n=1}^{\infty} |\alpha_n| \, |\lambda_n| \, (\Omega) \, \|e_n\|_E \right\},$$

where the infimum is taken over all sequences (λ_n) in $ba(\Sigma)$ and (e_n) in E and $(\alpha_n) \in \ell^1$ such that T admits a representation (1.1).

Let $\mathcal{L}(B(\Sigma), E)$ denote the Banach space of all bounded linear operators from $B(\Sigma)$ to E, equipped with the operator norm. Then in view of (1.1), we have

$$T = \sum_{n=1}^{\infty} \alpha_n \Phi_{\lambda_n} \otimes e_n \quad \text{in} \quad \mathcal{L}(B(\Sigma), E),$$

where $(\alpha_n \Phi_{\lambda_n} \otimes e_n)(u) = \alpha_n \Phi_{\lambda_n}(u) \, e_n$ for $u \in B(\Sigma)$.

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It is known that the space $\mathcal{N}(B(\Sigma),E)$ of all nuclear operators between $B(\Sigma)$ and E (equipped with the nuclear norm $\|\cdot\|_{nuc}$) is a Banach space (see [21, 3.1, Proposition, p. 51]).

Due to Diestel [5, Theorem 9] a bounded linear operator $T : B(\Sigma) \to E$ is nuclear if and only if m_T has an approximate Radon-Nikodym derivative with respect to its variation.

According to [18, Definition 2.1] we have

Definition 1.2. *A linear operator $T : B(\Sigma) \to E$ is said to be σ-smooth if $\|T(u_n)\|_E \to 0$ whenever (u_n) is a uniformly bounded sequence in $B(\Sigma)$ such that $u_n(\omega) \to 0$ for each $\omega \in \Omega$.*

By $\tau(B(\Sigma), ca(\Sigma))$ we denote the natural Mackey topology on $B(\Sigma)$. Note that $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ is a generalized DF-space, that is, $\tau(B(\Sigma), ca(\Sigma))$ is the finest locally convex topology agreeing with itself on norm-bounded sets in $B(\Sigma)$ (see [16], [18], [17], [11]).

The following characterization of σ-smooth operators $T : B(\Sigma) \to E$ will be useful (see [18, Proposition 2.2], [17, Proposition 3.1]).

Proposition 1.1. *For a bounded linear operator $T : B(\Sigma) \to E$, the following statements are equivalent:*

(i) *T is σ-smooth.*
(ii) *T is $(\tau(B(\Sigma), ca(\Sigma)), \|\cdot\|_E)$-continuous.*
(iii) *$m_T : \Sigma \to E$ is a countably additive measure.*

In this paper, we show that a linear operator $T : B(\Sigma) \to E$ is Bochner representable if and only if T is a nuclear σ-smooth operator and if and only if T is a nuclear operator between the locally convex space $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ and the Banach space E (see Corollary 2.5 below). We derive a formula for the trace of a Bochner representable operator $T : B(\mathcal{B}o) \to B(\mathcal{B}o)$ generated by a function $f \in L^1(\mathcal{B}o, C(\Omega))$, where Ω is a compact Hausdorff space (see Corollary 3.1 below).

2. Nuclearity of Bochner representable operators on $B(\Sigma)$

We will need the following result (see [16, Theorem 3], [20, Proposition 13 and Corollary 14]).

Proposition 2.1. *For a subset $\mathcal{M}$ of $ca(\Sigma)$, the following statements are equivalent:*

(i) *The family $\{\Phi_\lambda : \lambda \in \mathcal{M}\}$ is $\tau(B(\Sigma), ca(\Sigma))$-equicontinuous.*
(ii) *$\sup_{\lambda \in \mathcal{M}} \|\lambda\| < \infty$ and $\mathcal{M}$ is uniformly countably additive.*

Grothendieck carried over the concept of nuclear operators to locally convex spaces [12], [13] (see also [28, p. 289–293], [15, pp. 376–378], [24, Chap. 3, § 7], [27, § 47]). Following [24, Chap. 3, § 7], [27, § 47] and using Proposition 2.1 we have the following definition.

Definition 2.1. *A linear operator $T : B(\Sigma) \to E$ between the locally convex space $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ and a Banach space E is said to be nuclear if there exist a bounded*

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and uniformly countably additive sequence (λ_n) *in* $ca(\Sigma)$*, a bounded sequence* (e_n) *in* E *and a sequence* $(\alpha_n) \in \ell^1$ *such that*

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \left(\int_\Omega u(\omega)\, d\lambda_n \right) e_n \quad \text{for all} \quad \mathrm{u} \in \mathrm{B}(\Sigma). \tag{2.1}$$

Then $T : B(\Sigma) \to E$ is $(\tau(B(\Sigma), ca(\Sigma)), \|\cdot\|_E)$-compact, that is, $T(V)$ is relatively norm compact in E for some $\tau(B(\Sigma), ca(\Sigma))$-neighbourhood V of 0 in $B(\Sigma)$ (see [24, Chap. 3, § 7, Corollary 1], [27, Theorem 47.3]). Hence T is $(\tau(B(\Sigma), ca(\Sigma)), \|\cdot\|_E)$-continuous.

Let us put

$$\|T\|_{\tau-nuc} := \inf \left\{ \sum_{n=1}^{\infty} |\alpha_n|\, |\lambda_n|\, (\Omega)\, \|e_n\|_E \right\},$$

where the infimum is taken over all sequences (λ_n) in $ca(\Sigma)$ and (e_n) in E and $(\alpha_n) \in \ell^1$ such that T admits a representation (2.1).

According to [19, Theorem 2.1] and Proposition 1.1 we have the following characterization of nuclear σ-smooth operators $T : B(\Sigma) \to E$.

Theorem 2.2. *Assume that* $T : B(\Sigma) \to E$ *is a* σ*-smooth operator. Then the following statements are equivalent:*

- (i) T *is a nuclear operator between the Banach spaces* $B(\Sigma)$ *and* E.
- (ii) $|m_T|(\Omega) < \infty$ *and* m_T *has a* $|m_T|$*-Bochner integrable derivative, that is, there exists a function* $f \in L^1(|m_T|, E)$ *so that* $m_T(A) = \int_A f(\omega)\, d|m_T|$ *for all* $A \in \Sigma$.
- (iii) $|m_T|(\Omega) < \infty$ *and* T *is a* $|m_T|$*-Bochner integrable kernel, that is, there exists a function* $f \in L^1(|m_T|, E)$ *so that* $T(u) = \int_\Omega u(\omega) f(\omega)\, d|m_T|$ *for all* $u \in B(\Sigma)$.
- (iv) T *is a nuclear operator between the locally convex space* $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ *and the Banach space* E.

In this case, $\|T\|_{nuc} = \|T\|_{\tau-nuc} = |m_T|(\Omega)$.

Making us of [8, Sect.2, F, Theorem 30, p. 26] we have the following result.

Lemma 2.3. *For* $\mu \in ca(\Sigma)^+$ *and* $f \in L^1(\mu, E)$*, let us put*

$$\lambda(A) := \int_A \|f(\omega)\|_E\, d\mu, \quad \textit{for all} \quad A \in \Sigma,$$

and

$$h_f(\omega) := f(\omega)/\|f(\omega)\|_E \quad \textit{if} \quad f(\omega) \neq 0 \quad \textit{and} \quad h_f(\omega) := 0 \quad \textit{if} \quad f(\omega) = 0.$$

Then $h_f \in L^1(\lambda, E)$ *and*

$$\int_\Omega u(\omega) h_f(\omega)\, d\lambda = \int_\Omega u(\omega) f(\omega)\, d\mu, \quad \textit{for all} \quad u \in B(\Sigma).$$

In particular, $\int_A h_f(\omega)\, d\lambda = \int_A f(\omega)\, d\mu$ *for all* $A \in \Sigma$.

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Theorem 2.4. *Assume that $T : B(\Sigma) \to E$ is a Bochner representable operator. Then T is a nuclear operator between the locally convex space $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ and the Banach space E.*

Proof. There exists a measure $\mu \in ca(\Sigma)^+$ and a function $f \in L^1(\mu, E)$ so that

$$T(u) = \int_\Omega u(\omega) f(\omega)\, d\mu, \quad \text{for all} \ \ u \in B(\Sigma).$$

Hence

$$m_T(A) = \int_A f(\omega)\, d\mu \ \ \text{and} \ \ |m_T|(A) = \int_A \|f(\omega)\|_E\, d\mu, \quad \text{for all} \ \ A \in \Sigma,$$

where m_T is a countably additive measure (see [7, Theorem 4, p. 46]), and in view of Proposition 1.1 T is σ-smooth. Hence using Lemma 2.3 we get

$$m_T(A) = \int_A f(\omega)\, d\mu = \int_A h_f(\omega)\, d|m_T|, \quad \text{for all} \ \ A \in \Sigma,$$

where $h_f \in L^1(|m_T|, E)$. By Theorem 2.2 we derive that T is a nuclear operator between the locally convex space $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ and the Banach space E. □

In view of Theorem 2.4 and Theorem 2.2 we can obtain the following characterization of Bochner representable operators $T : B(\Sigma) \to E$.

Theorem 2.5. *For a linear operator $T : B(\Sigma) \to E$, the following statements are equivalent:*

(i) *T is a Bochner representable operator.*
(ii) *T is a nuclear operator between the locally convex space $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ and the Banach space E.*
(iii) *T is a σ-smooth nuclear operator between the Banach spaces $B(\Sigma)$ and E.*

As a consequence of Theorem 2.4 and Theorem 2.2, we get

Corollary 2.6. *Assume that $T : B(\Sigma) \to E$ is a Bochner representable operator. Then the mapping*

$$T^* : E' \ni e' \mapsto e' \circ m_T \in ca(\Sigma)$$

is a nuclear operator and $\|T^*\|_{nuc} = \|T\|_{nuc} = |m_T|(\Omega)$.

Proof. Let $\varepsilon > 0$ be given. In view of Theorem 2.4 and Theorem 2.2 there exist a bounded and uniformly countably additive sequence (λ_n) in $ca(\Sigma)$, a bounded sequence (e_n) in E and $(\alpha_n) \in \ell^1$ so that

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \Phi_{\lambda_n}(u)\, e_n, \quad \text{for all} \;\; u \in B(\Sigma)$$

and

$$\sum_{n=1}^{\infty} |\alpha_n|\, |\lambda_n|(\Omega)\, \|e_n\|_E \le |m_T|(\Omega) + \varepsilon. \tag{2.5}$$

One can show that for each $e' \in E'$, we have

$$e' \circ T = \sum_{n=1}^{\infty} \alpha_n\, e'(e_n) \Phi_{\lambda_n} \;\; \text{in} \;\; B(\Sigma)'.$$

Moreover, for each $e' \in E'$, we have $e' \circ m_T \in ca(\Sigma)$ and

$$(e' \circ T)(u) = \int_{\Omega} u(\omega)\, d(e' \circ m_T), \quad \text{for all} \;\; u \in B(\Sigma).$$

Let $i : E \to E''$ stand for the canonical isometry, that is, $i(e)(e') = e'(e)$ for $e \in E$, $e' \in E'$ and $\|i(e)\|_{E''} = \|e\|_E$. Hence for each $e' \in E'$, we get

$$T^*(e') = e' \circ m_T = \Phi^{-1}(e' \circ T) = \sum_{n=1}^{\infty} \alpha_n\, i(e_n)(e')\, \lambda_n.$$

This means that T^* is a nuclear operator and by (2.5) we get $\|T^*\|_{nuc} \le |m_T|(\Omega)$.

Now, we shall show that

$$|m_T|(\Omega) \le \|T^*\|_{nuc}.$$

Let $\varepsilon > 0$ be given. Since T^* is a nuclear operator, there exist a bounded sequence (e''_n) in E'', a bounded sequence (λ_n) in $ca(\Sigma)$ and $(\alpha_n) \in \ell^1$ so that

$$T^*(e') = \sum_{n=1}^{\infty} \alpha_n\, e''_n(e')\, \lambda_n \;\; \text{for} \;\; e' \in E'$$

and

$$\sum_{n=1}^{\infty} |\alpha_n| \|e''_n\|_{E''} |\lambda_n|(\Omega) \le \|T^*\|_{nuc} + \varepsilon. \tag{2.6}$$

Then for $A \in \Sigma$, we obtain

$$(e' \circ m_T)(A) = T^*(e')(A) = \sum_{n=1}^{\infty} \alpha_n\, e''_n(e')\, \lambda_n(A).$$

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Moreover, by the Hahn-Banach theorem for every $A \in \Sigma$, there exists $e'_A \in E'$ with $\|e'_A\|_{E'} = 1$ such that $\|m_T(A)\|_E = |(e'_A \circ m_T)(A)|$. Hence, if Π is a finite Σ-partition of Ω, then using (2.6) we have

$$\begin{aligned}\sum_{A\in\Pi} \|m_T(A)\|_E &= \sum_{A\in\Pi} |(e'_A \circ m_T)(A)| = \sum_{A\in\Pi} \left|\sum_{n=1}^{\infty} \alpha_n\, e''_n(e'_A)\, \lambda_n(A)\right| \\ &\le \sum_{A\in\Pi} \left(\sum_{n=1}^{\infty} |\alpha_n|\, |e''_n(e'_A)|\, |\lambda_n(A)|\right) \le \sum_{n=1}^{\infty} \left(|\alpha_n| \|e''_n\|_{E''} \sum_{A\in\Pi} |\lambda_n(A)|\right) \\ &\le \sum_{n=1}^{\infty} |\alpha_n| \|e''_n\|_{E''} |\lambda_n|(\Omega) \le \|T^*\|_{nuc} + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $|m_T|(\Omega) \le \|T^*\|_{nuc}$ and finally $\|T^*\|_{nuc} = |m_T|(\Omega) = \|T\|_{nuc}$. □

3. Traces of Bochner representable operators

Formulas for the traces of kernel operators on Banach function spaces (in particular, $L^p(\mu)$-spaces) have been the object of much study (see [14], [2], [4], [10], [22]).

Grothendieck [13, Chap. I, p. 165] showed that the notion of 'trace' can be defined for nuclear operators in Banach spaces with the approximation property (see [22, 4.6.2, Lemma, pp. 210–211]).

Recall that a Banach space $(X, \|\cdot\|_X)$ has the *approximation property* if for every compact subset K of X and every $\varepsilon > 0$ there exists a bounded finite rank operator $S : X \to X$ such that $\|x - S(x)\|_X \le \varepsilon$ for every $x \in K$ (see [23, Chap. 4, p. 72], [7, Definition 1, p. 238]).

Note that the Banach space $B(\Sigma)$ has the approximation property. Assume first that $B(\Sigma)$ is the Banach lattice of all bounded Σ-measurable real functions on Ω. Since $(B(\Sigma), \|\cdot\|_\infty)$ is an AM-space with the unit $\mathbb{1}_\Omega$, due to the Kakutani-Bohnenblust-M. and S. Krein theorem (see [1, Theorem 3.40]) $B(\Sigma)$ is lattice isometric to some $C(K)$-space for a unique (up to homeomorphism) compact Hausdorff space K in such a way that $\mathbb{1}_\Omega$ is identified with $\mathbb{1}_K$. This follows that $B(\Sigma)$ has the approximation property because $C(K)$ has the approximation property (see [23, Example 4.2]). For the Banach space $B(\Sigma)$ of complex-valued functions on Ω, one has to consider real and imaginary parts separate.

Assume that $T : B(\Sigma) \to B(\Sigma)$ is a nuclear operator, that is, there exist a bounded sequence (λ_n) in $ba(\Sigma)$, a bounded sequence (w_n) in $B(\Sigma)$ and $(\alpha_n) \in \ell^1$ so that

$$T = \sum_{n=1}^{\infty} \alpha_n \Phi_{\lambda_n} \otimes w_n \quad \text{in} \quad \mathcal{L}(B(\Sigma), B(\Sigma)). \tag{3.1}$$

Then the *trace* of T is given by

$$\operatorname{tr} T := \sum_{n=1}^{\infty} \alpha_n \Phi_{\lambda_n}(w_n) = \sum_{n=1}^{\infty} \alpha_n \int_\Omega w_n(\omega)\, d\lambda_n,$$

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and it does not depend on the special choice of the nuclear representation (3.1) of T (see [10, Chap. 5, Theorem 1.2], [22, Lemma, pp. 210–211]).

From now on we assume that $(\Omega, \mathcal{T})$ is a compact Hausdorff space and $\mathcal{B}o$ denotes the σ-algebra of Borel sets in Ω. Then $C(\Omega) \subset B(\mathcal{B}o)$.

Assume that a measure $\mu \in ca^+(\mathcal{B}o)$ is strictly positive, that is, for all $U \in \mathcal{T}$ with $U \neq \emptyset$, $\mu(U) > 0$. Then $L^1(\mu, C(\Omega)) \subset L^1(\mu, B(\mathcal{B}o))$.

Corollary 3.1. *Assume that $T : B(\mathcal{B}o) \to B(\mathcal{B}o)$ is a Bochner representable operator such that*

$$T(u) = \int_\Omega u(\omega)\, f(\omega)\, d\mu, \quad \textit{for all } u \in B(\mathcal{B}o),$$

where $f \in L^1(\mu, C(\Omega))$. Then T has a well-defined trace

$$\operatorname{tr} T = \int_\Omega f(\omega)(\omega)\, d\mu.$$

Proof. Let $L^1(\mu) \hat{\otimes} C(\Omega)$ denote the projective tensor product of $L^1(\mu)$ and $C(\Omega)$, equipped with the completed norm π (see [7, p. 227], [23, p. 17]). Note that for $z \in L^1(\mu) \hat{\otimes} C(\Omega)$, we have

$$\pi(z) = \inf \left\{ \sum_{n=1}^{\infty} |\alpha_n|\, \|v_n\|_1 \|w_n\|_\infty \right\},$$

where the infimum is taken over all sequences (v_n) in $L^1(\mu)$ and (w_n) in $C(\Omega)$ with $\lim_n \|v_n\|_1 = 0 = \lim_n \|w_n\|_\infty$ and $(\alpha_n) \in \ell^1$ such that $z = \sum_{n=1}^\infty \alpha_n\, v_n \otimes w_n$ in π-norm (see [23, Proposition 2.8, pp. 21–22]).

It is known that $L^1(\mu) \hat{\otimes} C(\Omega)$ is isometrically isomorphic to the Banach space $(L^1(\mu, C(\Omega)), \|\cdot\|_1)$ by the isometry J, defined by:

$$J(v \otimes w) := v\,(\cdot)\, w \;\text{ for } v \in L^1(\mu),\; w \in C(\Omega)$$

(see [7, Example 10, p. 228], [23, Example 2.19, p. 29]). Then there exist sequences (v_n) in $L^1(\mu)$ and (w_n) in $C(\Omega)$ with $\lim_n \|v_n\|_1 = 0 = \lim_n \|w_n\|_\infty$ and $(\alpha_n) \in \ell^1$ such that

$$J^{-1}(f) = \sum_{n=1}^{\infty} \alpha_n\, v_n \otimes w_n \;\text{ in } \big(L^1(\mu) \hat{\otimes} C(\Omega), \pi\big).$$

Thus it follows that

$$f = J\left(\sum_{n=1}^{\infty} \alpha_n v_n \otimes w_n\right) = \sum_{n=1}^{\infty} \alpha_n v_n(\cdot)\, w_n \;\text{ in }\; L^1(\mu, C(\Omega)),$$

and hence

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \left(\int_\Omega u(\omega)\, v_n(\omega)\, d\mu \right) w_n, \quad \text{for all} \ \ u \in B(\Sigma).$$

For $n \in \mathbb{N}$, let

$$\lambda_n(A) := \int_A v_n(\omega)\, d\mu, \quad \text{for all} \ \ A \in \Sigma.$$

Note that $\lambda_n \in ca(\Sigma)$ and $|\lambda_n|(\Omega) = \|v_n\|_1$ and hence $\lim \lambda_n(A) = 0$ for all $A \in \Sigma$. By the Nikodym convergence theorem (see [9, Theorem 8.6]), the family $\{\lambda_n : n \in \mathbb{N}\}$ is uniformly countably additive.

Since $\Phi_{\lambda_n}(u) = \int_\Omega u(\omega)\, d\lambda_n = \int_\Omega u(\omega)\, v_n(\omega)\, d\mu$ for all $u \in B(\Sigma)$ (see [3, Theorem 8C, p. 380]), we get

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \Phi_{\lambda_n}(u) w_n, \quad \text{for all} \ \ u \in B(\Sigma),$$

that is,

$$T = \sum_{n=1}^{\infty} \alpha_n \Phi_{\lambda_n} \otimes w_n \ \ \text{in} \ \ \mathcal{L}(B(\Sigma), B(\Sigma)).$$

Hence

$$\operatorname{tr} T = \sum_{n=1}^{\infty} \alpha_n \Phi_{\lambda_n}(w_n) = \sum_{n=1}^{\infty} \alpha_n \int_\Omega w_n(\omega)\, v_n(\omega)\, d\mu.$$

For $n \in \mathbb{N}$, let $f_n = \sum_{i=1}^{n} \alpha_i\, v_i(\cdot)\, w_i$. Hence $\int_\Omega \|f(\omega) - f_n(\omega)\|_\infty\, d\mu \to 0$. Thus we get,

$$\begin{aligned} &\left| \int_\Omega f(\omega)(\omega)\, d\mu - \sum_{i=1}^{n} \alpha_i \int_\Omega v_i(\omega)\, w_i(\omega)\, d\mu \right| \\ &\leq \int_\Omega \left| \left(f(\omega)(\omega) - \sum_{i=1}^{n} \alpha_i v_i(\omega)\, w_i(\omega) \right) \right| d\mu \leq \int_\Omega \|f(\omega) - f_n(\omega)\|_\infty\, d\mu. \end{aligned}$$

Let $g \in L^1(\mu, C(\Omega))$ be another function representing T, that is,

$$T(u)(t) = \int_\Omega u(\omega)\, f(\omega)(t)\, d\mu(\omega) = \int_\Omega u(\omega)\, g(\omega)(t)\, d\mu(\omega) \ \ \text{for} \ \ u \in B(\mathcal{B}o).$$

Denote $h(\omega)(t) := f(\omega)(t) - g(\omega)(t)$ for $\omega, t \in \Omega$. Then for every $A \in \mathcal{B}o$ and $u = \mathbb{1}_A$ we obtain

$$\int_A h(\omega)(t)\, d\mu(\omega) = 0 \ \ \text{for all} \ \ t \in \Omega.$$

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Hence for every $t \in \Omega$, $h(\cdot)(t) = 0$ μ-a.e and it follows that

$$\int_\Omega \left(\int_\Omega |h(\omega)(t)|\, d\mu(\omega) \right) d\mu(t) = 0. \tag{3.2}$$

We shall show that

$$\int_\Omega h(\omega)(\omega)\, d\mu(\omega) = 0.$$

For indirect proof suppose that $\left| \int_\Omega h(\omega)(\omega)\, d\mu(\omega) \right| > 0$. Then there exists $A \in \mathcal{B}o$, $\mu(A) \neq 0$ such that $h(\omega)(\omega) > 0$ or $h(\omega)(\omega) < 0$ for $\omega \in A$. Without loss of generality, let $h(\omega)(\omega) > 0$ for $\omega \in A$. Since for $\omega \in \Omega$ we have $h(\omega) \in C(\Omega)$, then there exists a neighbourhood H_ω of $\omega \in A$ such that

$$h(\omega)(t) > 0 \quad \text{for every} \quad t \in H_\omega.$$

Since μ is strictly positive, then for every $\omega \in A$, $\mu(H_\omega) > 0$ and hence

$$\int_{H_\omega} h(\omega)(t)\, d\mu(t) > 0.$$

Let $\omega_0 \in A$ be given. Then, we have

$$\int_\Omega |h(\omega_0)(t)|\, d\mu(t) \geq \int_{\bigcup H_\omega} |h(\omega_0)(t)|\, d\mu(t) \geq \int_{H_{\omega_0}} |h(\omega_0)(t)|\, d\mu(t) > 0.$$

Since ω_0 is arbitrary, it follows that

$$\int_\Omega \left(\int_\Omega |h(\omega)(t)|\, d\mu(t) \right) d\mu(\omega) > 0$$

and, in view of Hille's theorem (see [8, § 1, Theorem 36, p. 16]), this is in contradiction with (3.2). Hence we finally get

$$\int_\Omega h(\omega)(\omega)\, d\mu(\omega) = 0.$$

Thus this follows that the trace of T is well defined and $\operatorname{tr} T = \int_\Omega f(\omega)(\omega)\, d\mu$. □

Grothendieck [14] showed that if Ω is a compact Hausdorff space with a positive Borel measure μ on Ω and $k(\cdot,\cdot) \in C(\Omega \times \Omega)$, then the kernel operator $T_k : C(\Omega) \to C(\Omega)$ defined by:

$$T_k(u) := \int_\Omega u(\omega)\, k(\cdot, \omega)\, d\mu \quad \text{for} \quad u \in C(\Omega),$$

is nuclear and has a well-defined trace $\operatorname{tr} T_k = \int_\Omega k(\omega,\omega)\, d\mu$ (see [14], [22, 6.6.2, Theorem, p. 274]).

Now, we can extend this formula for the trace of kernel operators $T_k : B(\mathcal{B}o) \to B(\mathcal{B}o)$.

Let $k(\cdot,\cdot) \in C(\Omega \times \Omega)$. Hence for every $\omega \in \Omega$, $k(\cdot,\omega) \in C(\Omega)$. Let $C(\Omega, C(\Omega))$ denote the Banach space of all continuous functions $f : \Omega \to C(\Omega)$, equipped with the uniform norm $\|\cdot\|_\infty$.

Assume that $\mu \in ca(\mathcal{B}o)^+$. Let $\mathcal{L}^\infty(\mu, C(\Omega))$ denote the space of all μ-measurable functions $g : \Omega \to C(\Omega)$ such that $\mu - \operatorname{ess\,sup} \|g(\omega)\|_\infty < \infty$. In view of the Pettis measurability theorem (see [7, Theorem 2, p. 42]), we have

$$C(\Omega, C(\Omega)) \subset \mathcal{L}^\infty(\mu, C(\Omega)), \tag{3.3}$$

and the space $\mathcal{L}^\infty(\mu, C(\Omega))$ can be embedded in the space $L^1(\mu, C(\Omega))$ such that with each function from $\mathcal{L}^\infty(\mu, C(\Omega))$ is associated its μ-equivalence class in $L^1(\mu, C(\Omega))$.

It is well known (see [22, 6.1.4, p. 243]) that the function:

$$f : \Omega \ni \omega \mapsto k(\cdot,\omega) \in C(\Omega),$$

is bounded and continuous. Then in view of (3.3), $f \in \mathcal{L}^\infty(\mu, C(\Omega))$. Hence its μ-equivalence class belongs to $L^1(\mu, C(\Omega))$. Thus it follows that one can define the kernel operator $T_k : B(\mathcal{B}o) \to B(\mathcal{B}o)$ by

$$T_k(u) := \int_\Omega u(\omega)\, k(\cdot,\omega)\, d\mu, \quad \text{for all } \ u \in B(\mathcal{B}o).$$

For $t \in \Omega$, let $\Phi_t(u) = u(t)$ for all $u \in B(\mathcal{B}o)$. Then $\Phi_t \in C(\Omega)'$ and using Hille's theorem, for all $u \in B(\mathcal{B}o)$, $t \in \Omega$, we get

$$T_k(u)(t) = \int_\Omega u(\omega)\, \Phi_t(k(\cdot,\omega))\, d\mu = \int_\Omega u(\omega)\, k(t,\omega)\, d\mu.$$

As a consequence of Theorem 2.2 and Corollary 3.1, we get

Corollary 3.2. *The kernel operator $T_k : B(\mathcal{B}o) \to B(\mathcal{B}o)$ is nuclear σ-smooth and*

$$\operatorname{tr} T_k = \int_\Omega k(\omega,\omega)\, d\mu.$$

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