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NEARNESSES AND T_0-EXTENSIONS OF TOPOLOGICAL SPACES

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ABSTRACT. In [3], Reed establishes a bijection between the (equivalence classes of) principal T_1-extensions of a topological space X and the compatible, cluster-generated, Lodato nearnesses on X. We extend Reed's result to the T_0 case by obtaining a one-to-one correspondence between the principal T_0-extensions of a space X and the collections of sets (called "*t-grill sets*") which generate a certain class of nearnesses which we call "*t-bunch generated*" nearnesses. This correspondence specializes to principal T_0-compactifications. Finally, we show that there is a bijection between these t-grill sets and the filter systems of Thron [5], and that the corresponding extensions are equivalent.

The notion of nearness spaces was first introduced by Herrlich in [2]. It has since proven to be a useful tool in the classification of extensions of topological spaces. Reed [3], Bentley and Herrlich [1], and others have used nearnesses to classify the principal T_1-extensions of a T_1-space. In [3], Reed obtained a one-to-one correspondence between the principal T_1-extensions of a T_1-space X and the compatible, cluster-generated Lodato nearnesses on X. In this paper, we look at the question of characterizing T_0-extensions using nearnesses, namely we generalize Reed's result to classify the principal T_0-extensions of a T_0-space X. The nearness induced on X by a T_0-extension need be neither compatible nor cluster-generated nor Lodato. We show, however, that all nearnesses induced by extensions satisfy a condition which is a hybrid of the two conditions "compatible" and "Lodato". Furthermore, such a nearness, although not cluster-generated, is always generated by certain collections of grills. While non-equivalent T_0-extensions of X may induce the same nearness, we obtain a one-to-one correspondence between the principal T_0-extensions of X and these collections of generating grills. Finally, using a dualizing function, we observe that this correspondence is essentially the same as the correspondence that Thron obtains between principal T_0-extensions of X and certain collections of filters on X (cf. [5]).

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We begin by recalling some basic definitions concerning nearnesses, grills, and topological extensions. Throughout this paper, we use the notation of Reed [3].

Preliminary definitions. Let X be a set and let $\mathcal{T}$ be a topology on X.

1. A *nearness* on X is a collection ν of families of subsets of X which satisfy the following four conditions:

(i) $\bigcap \mathcal{A} \neq \varnothing \Rightarrow \mathcal{A} \in \nu$;

(ii) $\varnothing \in \mathcal{A} \Rightarrow \mathcal{A} \notin \nu$;

(iii) If $\mathcal{A} \in \nu$ and each set in $\mathcal{B}$ contains a set in $\mathcal{A}$, then $\mathcal{B} \in \nu$;

(iv) If $\mathcal{A} \vee \mathcal{B} \in \nu$, then $\mathcal{A} \in \nu$ or $\mathcal{B} \in \nu$, where $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}$ and $B \in \mathcal{B}\}$.

2. If ν is a nearness, then $c_\nu(A) = \{x \in X : \{\{x\}, A\} \in \nu\}$. ν is called *Lodato* if $c_\nu(\mathcal{A}) \in \nu \Rightarrow \mathcal{A} \in \nu$.

3. ν is *compatible with* $\mathcal{T}$ if $c_\nu(A) = A^-$, for all $A \subset X$.

4. Let $\chi \subset \nu$. ν is called χ*-generated* if every member of ν is contained in a member of χ.

5. A ν*-cluster* is a maximal element of ν.

6. A *grill* on X is a family σ of subsets of X satisfying the following three conditions:

(i) $A \in \sigma$ and $A \subset B \Rightarrow B \in \sigma$;

(ii) $A \cup B \in \sigma \Rightarrow A \in \sigma$ or $B \in \sigma$;

(iii) $\varnothing \notin \sigma$.

σ is called a *proper* grill if $\sigma \neq \varnothing$.

7. A ν*-bunch* is a grill σ which is a member of ν and which satisfies $c_\nu(A) \in \sigma \Rightarrow A \in \sigma$.

8. An *extension* $\kappa = (e, Y)$ of $(X, \mathcal{T})$ is a topological space Y and a dense embedding $e : X \to Y$. κ is called *strict* or *principal* if the collection $\{e(A)^- : A \subset X\}$ is a base for the closed sets of Y.

An extension of X induces a nearness on X in a natural way:

THEOREM 1 (Herrlich [2]). *Let X be a topological space, and let $\kappa = (e, Y)$ be an extension of X. Let $\nu_\kappa = \{\mathcal{A} \subset \mathcal{P}(X) : \bigcap e(\mathcal{A})^- \neq \varnothing\}$. Then ν_κ is a nearness on X. Furthermore, equivalent extensions of X induce the same nearness on X.*

The nearness ν_κ in the above theorem is generated by the sets $\tau(y) = \{A \subset X : y \in e(A)^-\}$, for $y \in Y$. The collection $\{\tau(y) : y \in Y\}$ is called the *trace system* induced by κ.

Now suppose that X is a T_1-space, and let ν be a compatible, cluster-generated, Lodato nearness on X. In [3], Reed shows how to construct a T_1-extension κ_ν of X, induced by ν in a natural manner. The construction is similar to that of Bentley and Herrlich [1]. Let Y_ν denote the set of ν-clusters. For $A \subset X$, let $A^\nu = \{\sigma \in Y_\nu : A \in \sigma\}$. Then $\{A^\nu : A \subset X\}$ forms a base for the closed sets of a topology on Y_ν. Define a map $e_\nu : X \to Y$ by $e_\nu(x) = \{A \subset X : x \in A^-\}$, and let $\kappa_\nu = (e_\nu, Y_\nu)$.

THEOREM 2 (Reed [3]). *Let X be a T_1-space and let ν be a compatible, cluster-generated, Lodato nearness on X. Then κ_ν is a principal T_1-extension of X.*

In [3], Reed shows that the maps $\kappa \mapsto \nu_\kappa$ and $\nu \mapsto \kappa_\nu$ are inverses on the sets of (equivalence classes of) principal T_1-extensions of a T_1-space X and the compatible, cluster-generated, Lodato nearnesses on X, thus obtaining the following lovely result:

THEOREM 3 (Reed [3]). *The map $\kappa \mapsto \nu_\kappa$ is a one-to-one correspondence between the principal T_1-extensions of a T_1-space X and the compatible, cluster-generated, Lodato nearnesses on X.*

If $\kappa = (e, Y)$ is a T_1-extension of X, then the clusters of ν_κ are the sets $\tau(y)$ for $y \in Y$.

We wish to find an appropriate analogue of Theorem 3 with "T_1" replaced by "T_0". If κ is a principal T_0-extension of X, then one can form $\nu_\kappa = \{\mathcal{A} : \bigcap e(\mathcal{A})^- \neq \emptyset\}$ as in Theorem 1 but this nearness may no longer have any of the properties of the nearnesses of Theorem 3, as the following example illustrates:

EXAMPLE 1. Let $\mathcal{N}_z$ denote the neighborhood filter of an element z in a topological space. It's easy to show that for any extension $\kappa = (e, Y)$ (principal or not) of a topological space X, the following condition holds:

$$\text{For all } y, z \in Y, \mathcal{N}_y \subset \mathcal{N}_z \Rightarrow \tau(z) \subset \tau(y).$$

(If κ is principal, then the converse also holds, cf. Reed [3].) Also, any cluster in the induced nearness must be of the form $\tau(y)$ for some $y \in Y$, since $\sigma \in \nu_\kappa \Rightarrow \exists y \in \bigcap e(\sigma)^- \Rightarrow \sigma \subset \tau(y)$. Now consider the trivial extension $\kappa = (1, X)$ where $X = \mathbb{Z}$ is the set of integers equipped with the right-interval topology (i.e. the basic open sets are the intervals $[n, \infty)$ for $n \in \mathbb{Z}$). Then X is T_0 but not T_1, and for $m \in X$, $\mathcal{N}_m$ is the filter of subsets of X which contain $[m, \infty)$. Hence, $m < n \Rightarrow \mathcal{N}_m$ is a proper subset of $\mathcal{N}_n$. By the above remarks, $m < n$ implies that $\tau(n)$ is a proper subset of $\tau(m)$, so that ν_κ is not cluster-generated. We next describe A^- and $c_\nu A$ to see that ν_κ is not compatible: First, $m \in A^-$ if and only if $[m, \infty) \cap A \neq \emptyset$. Thus, $A^- = \{m : m \leq a$ for some $a \in A\}$. Next, $m \in c_\nu A$ if and only if $\{m\}^- \cap A^- \neq \emptyset$ if and only if $(-\infty, m] \cap A^- \neq \emptyset$. But this holds as long as $A \neq \emptyset$ since $a \in A \Rightarrow (-\infty, a] \subset A^-$. Thus, for $A \neq \emptyset$, $c_\nu A = X$, so that ν_κ is not compatible. Finally, we see that ν_κ is not Lodato: Consider $\mathcal{A} = \{\{n\} : n \in \mathbb{Z}\}$. Then $c_\nu \mathcal{A} \in \nu_\kappa$ since $c_\nu \mathcal{A} = \{X\}$. But $\mathcal{A} \notin \nu_\kappa$, since $\bigcap \mathcal{A}^- = \bigcap \{(-\infty, n] : n \in \mathbb{Z}\} = \emptyset$. Thus, this extension possesses none of the properties of the nearnesses of Theorem 3.

The question then is what nearnesses are induced by T_0-extensions, or for that matter, by arbitrary extensions of a topological space? The answer is that while such nearnesses are in general neither compatible nor Lodato, they satisfy a condition related to both of these. In addition, while they are generally not cluster-generated, these nearnesses still possess relatively simple generating sets.

THEOREM 4. *Let $\kappa=(e, Y)$ be an (arbitrary) extension of a topological space X. Then the nearness ν_κ is t*-Lodato (*"topologically Lodato"*), *i.e. ν_κ satisfies the following condition*:

$$(*) \qquad \mathscr{A}^- \in \nu_\kappa \Rightarrow \mathscr{A} \in \nu_\kappa.$$

The proof is trivial: $\mathscr{A}^- \in \nu_\kappa \Rightarrow \bigcap e(\mathscr{A}^-)^- \neq \varnothing$. But, since e is an embedding, $e(\mathscr{A}^-)^- = e(\mathscr{A})^- \Rightarrow \bigcap e(\mathscr{A})^- \neq \varnothing \Rightarrow \mathscr{A} \in \nu_\kappa$. □

Note that the condition $(*)$ is a hybrid of the two conditions "Lodato" and "compatible". It appears to be the most one can ask for in terms of compatibility for arbitrary extensions.

We next examine the structure of the generating sets of nearnesses that are induced by extensions. Recall that, if ν is a nearness on a set X, then a *bunch* is a proper grill σ which is a member of ν and which satisfies $c_\nu A \in \sigma \Rightarrow A \in \sigma$.

DEFINITION 1. Let (X, ν) be a nearness space. If X possesses a topology (perhaps unrelated to ν), then a *t-bunch* is a proper grill σ which is a member of ν and which satisfies $A^- \in \sigma \Rightarrow A \in \sigma$. The nearness ν is called *bunch-generated* (resp. *t-bunch-generated*) if each element of ν is contained in some bunch (resp. t-bunch). Note that if ν is bunch-generated (resp. t-bunch-generated) then ν is Lodato (resp. t-Lodato).

THEOREM 5. *Let $\kappa=(e, Y)$ be an extension of a topological space X. Then ν_κ is a t-bunch-generated nearness on X.*

Proof. ν_κ is generated by $\{\tau(y): y \in Y\}$ and each $\tau(y)$ is clearly a t-bunch. □

A nearness induced by a non-T_1 extension is usually not bunch-generated. In Example 1, for instance, we see that ν_κ cannot be bunch-generated since it is not Lodato.

Theorems 4 and 5 describe which nearnesses are induced by topological extensions: If $\kappa=(e, Y)$ is any extension of any topological space X, then ν_κ is a t-bunch-generated (hence t-Lodato) nearness on X. To complete the analogy with Reed's result for T_1 spaces, we would like to find an inverse for the map $\kappa \mapsto \nu_\kappa$. Unfortunately, this map is no longer one-to-one, as the following example demonstrates:

EXAMPLE 2. Let $X=\mathbb{R}$ with the usual topology and let $\kappa_1=(1, X)$ be the trivial extension. Let $\kappa_2=(i, Y)$, where $Y=\mathbb{R}\cup\{z\}$, $i: X \to Y$ is inclusion, and Y has the following topology: For $x \in \mathbb{R}$, $x \neq 0$, $U \in \mathcal{N}_Y(x)$ if and only if

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$U \cap X \in \mathcal{N}_X(x)$. $V \in \mathcal{N}_Y(0)$ if and only if $V = U \cup \{z\}$, where $U \in \mathcal{N}_X(0)$. Finally, $V \in \mathcal{N}_Y(z)$ if and only if V contains $\{z\} \cup (0, \varepsilon)$ for some $\varepsilon > 0$. Then Y is a principal T_0-extension of X and $\mathcal{N}_Y(0)$ is a proper subset of $\mathcal{N}_Y(z)$. If $\tau_i(x)$ denotes the trace of $x \in X$ relative to κ_i, then $\tau_1(x) = \tau_2(x)$ for all x. Also $\tau(z) \subset \tau_2(0)$. Since ν_{κ_2} is generated by $\{\tau(z)\} \cup \{\tau_2(x) : x \in X\} = \{\tau(z)\} \cup \{\tau_1(x) : x \in X\}$, we have $\nu_{\kappa_1} = \nu_{\kappa_2}$.

In the above example, two different principal T_0-extensions induce the same nearness on the base space. Note, however, that the generating sets given by the respective trace systems are different. Thus, while the map $\kappa \mapsto \nu_\kappa$ is not one-to-one, there may be a one-to-one correspondence between extensions and certain generating sets of nearnesses.

DEFINITION 2. Let X be a topological space and let $\mathscr{C}$ be a collection of proper grills on X satisfying the following two conditions:

(i) $A^- \in \sigma \in \mathscr{C} \Rightarrow A \in \sigma$, and

(ii) For all $x \in X$, $\sigma_x = \{A \subset X : x \in A^-\} \in \mathscr{C}$.

Then $\mathscr{C}$ will be called a *t-grill set* and its members will be called *t-grills*.

If $\mathscr{C}$ is a t-grill set and we define $\nu_{\mathscr{C}} = \{\mathscr{A} : \mathscr{A} \subset \sigma$ for some $\sigma \in \mathscr{C}\}$, then $\nu_{\mathscr{C}}$ is a nearness generated by $\mathscr{C}$ and the elements of $\mathscr{C}$ are t-bunches of $\nu_{\mathscr{C}}$. Hence, $\nu_{\mathscr{C}}$ is a t-bunch-generated (and so also t-Lodato) nearness. Conversely, if ν is any t-bunch-generated nearness, then any collection of t-bunches containing all σ_x is a t-grill-set.

If $\kappa = (e, Y)$ is an extension of a topological space X, then the trace system $\mathscr{C}_\kappa$ induced by κ is a t-grill set on X.

Let $\mathscr{C}$ be a t-grill set on a T_0-space X. We now describe how to construct an extension of X induced by $\mathscr{C}$: Let $Y_{\mathscr{C}} = \mathscr{C}$. For $A \subset X$, let $A^{\mathscr{C}} = \{\sigma \in \mathscr{C} : A \in \sigma\}$. Then $\{A^{\mathscr{C}} : A \subset X\}$ is a base for the closed sets of a topology on $Y_{\mathscr{C}}$. Lastly, define a map $e_{\mathscr{C}} : X \to Y_{\mathscr{C}}$ by $e_{\mathscr{C}}(x) = \sigma_x$.

LEMMA. *Let $\mathscr{C}$ be a t-grill set on a T_0-space X and let $A \subset X$. Then $A^{\mathscr{C}} = (A^-)^{\mathscr{C}} = e_{\mathscr{C}}(A)^-$.*

Proof. The proof is essentially the same as that of Reed [3] Lemma 1.16, but we include it here for completeness. The first equality follows from the definition of a grill and condition (i) of Definition 2. To see that $A^{\mathscr{C}} \subset e_{\mathscr{C}}(A)^-$, let $\sigma \in A^{\mathscr{C}}$. Let $B \subset X$ with $e_{\mathscr{C}}(A) \subset B^{\mathscr{C}}$. Then $A \subset B^- \Rightarrow \sigma \in (B^-)^{\mathscr{C}} = B^{\mathscr{C}}$. Thus σ is an element of every basic closed set containing $e_{\mathscr{C}}(A)$, and so $\sigma \in e_{\mathscr{C}}(A)^-$. For the opposite containment, note that $e_{\mathscr{C}}(A) \subset A^{\mathscr{C}}$. Since $A^{\mathscr{C}}$ is closed, this implies that $e_{\mathscr{C}}(A)^- \subset A^{\mathscr{C}}$. □

THEOREM 6. *Let $\mathscr{C}$ be a t-grill set on a T_0-space X. Then $\kappa_{\mathscr{C}} = (e_{\mathscr{C}}, Y_{\mathscr{C}})$ is a principal T_0-extension of X. Furthermore, $\mathscr{C} = \mathscr{C}_{\kappa_{\mathscr{C}}}$.*

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Proof. (1) $e_{\mathscr{C}}$ is one-to-one: It's easy to see that X is T_0 if and only if $\sigma_x = \sigma_y \Rightarrow x = y$. Thus, $e_{\mathscr{C}}$ is one-to-one.

(2) $e_{\mathscr{C}}$ is a dense embedding: If K is closed in X, then $e_{\mathscr{C}}^{-1}(K^{\mathscr{C}}) = K$. Thus, $e_{\mathscr{C}}$ is continuous. Also, for K closed, $e_{\mathscr{C}}(K) = e_{\mathscr{C}}(K)^- \cap e_{\mathscr{C}}(X)$, which implies that $e_{\mathscr{C}}$ is a closed map. Finally, $e_{\mathscr{C}}(X)$ is dense in $Y_{\mathscr{C}}$, since $e_{\mathscr{C}}(X)^- = X^{\mathscr{C}} = Y_{\mathscr{C}}$.

(3) $Y_{\mathscr{C}}$ is T_0: Let $\sigma \neq \sigma'$, say $A \in \sigma \backslash \sigma'$. Then $Y_{\mathscr{C}} \backslash A^{\mathscr{C}}$ is a neighborhood of σ' which misses σ.

(4) $Y_{\mathscr{C}}$ is principal: This follows from the lemma.

(5) $\mathscr{C} = \mathscr{C}_{\kappa_{\mathscr{C}}}$: Let $\sigma \in \mathscr{C}$. Then $A \in \sigma$ if and only if $\sigma \in A^{\mathscr{C}}$ if and only if $\sigma \in e_{\mathscr{C}}(A)^-$, i.e. $\sigma = \tau(\sigma)$. □

THEOREM 7. *Let $\kappa = (e, Y)$ be a principal T_0-extension of a T_0-space X. Then κ is equivalent to $\kappa_{\mathscr{C}_\kappa}$.*

Proof. The trace map $\tau: Y \to Y_{\mathscr{C}_\kappa}$ gives the required equivalence:

(1) τ is a bijection: τ is certainly onto. Suppose $y, z \in Y$ with $y \neq z$. Since Y is principal T_0, choose $A \subset X$ such that $y \in e(A)^-$ and $z \notin e(A)^-$. Then $A \in \tau(y) \backslash \tau(z)$ which implies that τ is one-to-one.

(2) τ is continuous: For $A \subset X$, $\tau^{-1}(A^{\mathscr{C}_\kappa}) = e(A)^-$.

(3) τ is a closed map: $\tau(e(A^-)) = A^{\mathscr{C}_\kappa}$ for $A \subset X$.

(4) $\tau e = e_{\mathscr{C}_\kappa}$: $A \in \tau e(x)$ if and only if $e(x) \in e(A)^-$ if and only if $x \in A^-$ if and only if $A \in e_{\mathscr{C}_\kappa}(x)$. □

Theorems 6 and 7 establish a bijection between the set of (equivalence classes of) principal T_0-extensions of X and the t-grill sets on X. In particular, every t-bunch generated nearness ν on X is of the form ν_κ for some principal T_0-extension κ of X. Conversely, every principal T_0-extension κ of X is equivalent to $\kappa_{\mathscr{C}}$ where $\mathscr{C}$ is a generating set of some t-bunch generated nearness on X. Note, however, that the bijection is not between extensions and nearnesses, but rather it is between extensions and what are essentially "bases" of nearnesses.

Referring back to Reed's result (Theorem 3), we see that a compatible, cluster-generated, Lodato nearness is a t-bunch-generated nearness (the clusters are t-bunches), and so our result generalizes Theorem 3. In particular, the t-grill sets which give rise to T_1-extensions are those which consist entirely of maximal elements. Of course, a cluster-generated, compatible, Lodato nearness on a T_1-space X will in general have other generating sets which will induce principal T_0-extensions that are not T_1.

The correspondence established in Theorems 6 and 7 also specializes to a one-to-one correspondence between the principal T_0-compactifications of a T_0-space X and the *contigual t-grill sets* on X. A contigual t-grill set is a t-grill set whose induced nearness is contigual. In other words, a t-grill set $\mathscr{C}$ is contigual if it satisfies the following condition: If $\mathscr{A} \subset \mathscr{P}(X)$ has the property that each finite subcollection $\mathscr{B}$ of $\mathscr{A}$ is contained in some element $\sigma_{\mathscr{B}}$ of $\mathscr{C}$, then there exists $\sigma \in \mathscr{C}$ such that $\mathscr{A} \subset \sigma$. Thus a principal T_0-compactification always induces a contigual t-bunch generated nearness on the base space.

One can consider similar questions for spaces that are not T_0, but the map $\kappa \mapsto \mathscr{C}_\kappa$ is of course no longer one-to-one. A one-to-one correspondence can be obtained by considering "indexed" t-grill sets (i.e. pairs (σ, S_σ) where each σ is an element of some t-grill set and S_σ is a non-empty set which essentially counts the number of points in the extension having trace σ), but we will not discuss that construction here.

A central observation of this note is that t-bunch-generated nearnesses seem to be the appropriate object to examine if one wishes to study the (principal) T_0-extensions of a T_0-space X.

Finally, we observe that our t-grill sets are simply the duals of the filter systems used by Thron in [5] to characterize the principal T_0-extensions of a T_0-space:

A *filter system* in the sense of Thron is a collection θ of proper open filters on a T_0-space X such that θ contains all the neighborhood filters. Thron constructs an extension $l_\theta = (f_\theta, Z_\theta)$ as follows: $Z_\theta = \theta$, $f_\theta(x) = \mathcal{N}_x$ for $x \in X$, and the topology on Z_θ has as open base the family of all sets $A^\theta = \{\mathscr{F} \in \theta : A \in \mathscr{F}\}$, where A ranges over all subsets of X. He then shows that this is a principal extension of X. Conversely, suppose $l = (f, Z)$ is an extension of X, and for $z \in Z$, let $\mathscr{F}(z)$ be the filter generated by $f^{-1}(\mathcal{N}_z)$. Then the family $\theta_l = \{\mathscr{F}(z) : z \in Z\}$ is a filter system on X. On the set of principal T_0-extensions of a T_0-space X, the map $l \mapsto \theta_l$ is an inverse for the map $\theta \mapsto l_\theta$, and so Thron has established the following result:

THEOREM 8 (Thron [5]). *Let X be a T_0-space. The map $\theta \mapsto l_\theta$ is a one-to-one correspondence between the filter systems on X and the principal T_0-extensions of X.*

In [4], Thron introduces the dual map $d : \mathscr{P}^2(X) \to \mathscr{P}^2(X)$ given by:

$$d(\mathscr{A}) = \{B : \sim B \notin \mathscr{A}\}.$$

For more background on the dual map, see [4].

THEOREM 9. *Let X be a T_0-space. The dual map provides a one-to-one correspondence between the t-grill sets on X and the filter systems on X. Furthermore, d is an equivalence between the corresponding extensions $\kappa_{\mathscr{C}}$ and l_θ, where $\theta = d(\mathscr{C})$.*

Proof. We recall first that the dual of a filter is a grill and vice-versa, and furthermore, d is its own inverse (cf. [4]). To see that d is a one-to-one correspondence between the t-grill sets on X and the filter systems on X, note first that a grill σ satisfies the condition $A^- \in \sigma \Rightarrow A \in \sigma$ if and only if $d(\sigma)$ is

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an open filter. Also, it's easy to see that for all $x \in X$, $d(\sigma_x) = \mathcal{N}_x$. This establishes the first claim.

Now, let $\mathscr{C}$ be a t-grill set and let $\theta = d(\mathscr{C})$. Let $\kappa_{\mathscr{C}} = (e_{\mathscr{C}}, Y_{\mathscr{C}})$ and $l_\theta = (f_\theta, Z_\theta)$. By the above remarks, $de = f$. To see that d is a homeomorphism between $Y_{\mathscr{C}}$ and Z_θ, note that $d(A^{\mathscr{C}}) = Z_\theta \backslash A^\theta$ and $d(A^\theta) = Y_{\mathscr{C}} \backslash A^{\mathscr{C}}$. Hence, d is an equivalence between $\kappa_{\mathscr{C}}$ and l_θ. $\square$

Table 1, below, indicates the various correspondences discussed in this note.

The author wishes to thank Ellen Reed for introducing her to nearness spaces, and for suggesting the question of generalizing Reed's result for T_1-spaces to the T_0 case.

TABLE 1. In the following table, X is a T_0-space, "∪" indicates that a collection is contained in the one above it, "↔" indicates a one-to-one correspondence, and "←" or "→" indicates a surjection.

Nearnesses		*Extensions*		*t-grill sets*
t-Lodato nearnesses (on X)			filter systems (on X)	
∪		(Thron) ↗↙		↖↘
t-bunch generated nearnesses	←	principal T_0-extensions (of X)	↔	t-grill sets (on X)
∪		∪		∪
cluster-generated compatible Lodato nearnesses	(Reed) ↔	principal T_1-extensions	↔	t-grill sets all of whose elements are maximal
contigual t-bunch generated nearnesses	←	principal T_0-compactifications	↔	contigual t-grill sets

REFERENCES

1. H. L. Bentley and H. Herrlich, *Extensions of Topological Spaces*, Topology, Lecture Notes in Pure and Appl. Math. 24, Dekker, New York, 1976, 129–184.
2. H. Herrlich, *A Concept of Nearness*, Gen. Top. Appl. **4** 191–212 (1974).
3. E. E. Reed, *Nearnesses, proximities, and T_1-compactifications*, Trans. Amer. Math. Soc. **236** 193–207 (1978).
4. W. J. Thron, *Proximity Structures and Grills*, Math. Ann. **206** 35–62 (1973).
5. W. J. Thron, *Topological Structures*, Holt, Rinehart, and Winston, New York, 1966.

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