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RESEARCH ARTICLE

$P = W$ for Lagrangian fibrations and degenerations of hyper-Kähler manifolds

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Abstract
We identify the perverse filtration of a Lagrangian fibration with the monodromy weight filtration of a maximally unipotent degeneration of compact hyper-Kähler manifolds.

1. Throughout, we work over the complex numbers $\mathbb{C}$. Let M be an irreducible holomorphic symplectic variety or, equivalently, a projective hyper-Kähler manifold. Assume that it admits a (holomorphic) Lagrangian fibration $\pi : M \to B$. The perverse t-structure on the constructible derived category $D_c^b(B, \mathbb{Q})$ induces a perverse filtration on the cohomology of M,

$$P_\bullet H^*(M, \mathbb{Q}).$$

We refer to [1, 9] for the conventions of the perverse filtration.

2. Let $f : \mathcal{M} \to \Delta$ be a projective degenerating family of hyper-Kähler manifolds over the unit disk. For $t \in \Delta^*$, let N denote the logarithmic monodromy operator on $H^2(\mathcal{M}_t, \mathbb{Q})$. The degeneration $f : \mathcal{M} \to \Delta$ is called of type III if

$$N^2 \neq 0, \quad N^3 = 0.$$

By [5, Proposition 7.14], this is equivalent to having maximally unipotent monodromy. See the rest of [5] and also [3, 8] for more discussions on degenerations of hyper-Kähler manifolds.

Let

$$\left(H^*_{\lim}(\mathbb{Q}), W_\bullet H^*_{\lim}(\mathbb{Q}), F_\bullet H^*_{\lim}(\mathbb{C})\right)$$

denote the limiting mixed Hodge structure[1] associated with $f : \mathcal{M} \to \Delta$. In this short note, we prove the following result relating the perverse and the monodromy weight filtrations.

[1]Similar to the perverse filtration, we consider the Hodge filtration as an increasing filtration.

3. Theorem. *For any Lagrangian fibration $\pi : M \to B$, there exists a type III projective degeneration of hyper-Kähler manifolds $f : \mathcal{M} \to \Delta$ with $\mathcal{M}_t$ deformation equivalent to M for all $t \in \Delta^*$, such that*

$$P_k H^*(M, \mathbb{Q}) = W_{2k} H^*_{\lim}(\mathbb{Q}) = W_{2k+1} H^*_{\lim}(\mathbb{Q}) \tag{1}$$

through an identification of the cohomology algebras $H^*(M, \mathbb{Q}) = H^*_{\lim}(\mathbb{Q})$.

Because M and $\mathcal{M}_t$ are deformation equivalent, and hence diffeomorphic, they share the same cohomology. The limiting mixed Hodge structure can be viewed as supported on the cohomology of $\mathcal{M}_t$, which provides the required identification $H^*(M, \mathbb{Q}) = H^*_{\lim}(\mathbb{Q})$. This identification will be built into the construction of the degeneration $f : \mathcal{M} \to \Delta$.

4. Theorem 3 was previously conjectured by the first author in [4, Conjecture 1.4] and proven in the case of $K3$ surfaces.

The interaction between the perverse and the weight filtrations for certain (noncompact) hyper-Kähler manifolds was first discovered by de Cataldo, Hausel, and Migliorini [1], which is now referred to as the $P = W$ conjecture. More precisely, the $P = W$ conjecture identifies the perverse filtration of a Hitchin fibration with the weight filtration of the mixed Hodge structure of the corresponding character variety through Simpson's nonabelian Hodge theory [11]. Theorem 3 can be viewed as a direct analogue of this conjecture.

5. Theorem 3 also offers conceptual explanations to the main results in [9]. As is remarked in [4, Introduction], a recent result of Soldatenkov [12, Theorem 3.8] shows that limiting mixed Hodge structure for type III degenerations is of Hodge-Tate type.[2] In particular, we have

$$\dim_{\mathbb{Q}} \mathrm{Gr}^W_{2i} H^{i+j}_{\lim}(\mathbb{Q}) = \dim_{\mathbb{C}} \mathrm{Gr}^F_i H^{i+j}_{\lim}(\mathbb{C}).$$

Coupled with the equalities (by (1) and the definition of the limiting Hodge filtration)

$$\dim_{\mathbb{Q}} \mathrm{Gr}^P_i H^{i+j}(M, \mathbb{Q}) = \dim_{\mathbb{Q}} \mathrm{Gr}^W_{2i} H^{i+j}_{\lim}(\mathbb{Q}),$$
$$\dim_{\mathbb{C}} \mathrm{Gr}^F_i H^{i+j}_{\lim}(\mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Gr}^F_i H^{i+j}(\mathcal{M}_t, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Gr}^F_i H^{i+j}(M, \mathbb{C}),$$

this yields the 'Perverse = Hodge' equality in [9, Theorem 0.2],

$$\dim_{\mathbb{Q}} \mathrm{Gr}^P_i H^{i+j}(M, \mathbb{Q}) = \dim_{\mathbb{C}} \mathrm{Gr}^F_i H^{i+j}(M, \mathbb{C}).$$

See [9, Section 0.4] for various applications of this equality.

Moreover, the $P = W$ identity (1) implies the multiplicativity of the perverse filtration

$$\cup : P_k H^d(M, \mathbb{Q}) \times P_{k'} H^{d'}(M, \mathbb{Q}) \to P_{k+k'} H^{d+d'}(M, \mathbb{Q})$$

through the general fact that the monodromy weight filtration is multiplicative. The latter may follow from a combination of results of Fujisawa and Steenbrink. Fujisawa [2, Lemma 6.16] proved that the wedge product on the relative logarithmic de Rham complex of a projective semistable degeneration induces a cup product on the hypercohomology groups that respects a particular weight filtration. In a much earlier work [13, Section 4], Steenbrink identified the hypercohomology of the relative logarithmic de Rham complex with the cohomology of the nearby fibre, in such a way that the cup product matches the topological cup product and the weight filtration corresponds to the monodromy weight filtration. Alternatively, as the referee pointed out,[3] monodromy acts on cohomology by algebra automorphisms. The logarithmic monodromy operator then acts on the cohomology algebra as a derivation, which yields the multiplicativity of the monodromy weight filtration. This recovers [9, Theorem A.1].

[2]This parallels the fact that the mixed Hodge structure of character varieties is of Hodge-Tate type; see [10].
[3]We thank the referee for suggesting this simpler argument.

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Because the proof of Theorem 3 uses the same ingredients as in [9], our new way of deriving these results is not logically independent.

6. We now prove Theorem 3 and we make free use of the statements in [9]. To fix some notation, let $\pi : M \to B$ be a Lagrangian fibration with $\dim M = 2\dim B = 2n$. The second cohomology group $H^2(M,\mathbb{Z})$ (respectively $H^2(M,\mathbb{Q})$) is equipped with the Beauville-Bogomolov-Fujiki quadratic form $q_M(-)$ of signature $(3, b_2(M)-3)$, where $b_2(M)$ is the second Betti number of M.

Let $\eta \in H^2(M,\mathbb{Q})$ be a π-relative ample class, and let $\beta \in H^2(M,\mathbb{Q})$ be the pullback of an ample class on B. We have $q_M(\beta) = 0$ and, by taking $\mathbb{Q}$-linear combinations of η and β, we may assume $q_M(\eta) = 0$. Note that in this case, we have $b_2(M) \geq 4$.

7. Consider the following operators on the cohomology $H^*(M,\mathbb{Q})$:

$$L_\eta(-) = \eta \cup -, \quad L_\beta(-) = \beta \cup -.$$

In [9, Section 3.1], it was shown that L_η and L_β form $\mathfrak{sl}_2$-triples $(L_\eta, H_\eta, \Lambda_\eta)$ and $(L_\beta, H_\beta, \Lambda_\beta)$, which generate an $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-action on $H^*(M,\mathbb{Q})$. The action induces a weight decomposition

$$H^*(M,\mathbb{Q}) = \bigoplus_{i,j} P^{i,j} \tag{2}$$

with

$$H_\eta|_{P^{i,j}} = (i-n)\,\mathrm{id}, \quad H_\beta|_{P^{i,j}} = (j-n)\,\mathrm{id}.$$

A key observation in [9, Proposition 1.1] is that (2) provides a canonical splitting of the perverse filtration $P_\bullet H^*(M,\mathbb{Q})$. More precisely, we have

$$P_k H^d(M,\mathbb{Q}) = \bigoplus_{\substack{i+j=d \\ i\leq k}} P^{i,j}. \tag{3}$$

8. The $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-action above is part of a larger Lie algebra action on $H^*(M,\mathbb{Q})$ introduced by Looijenga-Lunts [7, Section 4] and Verbitsky [14, 15]. The Looijenga-Lunts-Verbitsky algebra

$$\mathfrak{g} \subset \mathrm{End}\big(H^*(M,\mathbb{Q})\big)$$

is defined to be the Lie subalgebra generated by all $\mathfrak{sl}_2$-triples $(L_\omega, H, \Lambda_\omega)$ with $\omega \in H^2(M,\mathbb{Q})$ such that $L_\omega(-) = \omega \cup -$ satisfies hard Lefschetz.

Given a $\mathbb{Q}$-vector space V equipped with a quadratic form q, we define the Mukai extension

$$\widetilde{V} = V \oplus \mathbb{Q}^2, \quad \tilde{q} = q \oplus \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right).$$

Looijenga–Lunts [7, Proposition 4.5] and Verbitsky [15, Theorem 1.4] showed independently that

$$\mathfrak{g} \simeq \mathfrak{so}(\widetilde{H}^2(M,\mathbb{Q}), \tilde{q}_M), \quad \mathfrak{g}_{\mathbb{R}} \simeq \mathfrak{so}(4, b_2(M)-2).$$

Here the statement with $\mathbb{Q}$-coefficients is taken from [3, Theorem 2.7]. Moreover, there is a weight decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ with natural isomorphisms

$$\mathfrak{g}_{-2} \simeq H^2(M,\mathbb{Q}), \quad \mathfrak{g}_0 \simeq \mathfrak{so}(H^2(M,\mathbb{Q}), q_M) \oplus \langle H \rangle, \quad \mathfrak{g}_2 \simeq H^2(M,\mathbb{Q}). \tag{4}$$

Another relevant Lie algebra is generated by the $\mathfrak{sl}_2$-triples associated with η, β and a third element $\rho \in H^2(M,\mathbb{Q})$ satisfying

$$q_M(\rho) > 0, \quad q_M(\eta,\rho) = q_M(\beta,\rho) = 0.$$

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Such a ρ exists by the signature $(3, b_2(M) - 3)$ of q_M. Let $\mathfrak{g}_\rho \subset \mathfrak{g}$ denote this Lie subalgebra and let

$$V_\rho = \langle \eta, \beta, \rho \rangle \subset H^2(M, \mathbb{Q}).$$

By [9, Corollary 2.6] complemented with the argument in [3, Theorem 2.7], we have

$$\mathfrak{g}_\rho \simeq \mathfrak{so}(\widetilde{V}_\rho,\ \tilde{q}_M|_{\widetilde{V}_\rho}). \tag{5}$$

The $\mathfrak{g}_\rho$-action on $H^*(M, \mathbb{Q})$ induces the same weight decomposition as (2); see [9, Section 3.1].

9. Recall the natural isomorphism $\bigwedge^2 H^2(M, \mathbb{Q}) \simeq \mathfrak{so}(H^2(M, \mathbb{Q}), q_M)$ defined by

$$a \wedge b \mapsto \frac{1}{2} q_M(a, -)\, b - \frac{1}{2} q_M(b, -)\, a.$$

As in [12, Lemma 4.1], we obtain a nilpotent operator $N_{\beta,\rho} = \beta \wedge \rho \in \mathfrak{so}(H^2(M, \mathbb{Q}), q_M)$ whose action on $H^2(M, \mathbb{Q})$ satisfies

$$\mathrm{Im}(N_{\beta,\rho}) = \langle \beta, \rho \rangle, \quad \mathrm{Im}(N_{\beta,\rho}^2) = \langle \beta \rangle, \quad N_{\beta,\rho}^3 = 0.$$

By [6, Lemma 3.9] and the assumption $q_M(\beta, \rho) = 0$, we can further identify $N_{\beta,\rho}$ with the commutator $[L_\beta, \Lambda_\rho] \in \mathfrak{g}_0$ through the isomorphisms (4). Note that $N_{\beta,\rho} = [L_\beta, \Lambda_\rho] \in \mathfrak{g}_\rho$.

In the two remaining sections, we show that the nilpotent operator $N_{\beta,\rho}$ induces an $\mathfrak{sl}_2$-triple whose weight decomposition splits both the perverse filtration $P_\bullet H^*(M, \mathbb{Q})$ and the monodromy weight filtration of a degeneration $f : \mathcal{M} \to \Delta$. This completes the proof of Theorem 3.

10. The construction of a degeneration $f : \mathcal{M} \to \Delta$ with logarithmic monodromy $N_{\beta,\rho}$ is precisely [12, Theorem 4.6]. Whereas the original statement requires $b_2(M) \geq 5$ to ensure the existence of an element $\beta \in H^2(M, \mathbb{Q})$ with $q_M(\beta) = 0$, in our situation β is readily given by the Lagrangian fibration $\pi : M \to B$. From the proof of [12, Theorem 4.6], it suffices to find an element $h \in H^2(M, \mathbb{Z})$ satisfying

$$q_M(h) > 0, \quad q_M(\beta, h) = q_M(\rho, h) = 0$$

in order to obtain nilpotent orbits $(N_{\beta,\rho}, x)$ with $x \in \widehat{\mathcal{D}}_h$ as in [12, Definition 4.3].[4] These nilpotent orbits eventually provide the required degeneration $f : \mathcal{M} \to \Delta$ through global Torelli. Now because q_M is of signature $(3, b_2(M) - 3)$ and $q_M|_{V_\rho}$ is only of signature $(2, 1)$ (recall that $b_2(M) \geq 4$), such an h exists.

By Jacobson-Morozov, the nilpotent operator $N_{\beta,\rho} \in \mathfrak{g}_\rho$ is part of an $\mathfrak{sl}_2$-triple that we denote $(L_N = N_{\beta,\rho}, H_N, \Lambda_N)$. Consider the action of this $\mathfrak{sl}_2$ on $H^*(M, \mathbb{Q})$ and the associated weight decomposition

$$H^*(M, \mathbb{Q}) = \bigoplus_{d,m} W_m^d \tag{6}$$

with $H_N|_{W_m^d} = m\,\mathrm{id}$. By the definition of the monodromy weight filtration, we have

$$W_k H_{\mathrm{lim}}^d(\mathbb{Q}) = \bigoplus_{d-m \leq k} W_m^d. \tag{7}$$

11. Finally, we match the perverse decomposition (2) with the weight decomposition (6). Because both decompositions are defined over $\mathbb{Q}$, it suffices to work with $\mathbb{C}$-coefficients.

We recall some basic facts about $\mathfrak{so}(5, \mathbb{C})$-representations. Let V be a $\mathbb{C}$-vector space admitting three $\mathfrak{sl}_2$-actions (L_1, H, Λ_1), (L_2, H, Λ_2) and (L_3, H, Λ_3) that generate an $\mathfrak{so}(5, \mathbb{C})$-action. More concretely, the operators

$$L_s,\ \Lambda_s,\ K_{st} = [L_s, \Lambda_t],\ H, \quad \text{for } s, t \in \{1, 2, 3\}$$

[4] Here $\widehat{\mathcal{D}}_h$ is the extended polarised period domain with respect to $h \in H^2(M, \mathbb{Z})$.

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satisfy the relations (2.1) in [14]. We consider the Cartan subalgebra

$$\mathfrak{h} = \langle H, -\sqrt{-1}K_{23}\rangle \subset \mathfrak{so}(5,\mathbb{C})$$

and the associated weight decomposition

$$V = \bigoplus_{i,j} V^{i,j}$$

with

$$H|_{V^{i,j}} = (i + j - 2n)\,\mathrm{id}, \quad (-\sqrt{-1}K_{23})|_{V^{i,j}} = (i - j)\,\mathrm{id}.$$

We define a nilpotent operator

$$L_N = \left[\frac{1}{2}L_2 - \frac{\sqrt{-1}}{2}L_3,\ \Lambda_1\right] = -\frac{1}{2}K_{12} + \frac{\sqrt{-1}}{2}K_{13} \in \mathfrak{so}(5,\mathbb{C}),$$

which induces an $\mathfrak{sl}_2$-triple (L_N, H_N, Λ_N) with

$$\Lambda_N = \left[-\frac{1}{2}L_2 - \frac{\sqrt{-1}}{2}L_3,\ \Lambda_1\right] = \frac{1}{2}K_{12} + \frac{\sqrt{-1}}{2}K_{13}, \quad H_N = \sqrt{-1}K_{23}.$$

In particular, we have $H_N|_{V^{i,j}} = (j - i)\,\mathrm{id}$. The weight decomposition with respect to this $\mathfrak{sl}_2$-action then takes the form

$$V = \bigoplus_m V_m^d, \quad V_m^d = \bigoplus_{\substack{i+j=d \\ j-i=m}} V^{i,j}$$

with $H_N|_{V_m^d} = m\,\mathrm{id}$.

In our geometric situation, let V be the total cohomology $H^*(M,\mathbb{C})$. We consider the three operators L_1, L_2, L_3 determined by

$$L_1 = L_\rho, \quad \frac{1}{2}L_2 + \frac{\sqrt{-1}}{2}L_3 = L_\eta, \quad \frac{1}{2}L_2 - \frac{\sqrt{-1}}{2}L_3 = L_\beta,$$

which induce a representation of $\mathfrak{so}(5,\mathbb{C})$ by (5). In particular, we have $V^{i,j} = P_{\mathbb{C}}^{i,j}$. Moreover, the nilpotent operator L_N is exactly $N_{\beta,\rho} = [L_\beta, \Lambda_\rho]$. We conclude from (3) and (7) that

$$P_k H^d(M,\mathbb{C}) = \bigoplus_{\substack{i+j=d \\ i\le k}} P_{\mathbb{C}}^{i,j} = \bigoplus_{\substack{i+j=d \\ j-i=m \\ d-m\le 2k}} V^{i,j} = \bigoplus_{d-m\le 2k} V_m^d = \bigoplus_{d-m\le 2k} W_{m,\mathbb{C}}^d = W_{2k}H_{\lim}^d(\mathbb{C}).$$

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