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CYCLICALLY SEPARATED GROUPS

BRIAN HARTLEY, JOHN C. LENNOX
AND AKBAR H. RHEMTULLA

We call a group G cyclically separated if for any given cyclic subgroup B in G and subgroup A of finite index in B, there exists a normal subgroup N of G of finite index such that $N \cap B = A$. This is equivalent to saying that for each element $x \in G$ and integer $n \geq 1$ dividing the order $o(x)$ of x, there exists a normal subgroup N of G of finite index such that Nx has order n in G/N. As usual, if x has infinite order then all integers $n \geq 1$ are considered to divide $o(x)$. Cyclically separated groups, which are termed "potent groups" by some authors, form a natural subclass of residually finite groups and finite cyclically separated groups also form an interesting class whose structure we are able to describe reasonably well. Construction of finite soluble cyclically separated groups is given explicitly. In the discussion of infinite soluble cyclically separated groups we meet the interesting class of *Fitting isolated* groups, which is considered in some detail. A soluble group G of finite rank is Fitting isolated if, whenever $H = K/L$ $(L \lhd K \leq G)$ is a torsion-free section of G and $F(H)$ is the Fitting subgroup of H then $H/F(H)$ is torsion-free abelian. Every torsion-free soluble group of finite rank contains a Fitting isolated subgroup of finite index.

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1.

1.1. INTRODUCTION

We call a group G cyclically separated (CS-group) if for any given cyclic subgroup $B \leq G$ and subgroup A of finite index in B, there exists a normal subgroup N of G of finite index such that $N \cap B = A$. This is clearly equivalent to saying that for each element x in G and integer $n \geq 1$ dividing the order $o(x)$ of x, there exists a normal subgroup N of G of finite index such that Nx has order n. If x is of infinite order, then, as is usual, all integers $n \geq 1$ are considered to be divisors of $o(x)$. We shall more often be using this second version of the definition of cyclically separated groups in this paper. Note that it is sufficient to restrict attention to prime divisors of $o(x)$. Cyclically separated groups form a natural subclass of residually finite groups. They have been investigated by Allenby and Tang in [1] and in some more recent papers to appear and by Poland in [11]. Cyclically separated groups are referred to as "Potent" groups by these authors. The usefulness of this concept was demonstrated in [1] for one relator groups and it is likely to receive more attention in future.

From now on we shall denote cyclically separated groups by CS-groups.

The arrangement of the paper is as follows. The notation used is described in Section 1.2. The basic closure properties of CS-group are given in Section 1.3. Finite CS-groups are discussed in Section 2 and general soluble CS-groups in Section 3. In Section 4 we establish an interesting property of torsion-free soluble groups with finite rank. Basically it states that every such group G has a subgroup H of finite index in which the Fitting subgroup of every torsion-free section of H is isolated. This is needed in the proof of Theorem D of Section 3. Although the result can be established by representing G as a subgroup of $GL(n, Q)$ and then using methods developed by B.A.F. Wehrfritz, we have chosen to give an elementary, though slightly lengthy, proof.

1.2. NOTATION

As stated earlier, a cyclically separated group will be called a CS-group. We shall write CS^* to denote the class of CS-groups all of whose subgroups and torsion-free quotients are CS-groups. $\overline{CS}$ will denote the largest subgroup and quotient closed subclass of CS-groups. Thus we

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have the chain $CS \supset CS^* \supset \overline{CS}$, with all inclusions proper. All standard notation used is from Robinson [12] except that we write Q instead of H for 'quotient' closure operation and $Z(G)$ for $\zeta(G)$ to denote the centre of G . Notation pertaining to finite groups is mainly from Gorenstein [5] if it is not in [12]. If B is a subgroup of a group G then we write ${}^{G}\sqrt{B}$ to denote the set $\{x \in G;\ x^n \in B$ for some positive integer $n\}$. Where we have used this notation, ${}^{G}\sqrt{B}$ turns out to be a subgroup. Finally we write $P \rtimes H$ to denote the split extension of P by H . Thus $G = P \rtimes H$ if $G = PH$, $P \cap H = 1$ and $P \lhd G$.

1.3. CLOSURE PROPERTIES

It follows from the definition that the class of CS-groups is subgroup closed. A quotient of a CS-group need not be potent; however finite CS-groups form a Q-closed class (Proposition 1).

LEMMA 1. *The class of* CS-*groups is closed under (restricted) direct product.*

Proof. Clearly it is enough to consider direct products with finitely many factors and hence we only need show that $G = G_1 \times G_2$ is a CS-group if G_1 and G_2 are. Let $x = x_1x_2 \in G$ $(x_i \in G_i)$. Every divisor n of $o(x)$ may be written in the form $n = n_1n_2$ where n_i divides $o(x_i)$, $i = 1, 2$ and n_1 and n_2 are coprime. We can find subgroups $N_i \lhd G_i$ $(i = 1, 2)$ such that $|G/N_i| < \infty$ and N_ix_i has order n_i . Putting $N = N_1 \times N_2$ we see that Nx has order n .

The class of CS-groups is not closed under any of the operations L, R, N_0, P and Q . But it is R_0-closed by Lemma 1 since it is subgroup closed.

2. Finite CS-groups

In this section we restrict our attention to finite groups. A finite simple CS-group must have all of its elements of prime order and so is either a cyclic group of prime order or A_5 by a theorem of Suzuki [13]. In fact, we have further the following interesting consequences of that

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result.

LEMMA 2. *If G is a finite CS-group then every non-abelian composition factor of G is isomorphic to A_5, the alternating group of degree five.*

Proof. Suppose the result is false. Choose $H \leq G$ minimal subject to the condition that H/K is a non-abelian composition factor of G not isomorphic to A_5. Then not all the elements of H/K are of prime order by Suzuki [13] as remarked above. Choose Kx in H/K with $o(Kx)$ a composite number divisible by a prime p. Since H is a CS-group, there exists $N \lhd H$ with x^p in N but x not in N. Then $H = KN$ with $N < H$. Thus $N/K \cap N \simeq H/K$. By minimality of H, $N/K \cap N \simeq A_5$, a contradiction.

As remarked earlier, the class of finite CS-groups is quotient closed. The proof of this rather surprising result is a little indirect.

LEMMA 3. *Let G be a finite group. Then G is a CS-group if and only if*

(a) every non-abelian composition factor of G is isomorphic to A_5;

(b) whenever p is a prime and y is a p'-element of G, then y commutes with no non-trivial p-element of $\langle y^G \rangle$.

Proof. First suppose that G is a CS-group. Then, by Lemma 2, G satisfies *(a)*. If y is as given and y commutes with a p-element x of $\langle y^G \rangle$, then since $o(x)$ divides $o(xy)$, we can find a normal subgroup N of G such that $o(Nxy) = o(x)$. Then $y \in N$, so $\langle y^G \rangle \leq N$, whence $x \in N$. Therefore $x = 1$.

Now let G satisfy *(a)* and *(b)*; let $z \in G$ and let p be a prime divisor of $o(z)$. We have to find a normal subgroup N of G such that $o(Nz) = p$. Let $z = xy$ where x is a p-element and y is a p'-element and $[x, y] = 1$. There exists a normal subgroup M of G such that $y \in M$ and $\langle x \rangle \cap M = 1$, namely we can take $M = \langle y^G \rangle$. Thus $o(Mx) = o(x)$. Now *(a)* implies that G/M has a series of normal subgroups in each factor of which every p-element has prime order. Passing up such a series, we will eventually reach a quotient in which the image of x has order p.

PROPOSITION 1. *The class of finite* CS-*groups is* Q-*closed.*

Proof. We prove this by showing that if $N \lhd G$ and G satisfies *(a)* and *(b)* of Lemma 3, so does G/N. Clearly G/N satisfies *(a)*. So now let Ny be a non-trivial p'-element of G/N and Nx be a p-element of G/N such that Nx and Ny commute and $Nx \in \langle (Ny)^{G/N} \rangle = \langle y^G \rangle N/N$. We may choose y to be a p'-element such that $|\langle y^G \rangle| \le \left|\left\langle y_0^G \right\rangle\right|$ for every p'-element $y_0 \in Ny$. Let $L = \langle y^G \rangle$ and $J = \langle x, N \rangle \cap L$. Then $[y, J] \le J_0 = N \cap L$. Let P be a Sylow p-subgroup of J. Then by the Frattini argument $N_L(J) = J_0 N_L(P)$. Hence there is a p'-element $y_0 \in N_L(P)$ such that $yJ_0 = y_0 J_0$. Clearly $\left\langle y_0^G \right\rangle \le L = \langle y^G \rangle$, and by the choice of y we must have equality. So we may suppose without loss of generality that $y \in N_L(P)$. We have $[y, P] \le P \cap N$ and since y is a p'-element, $P = (P \cap N)C_P(y)$. However $C_P(y) = 1$ since G satisfies *(b)* of Lemma 3 and $P \le \langle y^G \rangle$. Hence $P \le N$ and $J = PJ_0 = J_0$ and $x \in N$. This shows that G/N satisfies *(b)*.

Now recall that a monolithic group is one which has a unique minimal normal subgroup, and that the groups in any class of finite groups that is closed under subgroups, quotients and direct products are precisely the subdirect products of monolithic groups in that class. For this reason we now concentrate our attention on monolithic CS-groups.

LEMMA 4. *Let G be a finite monolithic* CS-*group. If the monolith of G is non-abelian then $G \simeq A_5$.*

Proof. Let N be the monolith of G. By Lemma 2, $N = N_1 \times N_2 \times \dots \times N_t$ where $t \ge 1$ and $N_i \simeq A_5$ $(1 \le i \le t)$. If $t > 1$ then choose x in N_1 of order 3. Then $N = \langle x^G \rangle$ since N is the monolith and $[x, N_2] = 1$ so that x commutes with $3'$-elements in

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$\langle x^G \rangle$. This violates condition *(b)* of Lemma 3. Thus $t = 1$. Since N is the monolith, $C_G(N) = 1$ or $C_G(N) = N$. But $Z(N) = 1$ so that $C_G(N) = 1$ and $G \simeq A_5$ or $G \simeq S_5$ since G is a subgroup of Aut N . But S_5 is not a CS-group. Hence $G \simeq A_5$.

LEMMA 5. *Let G be a finite monolithic* CS-*group with abelian monolith W . Then*

(i) W is an elementary abelian p-group for some prime p, and the Fitting subgroup of G is a p-group P ;

(ii) $C_G(W) = P$.

Proof. *(i)* is clear. In order to show *(ii)* we first note that $W \leq Z(P)$ (where $Z(P)$ is the centre of P), since W is the monolith of G . Thus $C_G(W) \geq P$. If $C_G(W) > P$ then $C_G(W)$ contains a non-trivial p'-element y . Then $[W, \langle y^G \rangle] = 1$ and $W \leq \langle y^G \rangle$ again because W is the monolith of G . This contradicts condition *(b)* of Lemma 2. Thus $C_G(W) = P$.

LEMMA 6. *Using the notation of Lemma 5, we have $C_W(y) = 1$ for every non-trivial p'-element $y \in G$.*

Proof. Since W is the monolith of G , we have $\langle y^G \rangle \geq W$. Hence $C_W(y) = 1$ by Lemma 3.

Because of Lemmas 5 and 6 we now make the following definition.

DEFINITION. $G \in \mathfrak{X}_p$ if and only if G is a CS-group and there is an $\mathbb{F}_p G$-module W such that $C_W(y) = 0$ for all non-trivial p'-elements $y \in G$.

LEMMA 7. *(i) $\mathfrak{X}_p$ is s-closed.*

(ii) If G is a CS-*group then $G \in \mathfrak{X}_p$ if and only if $G/O_p(G) \in \mathfrak{X}_p$.*

(iii) If G is a monolithic CS-*group whose monolith is an abelian*

p-group then $G \in \mathfrak{X}_p$.

Proof. *(i)* is immediate.

(ii) If G is a CS-group then so is $G/O_p(G)$ by Proposition 1. If W is an $\mathbb{F}_pG$-module on which every non-trivial p'-element of G operates fixed point freely, and W_1 is any composition factor of W , then $O_p(G)$ operates trivially on W_1 , and we see that W_1 is effectively a $G/O_p(G)$ module of the type required to guarantee that $G/O_p(G) \in \mathfrak{X}_p$. The converse follows since every $G/O_p(G)$-module can be viewed as a G-module on which $O_p(G)$ operates trivially.

(iii) This follows from Lemma 6.

We wish to identify the groups which can occur as $G/O_p(G)$, where G is a monolithic CS-group whose monolith is a p-group. By Lemma 7 every such group belongs to $\mathfrak{X}_p$ and contains no non-trivial normal p-subgroups. We shall proceed to establish the converse and then characterise the structure of these groups completely.

LEMMA 8. *Let* $G = P \rtimes H$ *be the semidirect product of a normal* p*-subgroup* P *by a group* H . *Then* G *is a* CS*-group if and only if* H *is a* CS*-group and* $[P, \langle y^H \rangle] \cap C_P(y) = 1$ *for all* p'*-elements* $y \in H$.

Proof. That G being a CS-group implies the other conditions follows from Lemma 3 and the subgroup closure of the class of CS-groups. Conversely, assume that these conditions hold. Every element of G is conjugate to an element xy where $[x, y] = 1$, x is a p-element and y is a p'-element of H . Let x have order p^α and y have order m . We have to find a quotient of G in which the image of xy has order equal to any given prime divisor d of $p^\alpha m$. If $d|m$ we do this by first passing to the CS-group G/P and noting that m divides the order of Pxy . Otherwise $d = p$. Now clearly

$$\langle y^G \rangle = [P, \langle y^H \rangle]\langle y^H \rangle ,$$

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so $\langle y^G\rangle \cap P = [P, \langle y^H\rangle]$. Hence $\langle x\rangle \cap \langle y^G\rangle \cap P \leq [P, \langle y^H\rangle] \cap C_p(y) = 1$. Therefore, since x has prime power order, either $\langle x\rangle \cap P = 1$ or $\langle x\rangle \cap \langle y^G\rangle = 1$. In the first case we pass to G/P, noting that p divides the order of Px, and then obtain the required quotient. In the second case, let $M = \langle y^G\rangle$. Then $Mxy = Mx$ has order p. Now by Lemma 2, the only non-abelian composition factor of H is A_5. Hence H has a series of normal subgroups, in each factor of which every p-element has order 1 or p; and so G/M has such a series. Passing up it, we eventually reach a quotient in which the image of Mx has order p.

LEMMA 9. *Let H be a group. Then the following two conditions are equivalent:*

(i) $O_p(H) = 1$ *and* $H \in \mathfrak{X}_p$;

(ii) there exists a monolithic CS-*group G such that the monolith of G is a p-group and* $G/O_p(G) \simeq H$.

Proof. *(ii) ⇒ (i).* This was remarked before Lemma 8.

(i) ⇒ (ii). Let W be an $\mathbb{F}_pH$-module on which every non-trivial p'-element of H operates fixed point freely. Passing to a composition factor of W, we may assume W is irreducible. Let $G = W \rtimes H$. Since H is a CS-group by assumption, Lemma 8 shows that G is a CS-group and since $O_p(H) = 1$, $W = O_p(G)$. Hence $G/O_p(G) \simeq H$.

Now we go on to analyse the structure of $\mathfrak{X}_p$-groups.

LEMMA 10. *Let* $H \in \mathfrak{X}_p$. *Then*

(i) every p'-subgroup of H, whose order is the product of two not necessarily distinct primes, is cyclic;

(ii) if $q \neq p$, then the Sylow q-subgroups of G are cyclic or generalized quaternion.

Proof. Every p'-subgroup of H admits a faithful linear representation in which every non-trivial element operates fixed point freely. The assertions are standard consequences of this (see Gorenstein [5], 5.3.14, 5.4.11).

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THEOREM A. *Let H be a group such that $O_p(H) = 1$. Then $H \in \mathfrak{X}_p$ if and only if*

(i) $H = S \times T$ where S is soluble and either $T = 1$ or $p = 2$, $T \simeq A_5$ and $|S|$ is not divisible by 3 or 5;

(ii) $S = Q \rtimes R$ is the semidirect product of a cyclic normal $\{p, 2\}'$-subgroup Q by a nilpotent group R whose Sylow q-subgroups are cyclic or generalized quaternion if $q \neq p$.

Furthermore $(|Q|, |R|) = 1$ and every element of prime order not equal to p in R centralizes Q.

Proof. Suppose first that $H \in \mathfrak{X}_p$ and $O_p(H) = 1$. Let T denote the largest normal subgroup of H which is a direct product of copies of A_5. By Lemma 10 *(i)*, either $T = 1$ or $T \simeq A_5$. If $T \simeq A_5$ and $S = C_G(T)$, then G/S is isomorphic to a subgroup of Aut A_5 containing A_5, and hence to A_5 or S_5. We have seen that S_5 is not a CS-group, and in fact this follows from Lemma 4. So $G = S \times T$. If $T = 1$ then the same holds with $S = G$. Clearly S contains no non-trivial normal subgroup which is a direct product of copies of A_5. Let F be the Fitting subgroup of S. Then $F \geq C_S(F)$ for otherwise we may choose a subnormal subgroup C of $C_S(F)$ such that $C > C_S(F) \cap F$ and $C/C_S(F) \cap F$ is simple. This factor is not cyclic since in that case $C \leq F$. It cannot be non-abelian simple either for otherwise $E(C \cap F) = C$ where E is the last term of the derived series of C. Thus $E/E \cap F$ is simple non-abelian and $E \cap F$ is central in F. If $E \cap F \neq 1$ and $1 \neq x \in E \cap F$ is a q-element for some prime q, then E contains a non-trivial q'-element $y \notin F$. We have $(E \cap F)\langle y^E \rangle = E$ and $[x, y] = 1$. So $E/\langle y^E \rangle$ is abelian and hence $\langle y^E \rangle = E$. Thus E is not a CS-group by Lemma 3. It follows that $E \cap F = 1$ so E is a simple subnormal subgroup of S, a contradiction. Thus $F \geq C_S(F)$.

Let $F_1 = O_{2'}(F)$. By Lemma 10 and the fact that $O_p(S) = 1$, we have that F_1 is a cyclic group, so $S/C_S(F_1)$ is abelian. If $F_2 = O_2(F)$,

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then $S/C_S(F_2)$ embeds in $\text{Aut } F_2$. Now F_2 is cyclic or generalized quaternion, so $\text{Aut } F_2$ is a 2-group unless F_2 is quaternion of order 8 ([5], Chapter 5). Thus either $S/C_S(F_2)$ is a 2-group or F_2 is a quaternion of order 8 and S contains a 3-element y such that $[F_2, y] = F_2$ and y centralizes the centre $Z(F_2)$ of F_2 . This is not possible in a CS-group. So $S/C_S(F_2)$ is a 2-group in all cases, and hence F_2 is in the hypercentre of S . Also, since $C_S(F) = C_S(F_1) \cap C_S(F_2) \leq F$, we conclude that S/F is nilpotent. Hence S/F_1 is also nilpotent and the nilpotent residual Q of S is a cyclic $\{p, 2\}'$-group and S splits over it as

$$S = Q \rtimes R$$

say ([4], Theorem 5.15).

Now if $T \neq 1$ then $p = 2$, since not every elementary abelian 2-subgroup of H is cyclic; and $|S|$ is not divisible by 3 or 5 for similar reasons. The remaining assertion about R follows from Lemma 3 and Lemma 10.

Conversely let H have the structure given by *(i)* and *(ii)* and suppose $O_p(H) = 1$. Since Q is cyclic, R' centralizes Q . From *(ii)* we see that R' is cyclic. Thus S is supersoluble and metabelian and hence a CS-group (Theorem E). Let Q_0 denote the (cyclic) subgroup of S generated by the elements of prime order not equal to p in S . There exists an irreducible $\mathbb{F}_pQ_0$-module U_0 that is faithful for Q_0 and hence operated on fixed point freely by every non-trivial element of Q_0 . Now $A_5 \simeq \text{SL}(2, 4)$ and if V_2 denotes the 2-dimensional vector space on which this acts, then the elements of order 3 and 5 operate fixed point freely, and we can think of V_2 as an $\mathbb{F}_2A_5$-module. Let $U = U_0$ if $T = 1$ or $U = U_0 \otimes_{\mathbb{F}_2} V_2$ if $T \neq 1$, in which case $p = 2$. This is to be thought of as an $\mathbb{F}_2[Q_0 \times T]$-module in the usual way. Now since $(|Q_0|, 15) = 1$, if $T \neq 1$, every non-trivial p'-element of $Q_0 \times T$ has a non-trivial power in Q_0 or T and so operates fixed point freely on U. Let $V = U^H$ be the induced module. Then $V_{Q_0 \times T}$ is a direct sum of conjugates of U and so is fixed point free for every non-trivial p'-element of $Q_0 \times T$. But every non-trivial p'-element of $H = S \times T$ has a non-trivial power in $Q_0 \times T$. Hence $C_V(y) = 0$ if y is a non-trivial p'-element of H. Therefore, by definition, $H \in X_p$.

COROLLARY. *Let H be a group such that $O_p(H) = 1$. Then the following conditions are equivalent:*

(i) $H \simeq G/O_p(G)$ *for some monolithic* CS-*group* G *whose monolith is a* p-*group;*

(ii) $H \in X_p$;

(iii) $H = S \times T$ *as in the statement of Theorem* A.

This is obtained by combining Lemma 9 and Theorem A.

THEOREM B. *Let G be a finite soluble* CS-*group with Fitting subgroup* $F(G)$. *Then* $G/F(G)$ *is metabelian and supersoluble.*

Proof. G is a subdirect product of monolithic groups of the same type by Proposition 1. Each of these satisfies the conclusion of the theorem since a soluble X_p-group H with $O_p(H) = 1$ is supersoluble and metabelian. From this the theorem follows.

We can improve Theorem B somewhat by obtaining a fuller description of the monolithic soluble CS-groups. We know that such a group G belongs to some X_p and then we know G/P where $P = O_p(G)$, so now it is a matter of analysing P.

DEFINITION. A group G is a *special* X_p-*group* if $G/O_p(G) \in X_p$ and $G = O_{p'p}(G)$. Such a group G is a subdirect product of the X_p-group $G/O_p(G)$ and the p-group $G/O_{p'}(G)$. Hence it is a CS-group by Lemma 1 and belongs to X_p by Lemma 7. In fact these groups are exactly the subdirect products of an X_p-group H with $O_p(H) = 1$ and a finite p-group and so may be considered to be well understood.

LEMMA 11. *Let* G *be a finite soluble group. Then* $G \in \mathfrak{X}_p$ *if and only if* $G = P \rtimes H$ *where*

(i) P *is a* p*-group,*

(ii) H *is a special* $\mathfrak{X}_p$*-group,*

(iii) $P = [P, O_{p'}(H)]$ *, and*

(iv) if $y \in O_{p'}(H)$ *then* $[P, \langle y^H\rangle] \cap C_P(y) = 1$.

REMARK. These conditions are all satisfied if H is a group of prime order q operating fixed point freely on P , so P can be of arbitrarily large class if q is allowed to be arbitrarily large. Condition *(iv)* seems rather strong if $O_{p'}(H)$ has many prime divisors but even so, P can be quite complicated, as we show by example at the end of this section.

Proof. Suppose that G is a soluble $\mathfrak{X}_p$-group and let $P_1 = O_p(G)$. Then G/P_1 is an $\mathfrak{X}_p$-group with no non-trivial normal p-subgroup (Lemma 7), so the structure of it is given by Theorem A. In particular $G/P_1 = O_{p'p}(G/P_1)$. Let $O_{p'}(G/P_1) = Q_1P_1/P_1$ for some p'-group Q_1 . Then if $H = N_G(Q_1)$, the Frattini argument gives $G = P_1H$. Now $Q_1 \lhd H$ and H/Q_1 is a p-group, so $H = O_{p'p}(H)$. Also

$$H/O_p(H) = H/P_1 \cap H \simeq G/P_1 \in \mathfrak{X}_p .$$

Hence H is a special $\mathfrak{X}_p$-group.

Now let $P = [P_1, Q_1]$. Then $P_1 = PC_{P_1}(Q_1) = P(P_1 \cap H)$ so $G = PH$. We have $[P, \langle y^H\rangle] \cap C_P(y) = 1$ for each element $y \in Q_1$ from Lemma 3. So it remains only to show that $P \cap H = 1$, that is, $C_P(Q_1) = 1$. For each $y \in Q_1$, let $P(y) = [P, \langle y^H\rangle]$. We know that y operates fixed point freely on $P(y)$ and hence on any quotient thereof. Also $P = \prod P(y)$ over all $y \in Q_1$. Writing $y_1, \ldots, y_n$ for the element of Q_1 and

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putting $P(i) = \prod_{j=1}^{i} P(y_j)$, we obtain a series of normal subgroups of G each factor of which is transformed fixed point freely by some element of Q_1 . It follows that $C_P(Q_1) = 1$ as required.

The fact that any group satisfying *(i)-(iv)* is a CS-group follows from Lemma 8. Hence $G \in \mathfrak{X}_p$ by Lemma 7, since $G/O_p(G) \simeq H/O_p(H)$.

Since the monolithic soluble CS-groups are exactly the monolithic soluble groups in $\cup \mathfrak{X}_p$, p prime (Lemma 7), we have

THEOREM C. *Let* G *be a monolithic soluble group. Then* G *is a* CS-*group if and only if* G *has the structure described in Lemma* 11, *for some prime* p .

This may be viewed as giving a reasonably explicit construction for monolithic soluble CS-groups (apart from the rather mysterious condition *(iv)* of Lemma 11) and hence for all soluble CS-groups.

It remains to consider insoluble monolithic CS-groups with abelian monolith. These lie in $\mathfrak{X}_2$ by Lemma 7. Our description of these groups is somewhat less satisfactory.

LEMMA 12. *Let* G *be an insoluble* $\mathfrak{X}_2$*-group. Then there exists a normal abelian* 2*-subgroup* P *of* G *such that*

$$G/P = S/P \times T/P$$

where S *is a soluble* $\mathfrak{X}_2$*-group of order prime to* 15 , $T/P \simeq A_5$, *and* $C_P(y) = 1$ *for all non-trivial elements* y *of odd order in* T .

REMARK. Further conditions about the action of elements of G on $O_2(G)$ will be needed for a converse statement. These seem rather clumsy and not worth formulating.

Proof. Most of this is straightforward, with the exception of the statement that P is abelian. Let $P_1 = O_2(G)$. By Lemma 7 and Theorem A, $G/P_1 = S/P_1 \times T_1/P_1$, where S is a soluble $\mathfrak{X}_2$-group of order prime to 15 and $T_1/P_1 \simeq A_5$. Let T be the last term of the derived series of T_1. We have $T \lhd G$ and so $P = P_1 \cap T \lhd G$. Since T_1/P_1 is perfect, $T_1 = P_1T$ and so $G = ST$. Also $[S, T] \leq P_1 \cap T = P$. Hence $G/P = S/P \times T/P$.

Let y be any non-trivial element of odd order in T. Then $T = P\langle y^T\rangle$ and so $T/\langle y^T\rangle$ is a 2-group. Since T is perfect, $T = \langle y^T\rangle \supseteq P$. Therefore $C_P(y) = 1$ as T is a CS-group (Lemma 3).

It remains to show that P is abelian. Now any chief factor of T below P may be viewed as an irreducible module for A_5 over $\mathbb{F}_2$ on which the elements of order 3 and 5 operate fixed point freely. This explains the relevance of the next result.

LEMMA 13. *Write $A = A_5$. There are exactly three isomorphism classes of irreducible $\mathbb{F}_2A$-modules, represented by V_1, V_2 and V_3 say, of dimension* 1, 4, 4. *Of these exactly one* (*say* V_2) *is transformed fixed point freely by the elements of odd order in A, and* $\mathrm{Hom}_A(V_2 \otimes V_2, V_2) = 0$.

Proof. The number of isomorphism classes of irreducible $\mathbb{F}_2A$-modules is the number of orbits of the 2-regular conjugacy classes under the map on the set of conjugacy classes induced by $x \to x^2$, that is, three (see [3]). We have the trivial module V_1 of course. We obtain V_2 by identifying A with $\mathrm{SL}(2, 4)$ and letting V_2 denote the natural module for this group, viewed as an $\mathbb{F}_2A$-module. An element of order three operates on V_2 as a diagonal matrix with eigenvalues λ and λ^2 (where V_2 is thought of as an $\mathbb{F}_4A$-module) where λ is a primitive cube root of 1 in $\mathbb{F}_4$, and so is fixed point free. Also since 4 is the order of 2 mod 5, V_2 is irreducible when restricted to a subgroup of order 5. Finally if W is a 5-dimensional vector space and A permutes a basis of W according to its natural permutation representation, then $V_3 = [W, A]$ is a 4-dimensional irreducible $\mathbb{F}_2A$-module. An element of order 3 in A

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has fixed point set of dimension 3 in W and hence dimension 2 in V_3.

To study $V_2 \otimes_{\mathbb{F}_2} V_2$ it is convenient to pass to the algebraic closure k of $\mathbb{F}_2$. The number of isomorphism classes of irreducible kA-modules is equal to the number of 2-regular conjugacy classes of A, that is four. It is easy to see that $W_3 = V_3 \otimes_{\mathbb{F}_2} k$ is irreducible. So is the module W_1 obtained from the natural $\mathbb{F}_4[\mathrm{SL}(2, 4)]$-module by field extension. We also have the module $W_2 = W_1^{(2)}$ obtained from W_1 by applying the Frobenius automorphism $\alpha \to \alpha^2$ to the entries of the matrices in $\mathrm{SL}(2, 4)$. By considering the eigenvalues of an element of order 5 we see that W_1 is not isomorphic to W_2. So W_1, W_2 and W_3 represent the three isomorphism types of non-trivial irreducible kA-modules. Now an element a of order 3 in A has eigenvalues λ and λ^2, each with multiplicity one, in W_1 and W_2, and if b is an element of order 5 in A, then for a suitable primitive 5th root of 1, say μ, the eigenvalues of b in W_1 are μ, μ^{-1} and on W_2 are μ^2 and μ^{-2}. On $W_1 \otimes_k W_2$, the element a has two eigenvalues equal to 1 and b has eigenvalues μ, μ^2, μ^3, μ^4. Hence $W_1 \otimes_k W_2 \simeq W_3$. On $W_1 \otimes W_1$, the element a has eigenvalues $1, 1, \lambda, \lambda^2$ and b has eigenvalues $1, 1, \mu, \mu^2$. With suitably chosen notation

$$a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{pmatrix} \in \mathrm{SL}(2, 4) = A$$

and the elements

$$s(\alpha) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \quad (\alpha \in \mathbb{F}_4)$$

form a Sylow 2-subgroup S of A. If we identify W with column vectors and let $v_1 = \binom{1}{0}$, $v_2 = \binom{0}{1}$ then

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$$av_1 = \lambda v_1 \ ; \qquad av_2 = \lambda^2 v_2 \ ,$$

$$s(\alpha)v_1 = v_1 + \alpha v_2 \ ; \quad s(\alpha)v_2 = v_2 \ .$$

We claim that $W_1 \otimes W_1$ has a unique one-dimensional submodule and no 2-dimensional irreducible submodule. For any one-dimensional irreducible submodule would have to lie in $U = k(v_1 \otimes v_2) \oplus k(v_2 \otimes v_1)$, since this is the 1-eigenspace of a. Now

$$(s(\alpha)-1)(\theta v_1 \otimes v_2 + \theta' v_2 \otimes v_1) = \alpha(\theta+\theta')v_2 \otimes v_2 \ .$$

and this is non-zero if $\theta \neq \theta'$ and $\alpha \neq 1$. Thus U has a unique minimal S-submodule, namely $k(v_2 \otimes v_2)$. Also, because of the structure of W_1 and W_2, the only possible 2-dimensional A-submodule of $W_1 \otimes W_1$ is that spanned by $v_1 \otimes v_1$ and $v_2 \otimes v_2$, the λ^2- and λ-eigenvectors of a. We easily see, however, that this is not S invariant.

Let W_{11} denote the unique minimal submodule of $W_1 \otimes W_1$. Then $W_1 \otimes W_1/W_{11}$ cannot contain a 1-dimensional submodule, since then W_1 would have a 2-dimensional submodule with two trivial composition factors and this would be trivial itself. Therefore $W_1 \otimes W_1/W_{11}$ contains a unique irreducible submodule W_{12}/W_{11} which has dimension 2 (and must be isomorphic to W_2 by consideration of the eigenvalues of b). Thus $W_1 \otimes W_1$ is uniserial, with submodules

$$0 < W_{11} < W_{12} < W_1 \otimes W_1$$

and $W_2 \otimes W_2$ will be similarly uniserial.

These considerations show that if $1 \leq i, j, l \leq 2$ then $\mathrm{Hom}_{kA}(W_i \otimes W_j, W_l) = 0$. Hence

$$\mathrm{Hom}_{kA}((W_1 \oplus W_2) \otimes_k (W_1 \oplus W_2), W_1 \oplus W_2) = 0 \ .$$

Now we claim that $V_2 \otimes_{\mathbb{F}_2} k \simeq W_1 \oplus W_2$. Since any element of $\mathrm{Hom}_A(V_2 \otimes V_2, V_2)$ determines an element of $\mathrm{Hom}_{kA}(V_2 \otimes V_2 \otimes k, V_2 \otimes k)$, this will show that $\operatorname{Hom}_A(V_2 \otimes V_2, V_2) = 0$. Since $V_2 \otimes k$ is completely reducible (for this property is preserved by separable field extension) and since a operates fixed point freely on it, $V_2 \otimes k$ must be the direct sum of two 2-dimensional modules. The types of these modules must be a union of Galois conjugacy classes, so $W_1 \oplus W_2$ is the only possibility. We have now proved Lemma 13.

Conclusion of proof of Lemma 12. We have $T/P \simeq A_5$ and P is a finite 2-group on which every non-trivial element of odd order operates fixed point freely. Let $X = P/P'$ viewed as a T/P-module by conjugation, and $Y = P'/[P, P']$ viewed similarly. Identifying T/P with A_5, we see from Lemma 13 that X has a composition series in which each factor is isomorphic to V_2. It follows that Y has a series in which each factor is isomorphic to a homomorphic image of $V_2 \otimes V_2$ (see Robinson [*12*], p. 56). By Lemma 13, such an image, if non-trivial, must have an image which is not isomorphic to V_2. But this is impossible. Hence $Y = 0$ so $P' = [P, P']$ and finally $P' = 1$.

EXAMPLE 1. The following example illustrates the possible complexity of soluble monolithic CS-groups. The construction we wish to use is also described in [*8*]. A preliminary lemma, some parts of it well known, will be useful.

LEMMA 14. *Let R be a ring with 1 and let S be a nil subring of R. Then*

(i) $1 + S = \{1+s;\ s \in S\}$ *is a group under multiplication: it is nilpotent if S is nilpotent;*

(ii) *if K is an ideal of S then* $1 + K \lhd 1 + S$;

(iii) *if $J + K = S$ for some subring J, then* $(1+K)(1+J) = 1 + S$;

(iv) *if also $J \cap K = 0$, then* $1 + S = (1+K) \rtimes (1+J)$.

Proof. *(i)* If $s \in S$ then $(1+s)^{-1} = 1 - s + s^2 \dots + (-1)^n s^n$ for suitable n. Also $1 + S \geq 1 + S^2 \geq \dots$ is a central series of $1 + S$.

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(ii) We have, if $s \in S$ and $k \in K$, then

$$(1+s)^{-1}(1+k)(1+s) = 1 + (1+s)^{-1}k(1+s) \in 1 + K .$$

(iii) Let $s \in S$. Then $s = k + j$ for some $k \in K$, $j \in J$. Hence $1 + s = 1 + k + j$ and

$$(1+s)(1+j)^{-1} = (1+j)(1+j)^{-1} + k(1+j)^{-1} \in 1 + K .$$

(iv) This follows from *(ii)* and *(iii)*.

Now let p be a given prime and let $q_1, q_2, \ldots, q_n$ be n distinct primes each congruent to $1 \bmod p$. These exist by Dirichlet's Theorem on the primes in an arithmetic progression. Let $H_i = \langle a_i \rangle \rtimes \langle b_i \rangle$ be a non abelian group of order pq_i, where a_i has order q_i and b_i has order p. Then H_i has a faithful irreducible module V_i over $\mathbb{F}_p$, and $C_{V_i}(a_i) = 0$. Let S denote the set of $(n+1) \times (n+1)$ matrices $u = (u_{ij})$ with rows and columns indexed by $\{1, 2, \ldots, n+1\}$ such that

$$u_{ij} = 0 \text{ if } i \geq j ,$$

$$u_{ij} \in V_i \otimes \ldots \otimes V_{j-1} \text{ if } i < j \leq n+1 .$$

Then S is a ring under matrix addition and multiplication if we use "tensor multiplication" (as in the tensor algebra) on the components. Let $H = H_1 \times \ldots \times H_n$. Then each $V_i \otimes \ldots \otimes V_{j-1}$ is an H-module on which H_t operates trivially if $t < i$ or $t \geq j$, and operates on the V_t component of the tensors if $i \leq t \leq j-1$. We can now allow H to operate "componentwise" on S, and we see that H then operates by ring automorphisms. If I_{n+1} denotes the $(n+1) \times (n+1)$ matrix identity, then $\mathbb{F}_p I_{n+1} \oplus S$ is also a ring operated on by H. Hence $P = I_{n+1} + S$ is a nilpotent group. Its class is easily seen to be exactly n. Also H operates on P by automorphisms. Now let $N = \{1, \ldots, n+1\}$, and let

$$\Lambda = \{(i, j) \in N \times N : i < j\} .$$

If $\Gamma_1, \Gamma_2 \subseteq N \times N$ then we can define

$\Gamma_1 \circ \Gamma_2 = \{(i, j) \in N \times N : \exists k \in N \text{ such that } (i, k) \in \Gamma_1 \text{ and } (k, j) \in \Gamma_2\}$.

For $\Gamma \subseteq \Lambda$, let

$$S_\Gamma = \{(u_{ij}) \in S : u_{ij} = 0 \text{ if } (i, j) \notin \Gamma\} .$$

Then it is clear from the definition of the matrix multiplication that S_Γ is an ideal of S if $(\Lambda \circ \Gamma) \cup (\Gamma \circ \Lambda) \subseteq \Gamma$, and S_Γ is a subring if $\Gamma \circ \Gamma \subseteq \Gamma$.

In particular let $I = \{i_1, \ldots, i_r\}$ be a subset of N with $i_1 < i_2 < \ldots < i_r$, and define

$$\Gamma(I) = \{(i, j) \in \Lambda : i \le i_t \le j \text{ for some } t \text{ with } 1 \le t \le r\}$$

and $\Gamma(I)^* = \Lambda \backslash \Gamma(I)$. Clearly $(\Lambda \circ \Gamma(I)) \cup (\Gamma(I) \circ \Lambda) \subseteq \Gamma(I)$ and $\Gamma(I)^* \circ \Gamma(I)^* \subseteq \Gamma(I)^*$. So, writing S_I for $S_{\Gamma(I)}$ and S_I^* for $S_{\Gamma(I)^*}$, we have that S_I is an ideal of S and S_I^* is a subring. Clearly $S = S_I \oplus S_I^*$ and so from Lemma 14, $P = P_I \rtimes P_I^*$ where $P_I = I_{n+1} + S_I$ and $P_I^* = I_{n+1} + S_I^*$.

These subgroups are clearly H-invariant. Now if $a_I = a_{i_1} \ldots a_{i_r}$, then since $\langle a_I \rangle = \langle a_{i_1}, \ldots, a_{i_r} \rangle$ we see that a_I operates trivially on $V_i \otimes \ldots \otimes V_j$ if $(i, j) \in \Gamma(I)^*$, and operates fixed point freely if $(i, j) \in \Gamma(I)$. This is because if $i \le t \le j$ then $V_i \otimes \ldots \otimes V_j$, as $\langle a_t \rangle$-module, is a direct sum of copies of V_t . Thus it is clear that a_I fixes every element of P_I^* and fixes no non trivial element of P_I since it changes every off diagonal entry of such a matrix. Thus $C_{P_I}(a_I) = 0$ and it follows that $C_P(a_I) = P_I^*$. Hence at last $[P_I^*, a_I] \cap C_P(a_I) = 1$. Now H is a CS-group, either obviously or because it is supersoluble and metabelian; also $\langle a_I \rangle \lhd H$ and every p'-subgroup of H has the form $\langle a_I \rangle$. It follows from Lemma 7 that $G = P \rtimes H$ is a CS-group.

3. Infinite soluble CS-groups

We saw in Section 2 that the class of finite CS-groups is quotient closed. This is obviously not the case for infinite soluble groups; for all such groups are quotients of free soluble groups which, by virtue of being residually finite p-groups for all prime p, are CS-groups. We are thus led to define two subclasses of CS-groups. We call G a CS*-group if all subgroups of G and their torsion-free quotients are CS-groups. G is called a $\overline{\text{CS}}$-group if all quotients of all subgroups of G are CS-groups. F.C. Tang has raised the question whether torsion-free polycyclic groups are CS-groups. Example 2 at the end of this section shows this is not so. On the positive side we have the following results.

PROPOSITION 2. *Every poly-infinite-cyclic group is a* CS-*group.*

In fact the class of poly-infinite-cyclic groups can be replaced by a larger class, which also figures in the next theorem. We define a class $\mathcal{Y}$ of abelian groups by:

$A \in \mathcal{Y} \Leftrightarrow A$ contains a free abelian subgroup B of finite rank such that A/B is a periodic group with finite primary components.

Let $\mathcal{Y}_0$ be the class of torsion-free $\mathcal{Y}$-groups.

PROPOSITION 2'. *Every* $P\mathcal{Y}_0$*-group is a* CS-*group.*

Proposition 2 obviously follows from this.

THEOREM D. *A torsion free soluble group* G *has a* CS*-*group of finite index if and only if* G *is a* $P\mathcal{Y}$*-group of finite rank.*

THEOREM E. *Every abelian-by-nilpotent supersoluble group is a* $\overline{\text{CS}}$-*group.*

Since every polycyclic group has a torsion-free subgroup of finite index, it follows from Theorem D that every polycyclic group has a CS*-subgroup of finite index.

At the end of this section we give several examples to show

(i) not every poly-infinite-cyclic group is a CS*-group,

(ii) there exists a metabelian CS-group G with a subgroup A such that every torsion-free quotient of G is a CS-group but A has a torsion-free quotient that is not a CS-group.

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The proofs of Proposition 2' and Theorem D require a little preparation. The following elementary facts about $\mathcal{Y}$-groups can be established by routine arguments.

LEMMA 15. *(i)* $\mathcal{Y} = Q S \mathcal{Y}$.

(ii) $P\mathcal{Y}$ *is* PQS*-closed.*

(iii) Every abelian $P\mathcal{Y}$*-group belongs to* $\mathcal{Y}$.

If G is any group, let G^* denote the smallest normal subgroup of G such that G/G^* is torsion-free abelian. Clearly, if G has a finite series with torsion free abelian factors, then the series $G = G_0 \geq G_1 \geq G_2 \ldots$, defined by $G_{i+1} = G_i^*$ for $i \geq 0$, is a series of characteristic subgroups of G reaching the identity after finitely many steps. If furthermore $G \in P\mathcal{Y}$, then Lemma 15 shows that the factors of this series will be $\mathcal{Y}_0$-groups. Hence we have

LEMMA 16. *A group* G *belongs to* $P\mathcal{Y}_0$ *if and only if* G *has a finite series of characteristic subgroups with* $\mathcal{Y}_0$*-factors.*

We note in passing, though it will not affect our arguments, that every $\mathcal{Y}_0$-group has finite rank and so has a finite series with torsion-free factors of rank one; also an additive subgroup A of the rationals belongs to $\mathcal{Y}_0$ if and only if, for each prime p , there exists an integer $k(p)$ such that A contains no rational of the form $a/p^{k(p)}$ where a is a non-zero integer prime to p .

We also require

LEMMA 17. *Every* $P\mathcal{Y}$*-group is residually finite.*

This follows from [12], Theorem 9.31.

Proof of Proposition 2'. Let G be a $P\mathcal{Y}_0$-group. By Lemma 16, G has a finite characteristic series with $\mathcal{Y}_0$-factors. We use induction on the length of such a series, and so we may assume that G has a characteristic $\mathcal{Y}_0$-subgroup A such that G/A is a CS-group. It suffices to show that if $1 \neq x \in A$ and p is any prime, then there exists a normal subgroup B of G such that $|G : B| < \infty$ and Bx has order p .

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By Lemma 17, there exists a subgroup A_1 of A such that $|A : A_1| < \infty$, $x \notin A$ and $x^p \in A_1$. If $t = |A : A_1|$, then A/A^t is of finite rank and finite exponent and so is finite. Therefore, replacing A_1 by A^t, we may assume that A_1 is characteristic in A. Taking A_1 maximal subject to being a characteristic subgroup of finite index in A containing x^p but not containing x, we find that A_1x has order p exactly. By Lemmas 15 and 16, B/A_1 is residually finite, so there exists a normal subgroup B of G such that $|G : B| < \infty$ and $B \cap A = A_1$. Clearly Bx has order p, as required.

Proof of Theorem D. Let G be a torsion free soluble group containing a CS*-subgroup of finite index. Since the additive group of rationals is not a CS-group, but is a homomorphic image of a free abelian group of countably infinite rank, G cannot contain a free abelian subgroup of infinite rank. Hence every abelian subgroup of G has finite rank. By a theorem of Kargapolov [9], G has finite rank. Theorem F now shows that (see §4), G has a subgroup H of finite index such that $H/F(H)$ is torsion free abelian. Since G has a CS*-subgroup of finite index, we may assume H is a CS*-group. Now H has a finite series with torsion-free abelian factors. This can be refined to a similar series in which the factors have rank 1. Since H is a CS*-group, these factors are CS-groups, and so cannot contain non trivial elements of infinite p-height for any prime p. It is easy to see that a torsion free abelian group of rank one with no elements of infinite p-height for any p belongs to $\mathcal{Y}_0$. Therefore G is a finite extension of a $P\mathcal{Y}_0$-group and belongs to $P\mathcal{Y}$.

Conversely, suppose $G \in P\mathcal{Y}$ and G has finite rank. As above, but using the full force of Theorem F, we see that G has a subgroup H of finite index all of whose torsion free quotients belong to $P\mathcal{Y}_0$. By Proposition 2', H is a CS*-group.

Proof of Theorem E. Let G be an abelian-by-nilpotent supersoluble group. Since such groups satisfy the maximal condition on subgroups it suffices to obtain a contradiction from the assumption that B is not a CS-group while all its proper quotients are. Then $G \neq 1$ so G contains a non-trivial cyclic normal subgroup N. By Lemma 1, if $1 \neq M \lhd G$ then $M \cap N \neq 1$.

Case 1. N is infinite. Let $1 \neq x \in G$ and p divide $o(x)$. If $\langle x \rangle \cap N = 1$ then $o(Nx) = o(x)$ and since G/N is a CS-group we can find a finite quotient of G/N in which the image of Nx has order p. Otherwise $\langle x \rangle$ is infinite cyclic and if $N = \langle y \rangle$ and t is the order of $x \bmod N$, which is necessarily finite, then $x^t = y^m$ for some $m \geq 1$. Let $L = \langle y^{mp} \rangle \lhd G$. Then Lx has order tp in G/L which is a CS-group. Thus G/L has a finite quotient in which the image of x has order p.

Case 2. N is finite. Then we may suppose $|N| = p$, a prime. Since $M \cap N \neq 1$ if $1 \neq M \lhd G$, $M \geq N$ for all $1 \neq M \lhd G$. Thus N is the monolith of G. Let F denote the Fitting subgroup of G. Then $Z = Z(F)$ must be a finite p-group since N is contained in every characteristic subgroup of Z. Thus F is a finite p-group and hence G is finite. Let A be the nilpotent residual of G. If $A = 1$ then G is a finite p-group and a moment's thought shows that G is a CS-group, a contradiction. Hence $A \neq 1$. By hypothesis A is abelian. Also $N \leq A$ so that A is a finite p-group. Thus ([4], Theorem 5.15) G splits over A as $G = A \rtimes B$ say, where $C_B(A) = 1$ since $C_B(A) \lhd G$ and $N \nleq C_B(A)$. Hence $C_G(A) = A$. Let $B = P \times Q$ where P is a p-group and Q a p'-group. Since G is supersoluble, $Q' \leq F$ and clearly $F = AP$. Thus $Q' = 1$, and Q is abelian. If $1 \neq x \in Q$ then $A = C_A(x) \times [A, x]$ and since Q is abelian and $[P, x] = 1$, both factors are normal in G. Since N is the monolith of G, one of these factors must be trivial, and since $x \notin A = C_G(A)$, we have $C_A(x) = 1$. Hence $C_G(x) = B$.

Now let $y \in G$ and n be a divisor of $o(y)$. If y is a p-element then we can find a quotient of G in which the image of y has order n since G has a normal series with factors of prime exponent. Otherwise y is conjugate to an element zx where z is a p-element, $1 \neq x \in Q$ and $[x, z] = 1$. Hence by the previous paragraph, $y \in B$. So $o(y) = o(Ay)$. Since all proper quotients of G are CS-groups, we can find a quotient of G/A in which the image of y has order n. We have thus reached a final contradiction and so established the result.

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We now give the examples promised earlier.

EXAMPLE 2. Let

$$G = \langle a, b, c, t;\ a^t = b,\ b^t = a^{-1}b^{-1},\ [a, b] = c,$$
$$[a, c] = [b, c] = 1,\ t^9 = c\rangle .$$

This group is torsion free (see [2]). It is not a CS-group because there is no normal subgroup K of G such that Kt has order two in G/K. For if $t^2 \in K$ then $[t^2, b, a] = [b, a] = c^{-1} \in K$. Thus $c = t^9 \in K$ and hence $t \in K$. Observe that G is a quotient of the group

$$J = \langle a, b, c, t;\ a^t = b,\ b^t = a^{-1}b^{-1},\ [a, b] = c,\ [a, c] = [b, c] = 1\rangle ,$$

which is poly-infinite-cyclic and hence a CS-group by Proposition 2. Thus poly-infinite cyclic groups are CS but not CS*-groups.

EXAMPLE 3. Start with an infinite cyclic group $\langle x\rangle$ and form the module

$$A = \mathbb{Z}\langle x\rangle/(f(x)) ,$$

where $f(x) = 2x^2 + x + 7$. Then A is a finitely generated $\mathbb{Z}\langle x\rangle$-module and it is not p-divisible for any prime p since $f(x)$ does not reduce to a unit mod p for any prime p. It is easy to verify that $G = A \rtimes \langle x\rangle$ is a metabelian minimax CS-group; every torsion free quotient of G is a CS-group but the subgroup A of G has a torsion free quotient that is not a CS-group. The only torsion free quotients of G are G, G/A and 1.

EXAMPLE 4. The group $G = \langle a, b,\ ;\ a^b = a^2\rangle$ is a finitely generated torsion free soluble group with finite rank that has no CS-subgroups of finite index.

EXAMPLE 5. The group $\langle u, v;\ (u^2)^v = u^{-2},\ (v^2)^u = v^{-2}\rangle$ is metabelian and supersoluble. Thus by Theorem E it is a CS-group. It is torsion free but not poly-infinite-cyclic. Thus a torsion free polycyclic group does not have to be poly-infinite-cyclic to be a CS-group.

4.

The main purpose of this section is to establish Theorem F which was needed in the proof of Theorem D.

DEFINITION. Let G be a soluble group with finite rank. We say G is FI (Fitting isolated) if whenever $H = K/L$ $(L \lhd K \leq G)$ is a torsion free section of G, then $H/F(H)$ is torsion free abelian, where $F(H)$ denotes the Fitting subgroup of H. The class of FI-groups is clearly Q and S closed.

THEOREM F. *If G is any torsion free soluble group with finite rank, then G contains an* FI*-subgroup of finite index.*

Proof. We may assume that $G/F(G)$ is abelian. Let $1 = Z_0 < Z_1 < \ldots < Z_c = F(G)$ be the upper central series of $F(G) = F$, and $C_i = C_G(Z_i/Z_{i-1})$, $i = 1, \ldots, c$. Then $F = \bigcap_{i=1}^{c} C_i$, and each of the groups G/C_i can be thought of as an abelian group of matrices over Q. The torsion subgroup of each G/C_i is finite ([*12*], 9.33), and it follows that the torsion subgroup T/F of G/F is finite. Hence G/F splits over T/F, and there exists a subgroup G_1 of finite index in G, such that G_1/F is torsion free. Since $F = F(G_1)$, we may even assume that $G/F(G)$ is torsion free abelian.

The proof now falls into two parts.

(1) There exists a subgroup G_0 of G such that $G_0 \geq F(G)$, $|G : G_0| < \infty$ and $H/F(H)$ is torsion free, for every $H \leq G_0$.

(2) G_0 is FI.

We prove (1) by induction on $h(G)$ the Hirsch number of G. If $h(G) = 0$ there is nothing to do, so assume $h(G) > 0$. Let B be an abelian normal subgroup of minimal rank of G contained in the centre $Z(F(G))$ of $F(G)$, and let $A = {}^{G}\!\sqrt{B}$. Then $A \leq F(G)$ since $G/F(G)$ is torsion free, and hence, by the theory of isolators in nilpotent groups, [*6*], A is a subgroup of $Z(F(G))$. By induction, there is a subgroup G_1/A of G/A, containing $F(G/A)$ and hence $F(G)A/A$, such that $|G : G_1| < \infty$ and G_1/A satisfies (1).

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Let $C = C_G(A)$. Then $C \geq F(G)$, and we have seen above that the torsion subgroup of G/C is finite, so that G/C contains a torsion free subgroup G_2/C of finite index. Let $G_0 = G_1 \cap G_2$. Then $|G : G_0| < \infty$ and $G_0 \geq F(G)$.

Now let $H \leq G_0$. If $H \cap A = 1$, then $H \simeq HA/A$, and the fact that $H/F(H)$ is torsion free follows from the properties of G_1/A . Suppose that $H \cap A \neq 1$, and let h be an element of H such that $h^n \in F(H)$, for some $n \geq 1$. Since $H \cap A \leq F(H)$, we have $[H \cap A, h^n, \ldots, h^n] = 1$, and hence $C_{H\cap A}(h^n) \neq 1$. Hence $C_A(h^n)$ is a non trivial isolated subgroup of A , and is normal in G since G/C is abelian. Therefore $h^n \in C$. But $G_0/(G_0 \cap C)$ is torsion free. Hence $h \in C$. If $F/A \cap H = F(H/(A \cap H))$, we also know by induction that $h \in F$. Hence $h \in C_F(A \cap H)$, a nilpotent normal subgroup of H . Therefore $f \in F(H)$, as required.

Next we will prove (2). We have to show that if G is a torsion free solvable group with finite rank and

(*) $\qquad H/F(H)$ is torsion-free abelian for every $H \leq G$,

then G/N has the property (*) whenever $N \lhd G$ and G/N is torsion free. Suppose this is false, and let r be the smallest integer for which there exists a counterexample G with $h(G) = r$. Among all pairs (G, N) which furnish a counterexample with $h(G) = r$, choose one with $h(N)$ minimal. Then G/N contains a subgroup H/N such that $(H/N)/F(H/N)$ is not torsion free. We may clearly assume that $H = G$.

Let $F/N = F(G/N)$, and choose an element $t \in G\backslash F$ such that $t^m \in F$ for some $m > 0$. Let $G_1 = F(G)\langle t \rangle$, $N_1 = N \cap G_1$, $F_1 = F \cap G_1$. Then $G_1/N_1 \simeq G_1N/N$, and under this isomorphism, F_1/N_1 corresponds to $(F \cap G_1)N/N = F/N \cap G_1N/N = F(G_1N/N)$, as $G_1N/N \lhd G/N$. We have $t \in G_1$, $t \notin F_1$, $t^m \in F_1$. Thus (G_1, N_1) is a counterexample, so we may assume that $G = G_1$, that is $G = F(G)\langle t \rangle$.

Next we notice that, if $Z = Z(F(G))$, then

(1) $$C_Z(t) = 1 .$$

Clearly $C_Z(t) = Z(G)$. Let Y denote this subgroup, which is isolated in G since Z is. By the minimality of $h(N)$, there is no normal subgroup M of G such that $1 < M < N$ and G/M is torsion-free. Clearly $N \cap F(G) \neq 1$, and so $N \cap Z \neq 1$. Since Z is isolated in $F(G)$, [6], and $G/F(G)$ is torsion free Z is isolated in G , so $N \leq Z$. Hence $YN \leq Z$, and the isolator ${}^G\sqrt{YN}$ of YN in G is an abelian normal subgroup of G . Also $\sqrt{YN}/N = V$ is a torsion free abelian normal subgroup of G/N , and $V/C_V(G)$ is periodic, since $C_V(G) \geq YN/N$. Hence $[V, G] = 1$, that is $\sqrt{YN}/N \leq Z(G/N)$.

Now trivial arguments show that G/Y has the property (*). If $Y \neq 1$, then the minimality of $h(G)$ shows that $G/\sqrt{YN}$ has the property (*). Hence G/F is torsion free, a contradiction. This proves (1).

Clearly $Z/N \leq F/N$, and since $t^m \in F$, we have

$$[Z, t^m, \ldots, t^m] \leq N .$$

In particular, if $Z > N$, then $C_{Z/N}(t^m) = K/N \neq 1$. Now commutation with t^m induces an endomorphism ζ of K whose image lies in N and so has smaller rank than K . Hence $\ker \zeta = C_K(t^m) \neq 1$. Let $L = C_K(t^m)$. Then $t^m \in F(L\langle t\rangle)$, and by (*), $t \in F(L\langle t\rangle)$, that is $L\langle t\rangle$ is nilpotent. Therefore $[L, t, \ldots, t] = 1$, and $C_L(t) \neq 1$. This contradicts (1). We deduce that $Z = N$.

It clearly follows that $F(G)' \neq 1$. Let $U/F(G)'$ be the torsion subgroup of $F(G)/F(G)'$. Then $N \leq U$, as $N \cap F(G)' \neq 1$. Since $F(G) \leq F$, we deduce that

$$[F(G), t^m, \ldots, t^m] \leq U .$$

Let c be the nilpotency class of $F(G)$. Then

$$1 \neq \gamma_c(F(G)) \leq N ,$$

where $\{\gamma_i(X)\}$ is the lower central series of a group X.

If $x_1, \ldots, x_c \in F(G)$, then since $\gamma_c(F(G))$ is torsion free, the value of $[x_1, \ldots, x_c]$ only depends on the value of $x_1, \ldots, x_c$ modulo U. We obtain a well-defined $\langle t \rangle$-module epimorphism of $F(G)/U \otimes \ldots \otimes F(G)/U$ (with c factors) onto $\gamma_c(F(G))$, namely

$$x_1 U \otimes \ldots \otimes x_c U \to [x_1, \ldots, x_c]$$

(*cf.* [12], Part 1, p. 55). The tensor product is to be viewed as a $\langle t \rangle$-module via the diagonal action, the action on the individual factors being by conjugation. Since $F(G)/U$ has a finite series with t^m-trivial factors, so does the tensor product, and hence also $\gamma_c(F(G))$ (*cf.* [12], Part 1, p. 56). Hence $C_N(t^m) \neq 1$. Arguing as in the previous paragraph, we deduce that $C_N(t) \neq 1$, and obtain a contradiction to (1). This concludes the proof.

We note the following useful properties of FI-groups, which are extensions of facts well known for torsion free nilpotent groups.

LEMMA 16. *Let G be a FI-group, $N \triangleleft H \leq G$, and suppose that H/N is torsion free. Let $x, y \in H/N$. Then*

(i) if $x^r = y^s$ $(r, s \neq 0)$, then $\langle x, y \rangle$ is cyclic;

(ii) if $x^r = y^r$ $(r \neq 0)$ then $x = y$;

(iii) if $[x^r, y^s] = 1$ $(r, s \neq 0)$ then $[x, y] = 1$.

Proof. We may clearly suppose that $G = \langle x, y \rangle$.

(i) Clearly $x^r = y^s \in Z(G) \leq F(G)$. Since G is an FI-group it follows that $Z(G)$ is isolated in G; so $x, y \in Z(G)$ and G is torsion free abelian. Hence G is cyclic.

(ii) follows from *(i)*.

(iii) Let $z = x^r$. Then $y^s = (y^s)^z = (y^z)^s$. From (ii) we obtain $y = y^z$, that is $[x^r, y] = 1$. Repeating the argument we get

$[x, y] = 1$.

We make a final note before ending the paper. Recall the definition of a CS-group G . For any given cyclic group B and subgroup $A \leq B$ with B/A finite, there exists $N \lhd G$ with G/N finite and $N \cap B = A$. If we were to drop the condition that N be of finite index in G , then we get a larger class, which, for convenience, we shall denote by X . Then it can be shown that every group G in $X \cap L$ is a CS-group where L is any quotient closed subclass of residually finite groups. Also the examples constructed in ([7], Theorem 1) show that a finitely generated centre-by-metabelian X-group that is residually finite p for all but one specified prime p does not need to be a CS-group. If every element of a group G is of prime order then obviously G is an X-group. If in addition G is residually finite, then G is a CS-group. Thus groups in the Burnside variety B_p , p a prime, are X-groups and groups in the variety generated by A_5 are CS-groups, since they are residually finite as shown in ([9], Theorem 1).

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Department of Mathematics,
The University,
Manchester M13 9PL,
England;

Department of Pure Mathematics,
University College, Cardiff,
Cardiff CF1 1XL,
United Kingdom;

Department of Mathematics,
University of Alberta,
Edmonton,
Canada T6G 2G1.