J. Aust. Math. Soc. **96** (2014), 275–288
doi:10.1017/S1446788713000669

SYMMETRIC GRAPHS WITH 2-ARC TRANSITIVE QUOTIENTS

GUANGJUN XU✉ and SANMING ZHOU

(Received 27 November 2012; accepted 20 November 2013)

Communicated by B. Alspach

Abstract

A graph Γ is G-symmetric if Γ admits G as a group of automorphisms acting transitively on the set of vertices and the set of arcs of Γ, where an arc is an ordered pair of adjacent vertices. In the case when G is imprimitive on $V(\Gamma)$, namely when $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$, the quotient graph $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$ is always G-symmetric and sometimes even $(G, 2)$-arc transitive. (A G-symmetric graph is $(G, 2)$-arc transitive if G is transitive on the set of oriented paths of length two.) In this paper we obtain necessary conditions for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive (regardless of whether Γ is $(G, 2)$-arc transitive) in the case when $v - k$ is an odd prime p, where v is the block size of $\mathcal{B}$ and k is the number of vertices in a block having neighbours in a fixed adjacent block. These conditions are given in terms of v, k and two other parameters with respect to $(\Gamma, \mathcal{B})$ together with a certain 2-point transitive block design induced by $(\Gamma, \mathcal{B})$. We prove further that if $p = 3$ or 5 then these necessary conditions are essentially sufficient for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive.

2010 *Mathematics subject classification*: primary 05C25; secondary 05E18.

Keywords and phrases: symmetric graph, arc-transitive graph, 2-arc transitive graph, 3-arc graph, transitive block design.

1. Introduction

A graph $\Gamma = (V(\Gamma), E(\Gamma))$ is *G-symmetric* if Γ admits G as a group of automorphisms such that G is transitive on $V(\Gamma)$ and on the set of arcs of Γ, where an *arc* is an ordered pair of adjacent vertices. If in addition Γ admits a *nontrivial G-invariant partition*, that is, a partition $\mathcal{B}$ of $V(\Gamma)$ such that $1 < |B| < |V(\Gamma)|$ and $B^g := \{\alpha^g : \alpha \in B\} \in \mathcal{B}$ for any $B \in \mathcal{B}$ and $g \in G$ (where α^g is the image of α under g), then Γ is called an *imprimitive G-symmetric graph*. In this case the *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$ is defined to have vertex set $\mathcal{B}$ such that $B, C \in \mathcal{B}$ are adjacent if and only if there exists at least

Guangjun Xu was supported by the MIFRS and SFS scholarships of the University of Melbourne. Sanming Zhou was supported by a Future Fellowship (FT110100629) of the Australian Research Council.

one edge of Γ between B and C. It is readily seen that $\Gamma_{\mathcal{B}}$ is G-symmetric under the induced action of G on $\mathcal{B}$. We assume that $\Gamma_{\mathcal{B}}$ contains at least one edge, so that each block of $\mathcal{B}$ is an independent set of Γ. Denote by $\Gamma(\alpha)$ the neighbourhood of $\alpha \in V(\Gamma)$ in Γ, and define $\Gamma(B) = \bigcup_{\alpha \in B} \Gamma(\alpha)$ for $B \in \mathcal{B}$. For blocks $B, C \in \mathcal{B}$ adjacent in $\Gamma_{\mathcal{B}}$, let $\Gamma[B, C]$ be the bipartite subgraph of Γ induced by $(B \cap \Gamma(C)) \cup (C \cap \Gamma(B))$. Since $\Gamma_{\mathcal{B}}$ is G-symmetric, up to isomorphism $\Gamma[B, C]$ is independent of the choice of (B, C). Define $\Gamma_{\mathcal{B}}(\alpha) := \{C \in \mathcal{B} : \Gamma(\alpha) \cap C \neq \emptyset\}$ and $\Gamma_{\mathcal{B}}(B) := \{C \in \mathcal{B} : B$ and C are adjacent in $\Gamma_{\mathcal{B}}\}$, the latter being the neighbourhood of B in $\Gamma_{\mathcal{B}}$. Define

$$v := |B|, \quad k := |B \cap \Gamma(C)|, \quad r := |\Gamma_{\mathcal{B}}(\alpha)|, \quad b := \mathrm{val}(\Gamma_{\mathcal{B}})$$

to be the block size of $\mathcal{B}$, the size of each part of the bipartition of $\Gamma[B, C]$, the number of blocks containing at least one neighbour of a given vertex, and the valency of $\Gamma_{\mathcal{B}}$, respectively. These parameters depend on $(\Gamma, \mathcal{B})$ but are independent of $\alpha \in V(\Gamma)$ and adjacent $B, C \in \mathcal{B}$.

In [6] Gardiner and Praeger introduced a geometrical approach to imprimitive symmetric triples $(\Gamma, G, \mathcal{B})$, which involves $\Gamma_{\mathcal{B}}$, $\Gamma[B, C]$ and an incidence structure $\mathcal{D}(B)$ with point set B and block set $\Gamma_{\mathcal{B}}(B)$. A 'point' $\alpha \in B$ and a 'block' $C \in \Gamma_{\mathcal{B}}(B)$ are incident in $\mathcal{D}(B)$ if and only if $\alpha \in \Gamma(C)$; we call (α, C) a flag of $\mathcal{D}(B)$ and write $\alpha \mathrm{I} C$. It is clear that $\mathcal{D}(B) = (B, \Gamma_{\mathcal{B}}(B), I)$ is a 1-(v, k, r) design [6] with b blocks which admits G_B as a group of automorphisms acting transitively on its points, blocks and flags, where G_B is the setwise stabilizer of B in G. Note that $vr = bk$. Define $\overline{\mathcal{D}}(B) := (B, \Gamma_{\mathcal{B}}(B), \bar{\mathrm{I}})$ to be the complementary structure [12] of $\mathcal{D}(B)$ for which $\alpha \bar{\mathrm{I}} C$ if and only if $\alpha \notin \Gamma(C)$. Then $\overline{\mathcal{D}}(B)$ is 1-$(v, v-k, b-r)$ design with b blocks. Up to isomorphism $\mathcal{D}(B)$ and $\overline{\mathcal{D}}(B)$ are independent of B. The cardinality of $\{D \in \Gamma_{\mathcal{B}}(B) : \Gamma(D) \cap B = \Gamma(C) \cap B\}$, denoted by m, is independent of the choice of adjacent $B, C \in \mathcal{B}$ and is called the *multiplicity* of $\mathcal{D}(B)$.

An *s-arc* of Γ is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of $s + 1$ vertices of Γ such that α_i, α_{i+1} are adjacent for $i = 0, \ldots, s-1$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $i = 1, \ldots, s-1$. If Γ admits G as a group of automorphisms such that G is transitive on $V(\Gamma)$ and on the set of s-arcs of Γ, then Γ is called (G, s)-*arc transitive* [2]. A $(G, 1)$-arc transitive graph is precisely a G-symmetric graph, and a (G, s)-arc transitive graph is $(G, s-1)$-arc transitive.

This paper was motivated by the following questions asked in [7]: When does a quotient of a symmetric graph admit a natural 2-arc transitive group action? If there is such a quotient, what information does this give us about the original graph? These questions were studied in [7, 8, 10, 11, 13, 14, 16, 17], with a focus on the case where $v - k \geq 1$ or $k \geq 1$ is small. In the present paper we consider the more general case where $k = v - p$ for a prime $p \geq 3$. In this case we obtain necessary conditions for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive, regardless of whether Γ is $(G, 2)$-arc transitive. We prove further that when $p = 3$ or 5 such necessary conditions are essentially sufficient for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive.

A few definitions and notations are needed before stating our main result. Let G and H be groups acting on Ω and Λ respectively. The action of G on Ω is said to be *permutationally isomorphic* [5, page 17] to the action of H on Λ if there exist a bijection $\rho : \Omega \to \Lambda$ and a group isomorphism $\psi : G \to H$ such that $\rho(\alpha^g) = (\rho(\alpha))^{\psi(g)}$ for all $\alpha \in \Omega$ and $g \in G$. In the case when $G = H$ and the actions of G on Ω and Λ are permutationally isomorphic, we simply write $G^{\Omega} \cong G^{\Lambda}$.

Now we return to our discussion on imprimitive symmetric triples $(\Gamma, G, \mathcal{B})$. Define $G_{(B)} = \{g \in G_B : \alpha^g = \alpha$ for every $\alpha \in B\}$ to be the pointwise stabilizer of B in G, and $G_{[B]} = \{g \in G_B : C^g = C$ for every $C \in \Gamma_{\mathcal{B}}(B)\}$ the pointwise stabilizer of $\Gamma_{\mathcal{B}}(B)$ in G_B. As usual, by G_B^B we mean the group $G_B/G_{(B)}$ with its action restricted to B, and by $G_B^{\Gamma_{\mathcal{B}}(B)}$ we mean $G_B/G_{[B]}$ with its action restricted to $\Gamma_{\mathcal{B}}(B)$. (Thus, whenever we write $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)}$, we mean that the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally isomorphic.) Define $G_{(\mathcal{B})} = \{g \in G : B^g = B$ for every $B \in \mathcal{B}\}$.

Let Σ be a graph and Δ a subset of the set of 3-arcs of Σ. We say that Δ is *self-paired* if $(\tau, \sigma, \sigma', \tau') \in \Delta$ implies $(\tau', \sigma', \sigma, \tau) \in \Delta$. In this case the *3-arc graph* $\Xi(\Sigma, \Delta)$ is defined [10] to have arcs of Σ as its vertices such that two such arcs $(\sigma, \tau), (\sigma', \tau')$ are adjacent if and only if $(\tau, \sigma, \sigma', \tau') \in \Delta$. We denote by $n \cdot \Sigma$ the graph which is n vertex-disjoint copies of Σ, and by C_n the cycle of length n. We may view the complete graph K_n on n vertices as a degenerate design of block size two.

As shown in [12], when $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, the *dual design* $\mathcal{D}^*(B)$ of $\mathcal{D}(B)$ plays a significant role in the study of Γ, where $\mathcal{D}^*(B)$ is obtained from $\mathcal{D}(B)$ by interchanging the roles of points and blocks but retaining the incidence relation. Since in this case G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$, as observed in [12],

$$\lambda := |\Gamma(C) \cap \Gamma(D) \cap B| \tag{1.1}$$

is independent of the choice of distinct $C, D \in \Gamma_{\mathcal{B}}(B)$. Denote by $\overline{\mathcal{D}^*}(B)$ the complementary incidence structure of $\mathcal{D}^*(B)$, which is defined to have the same 'point' set $\Gamma_{\mathcal{B}}(B)$ as $\mathcal{D}^*(B)$ such that a 'point' $C \in \Gamma_{\mathcal{B}}(B)$ is incident with a 'block' $\alpha \in B$ if and only if $C \notin \Gamma_{\mathcal{B}}(\alpha)$. As observed in [12, Theorem 3.2], if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, then either $\lambda = 0$ or $\mathcal{D}^*(B)$ is a 2-(b, r, λ) design with v blocks, and either $\overline{\lambda} := v - 2k + \lambda = 0$ or $\overline{\mathcal{D}^*}(B)$ is a 2-$(b, b - r, \overline{\lambda})$ design with v blocks. Moreover, each of $\mathcal{D}^*(B)$ and $\overline{\mathcal{D}^*}(B)$ admits [12] G_B as a group of automorphisms acting 2-transitively on its point set and transitively on its block set. The first main result in this paper, Theorem 1.1 below, gives the parameters of $\mathcal{D}^*(B)$ and information about Γ, $\mathcal{D}^*(B)$ or/and the action of G_B on $\Gamma_{\mathcal{B}}(B)$ in the case when $k = v - p$ for a prime $p \geq 3$. Our proof of this result relies on the classification of finite 2-transitive groups (see, for example, [5]) and that of 2-transitive symmetric designs [9] (which in turn rely on the classification of finite simple groups). Without loss of generality, we may assume that $\Gamma_{\mathcal{B}}$ is connected.

THEOREM 1.1. *Let Γ be a G-symmetric graph with $V(\Gamma)$ admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - p \geq 1$ and $\Gamma_{\mathcal{B}}$ is connected with valency $b \geq 2$, where $p \geq 3$ is a prime and $G \leq \mathrm{Aut}(\Gamma)$. Suppose $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Then one of* (a)–(f) *in Table 1 occurs, and in* (c)–(f) *the parameters of the 2-(b, r, λ) design $\mathcal{D}^*(B)$ with v blocks are given in the third column of the table.*

Moreover, in (a), $\Gamma \cong (|V(\Gamma)|/2) \cdot K_2$, $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)}$ *is 2-transitive of degree $p + 1$, and any connected $(p + 1)$-valent $(G, 2)$-arc transitive graph can occur as $\Gamma_{\mathcal{B}}$ in* (a).

https://doi.org/10.1017/S1446788713000669 Published online by Cambridge University Press

Table 1. Theorem 1.1.

Case	$\overline{\mathcal{D}^*}(B)$	(v, b, r, λ)	Conditions
(a)		$(p+1, p+1, 1, 0)$	
(b)		$(2p, 2, 1, 0)$	
(c)	$\mathrm{PG}_{n-1}(n, q)$	$\left(\frac{q^{n+1}-1}{q-1}, \frac{q^{n+1}-1}{q-1}, q^n, q^n - q^{n-1}\right)$	$p = \frac{q^n-1}{q-1}, n \geq 2$ q a prime power $\frac{q^n-1}{q-1}$ is a prime
(d)	2-(11, 5, 2)	$(11, 11, 6, 3)$	$p = 5$
(e)		$(pa, a, a-1, p(a-2))$	$a \geq 3$
(f)		$\left(pa, ps+1, \frac{(ps+1)(a-1)}{a}, p(a-2) + \frac{ps-a+1}{as}\right)$	$a \geq 2, s \geq 1$ a a divisor of $ps+1$ s a divisor of $\frac{ps-a+1}{a}$ $\frac{a-1}{p-a} \leq s \leq a-1 \leq p-2$

In (b)*, we have* $\Gamma \cong n \cdot \Gamma[B, C]$*, where* $n = |V(\Gamma)|/2p$*,* $\Gamma_{\mathcal{B}} \cong C_n$*, and* $G/G_{(\mathcal{B})} = D_{2n}$*.*

In (c)*,* $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)}$ *is isomorphic to a* 2*-transitive subgroup of* $\mathrm{P\Gamma L}(n+1, q)$*, and* G *is faithful on* $\mathcal{B}$*.*

In (d)*, we have* $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(2, 11)$*.*

In (e)*,* $V(\Gamma)$ *admits a* G*-invariant partition* $\mathcal{P}$ *with block size* p *which is a refinement of* $\mathcal{B}$ *such that* $\Gamma_{\mathcal{P}} \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ *for a self-paired* G*-orbit* Δ *on the set of* 3*-arcs of* $\Gamma_{\mathcal{B}}$*. Moreover,* $\hat{\mathcal{B}} = \{\hat{B} : B \in \mathcal{B}\}$ *(where* $\hat{B}$ *is the set of blocks of* $\mathcal{P}$ *contained in* B*) is a* G*-invariant partition of* $\mathcal{P}$ *such that* $(\Gamma_{\mathcal{P}})_{\hat{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$ *and the parameters with respect to* $(\Gamma_{\mathcal{P}}, \hat{\mathcal{B}})$ *are given by* $v_{\hat{\mathcal{B}}} = b_{\hat{\mathcal{B}}} = a$ *and* $k_{\hat{\mathcal{B}}} = r_{\hat{\mathcal{B}}} = a - 1$*.*

In (f)*, if* $s = 1, 2$*, then all possibilities are given in Tables 2–3 respectively, where* $G_B^{\Gamma_{\mathcal{B}}(B)}$ *is isomorphic to the group or a* 2*-transitive subgroup of the group in the first column (with natural actions).*

Remark 1.2. (1) In (e), denote by s the valency of $\Gamma_{\mathcal{P}}[\hat{B}, \hat{C}]$ for adjacent $B, C \in \mathcal{B}$, and by t the number of blocks of $\mathcal{P}$ contained in C which contain at least one neighbour of a fixed vertex in $B \cap \Gamma(C)$. Since $r_{\hat{\mathcal{B}}} = a - 1$, the parameters with respect to $\mathcal{P}$ satisfy $b_{\mathcal{P}} = (a-1)s$ and $r_{\mathcal{P}} = (a-1)t$. Since $v_{\mathcal{P}} r_{\mathcal{P}} = b_{\mathcal{P}} k_{\mathcal{P}}$ and $v_{\mathcal{P}} = p$, we have $pt = k_{\mathcal{P}} s$. Since $1 \leq t \leq s \leq a-1$, $1 \leq k_{\mathcal{P}} \leq p$ and p is a prime, we have either: (i) $k_{\mathcal{P}} = p$ and $s = t$; or (ii) $s = pc$ and $t = k_{\mathcal{P}} c$ for some integer c with $1 \leq c \leq \lfloor (a-1)/p \rfloor$.

Since $v - 2k + \lambda = 0$ in (e), examples of $(\Gamma, G, \mathcal{B})$ in this case can be constructed using [12, Construction 3.8] by first lifting a $(G, 2)$-arc transitive graph to a G-symmetric 3-arc graph and then lifting the latter to a G-symmetric graph Γ by the standard covering graph construction [2].

https://doi.org/10.1017/S1446788713000669 Published online by Cambridge University Press

TABLE 2. Possibilities when $s = 1$ in case (f).

$G_B^{\Gamma_{\mathcal{B}}(B)}$	$\mathcal{D}^*(B)$	(v, b, r, λ)	Conditions
A_{p+1}	$\overline{\mathcal{D}^*}(B) \cong K_{p+1}$		$a = \dfrac{p+1}{2}$
$\leq \mathrm{AGL}(n, 2)$		$\begin{pmatrix} 2^m(2^n-1) \\ 2^n \\ 2^n - 2^{n-m} \\ (2^m-1)(2^n-2^{n-m}-1) \end{pmatrix}$	$1 \leq m \leq n-1$ $p = 2^n - 1$ a Mersenne prime $r^* = (2^n-1)(2^m-1)$
$\leq \mathrm{PGL}(2, p)$			$a - 1$ a divisor of $p - 1$
$\mathrm{Sp}_4(2)$	2-(6, 3, 2)		$p = 5$
M_{11}	2-(12, 6, 5)		$p = 11$ $\mathcal{D}^*(B)$ is a Hadamard 3-subdesign of the Witt design W_{12} (3-(12, 6, 2) design)

TABLE 3. Possibilities when $s = 2$ in case (f).

$G_B^{\Gamma_{\mathcal{B}}(B)}$	$\mathcal{D}^*(B)$	(v, b, r, λ)	Conditions
$\leq \mathrm{AGL}(n, 3)$		$\begin{pmatrix} \dfrac{(3^n-1)3^j}{2} \\ 3^n \\ 3^{n-j}(3^j-1) \\ \dfrac{(3^n-1)(3^j-2)}{2} + \dfrac{3^{n-j}-1}{2} \end{pmatrix}$	$n \geq 3$ odd $p = \dfrac{3^n-1}{2}$ $1 \leq j \leq n-1$
$\leq \mathrm{PGL}(n, 2)$		$\begin{pmatrix} a(2^{n-1}-1) \\ 2^n - 1 \\ \dfrac{(2^n-1)(a-1)}{a} \\ (2^{n-1}-1)(a-2) + \dfrac{2^n-1-a}{2a} \end{pmatrix}$	a an odd divisor of $2p+1$ $3 \leq a \leq \dfrac{2p+1}{3}$ $p = 2^{n-1} - 1$ a Mersenne prime ($n - 1 \geq 3$ a prime)
A_7	$\overline{\mathcal{D}^*}(B) \cong \mathrm{PG}(3, 2)$	(35, 15, 12, 22)	

https://doi.org/10.1017/S1446788713000669 Published online by Cambridge University Press

(2) The condition $(v, b, r, \lambda) = (pa, a, a-1, p(a-2))$ in (e) is sufficient for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive. In fact, in this case for any $B \in \mathcal{B}$ and $\alpha \in B$, there exists exactly one block $A \in \Gamma_{\mathcal{B}}(B)$ which contains no neighbour of α. Thus, for any distinct $C, D \in \Gamma_{\mathcal{B}}(B) \setminus \{A\}$, there exist $\beta \in C$ and $\gamma \in D$ which are adjacent to α in Γ. Since Γ is G-symmetric, there exists $g \in G_\alpha$ such that $\beta^g = \gamma$. So $(B, C)^g = (B, D)$. Since g fixes α, it must fix the unique block of $\Gamma_{\mathcal{B}}(B)$ having no neighbour of α, that is, $A^g = A$. It follows that $G_{A,B}$ is transitive on $\Gamma_{\mathcal{B}}(B) \setminus \{A\}$ and hence $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive.

(3) In (f), it seems challenging to determine $G_B^{\Gamma_{\mathcal{B}}(B)}$ and $\mathcal{D}^*(B)$ when s is not specified.

We appreciate Yuqing Chen for constructing the following example for the third row of Table 2. Denote $F = \mathrm{GF}(2^n)$ and let H be a subgroup of the additive group $E = (\mathrm{GF}(2^n), +)$ of order 2^{n-m} (where $2^n - 1$ is not necessarily a Mersenne prime). Then $E \rtimes F^*$ acts on E as a 2-transitive subgroup of $\mathrm{AGL}(n, 2)$. The incidence structure whose point set is E and blocks are the complements in E of the $E \rtimes F^*$-orbits of H is a 2-$(2^n, 2^n - 2^{n-m}, (2^m - 1)(2^n - 2^{n-m} - 1))$ design admitting $E \rtimes F^*$ as a 2-point transitive group of automorphisms.

In the case when $k = v - 3 \geq 1$ or $k = v - 5 \geq 1$, Theorem 1.1 enables us to obtain necessary and sufficient conditions for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive. This will be given in Theorem 3.2 and Corollary 3.3 in Section 3, respectively.

We will use standard notation and terminology on block designs [1, 4] and permutation groups [5]. The set of arcs of a graph Σ is denoted by $\mathrm{Arc}(\Sigma)$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Suppose $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Then G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ and hence λ defined in (1.1) is independent of the choice of distinct $C, D \in \Gamma_{\mathcal{B}}(B)$. It is known [12, Section 3] that either $\lambda = 0$ or $\mathcal{D}^*(B)$ is a 2-(b, r, λ) design of v 'blocks' with G_B doubly transitive on its points and transitive on its blocks and flags. Since $k = v - p \geq 1$,

$$vr = b(v - p) \tag{2.1}$$

and by [12, Corollary 3.3],

$$\lambda(b - 1) = (v - p)(r - 1). \tag{2.2}$$

Consider the case $\lambda = 0$ first. In this case we have $r = 1$ as $v - p \geq 1$. Thus $v = b(v - p)$ and so $v = p + (p/(b - 1))$. Since v is an integer and p is a prime, we have $b = p + 1$ or 2, and therefore $(v, b, r, \lambda) = (p + 1, p + 1, 1, 0)$ or $(2p, 2, 1, 0)$. In the former case, we have $k = v - p = 1$ and $\Gamma \cong (|V(\Gamma)|/2) \cdot K_2$. Moreover, the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally isomorphic. Thus $G_{(B)} = G_{[B]}$ and $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)}$ is 2-transitive of degree $p + 1$. On the other hand, for any connected $(p + 1)$-valent $(G, 2)$-arc transitive graph Σ, define Γ to be the graph with vertex set $\mathrm{Arc}(\Sigma)$ and edges joining (σ, τ) to (τ, σ) for all $(\sigma, \tau) \in \mathrm{Arc}(\Sigma)$. Then Γ is G-symmetric admitting $\mathcal{B} = \{B(\sigma) : \sigma \in V(\Sigma)\}$ (where $B(\sigma) = \{(\sigma, \tau) : \tau \in \Sigma(\sigma)\}$) as a G-invariant partition such

https://doi.org/10.1017/S1446788713000669 Published online by Cambridge University Press

that $(v, b, r, \lambda) = (p+1, p+1, 1, 0)$ and $\Gamma_{\mathcal{B}} \cong \Sigma$. (This simple construction was used in [7, Example 2.4] for trivalent Σ. It is a very special case of the flag graph construction [15, Theorem 4.3].) In the case where $(v, b, r, \lambda) = (2p, 2, 1, 0)$, $\Gamma[B, C]$ is a bipartite $G_{B,C}$-edge transitive graph with p vertices in each part of its bipartition, $\Gamma \cong n \cdot \Gamma[B, C]$, where $n = |V(\Gamma)|/2p$, $\Gamma_{\mathcal{B}} \cong C_n$, and therefore $G/G_{(\mathcal{B})} = D_{2n}$.

Assume $\lambda \geq 1$ from now on. Denote by r^* the replication number of $\mathcal{D}^*(B)$, that is, the number of 'blocks' containing a fixed 'point'. We distinguish between the following two cases.

Case 1: v is not a multiple of p. In this case, v and $v - p$ are coprime. Thus, by (2.1), v divides b and $v - p$ divides r. On the other hand, as noticed in [12], by the well-known Fisher's inequality we have $b \leq v$ and $r \leq v - p$. Thus $v = b$ and $r = v - p = k$. From (2.2) we then have $\lambda = (v-p)(v-p-1)/(v-1) = (v-2p) + p(p-1)/(v-1)$. Note that $v \neq p+1$, for otherwise $\lambda = 0$, which contradicts our assumption $\lambda \geq 1$. Since λ is an integer, $v - 1$ is a divisor of $p(p-1)$. Since p is a prime and $v - 1 \geq p + 1$, it follows that p is a divisor of $v - 1$. Set $a = (v-1)/p$. Then $a \geq 2$ is a divisor of $p - 1$ and $(v, b, r, \lambda) = (pa+1, pa+1, p(a-1)+1, p(a-2) + ((p+a-1)/a))$. Hence $\mathcal{D}^*(B)$ is a 2-transitive symmetric 2-$(pa+1, p(a-1)+1, p(a-2)+(p+a-1)/a)$ design. Thus, by the classification of 2-transitive symmetric designs [9] (see also [1, Theorem XII-6.22]), $\mathcal{D}^*(B)$ or $\overline{\mathcal{D}^*}(B)$ is isomorphic to one of the following:

- $\mathrm{PG}_{n-1}(n, q)$ (where $n \geq 2$ and q is a prime power);
- the unique 2-$(11, 5, 2)$ design;
- the unique symmetric 2-$(176, 50, 14)$ design;
- the unique 2-$(2^{2m}, 2^{m-1}(2^m - 1), 2^{m-1}(2^{m-1} - 1))$ design (where $m \geq 2$).

Since $\mathrm{PG}_{n-1}(n, q)$ has $(q^{n+1} - 1)/(q - 1)$ points and block size $(q^n - 1)/(q - 1)$, while $\mathcal{D}^*(B)$ has $pa + 1$ 'points' and block size $p(a - 1) + 1$, by comparing these parameters one can show that $\mathcal{D}^*(B) \not\cong \mathrm{PG}_{n-1}(n, q)$. In the same fashion we can see that none of the 2-transitive symmetric designs above can occur as $\mathcal{D}^*(B)$. On the other hand, since p is a prime, $\overline{\mathcal{D}^*}(B)$ cannot be isomorphic to the unique symmetric 2-$(176, 50, 14)$ design or the unique 2-$(2^{2m}, 2^{m-1}(2^m - 1), 2^{m-1}(2^{m-1} - 1))$ design. We are left with the case where $\overline{\mathcal{D}^*}(B) \cong \mathrm{PG}_{n-1}(n, q)$ or $\overline{\mathcal{D}^*}(B)$ is isomorphic to the unique 2-$(11, 5, 2)$ design.

It is easy to verify that $\overline{\mathcal{D}^*}(B) \cong \mathrm{PG}_{n-1}(n, q)$ only if $p = (q^n - 1)/(q - 1)$ and $a = q$. In this case, $G_B^{\Gamma_{\mathcal{B}}(B)}$ is isomorphic to a 2-transitive subgroup of $\mathrm{P\Gamma L}(n+1, q)$ since $G_B^{\Gamma_{\mathcal{B}}(B)} \leq \mathrm{Aut}(\overline{\mathcal{D}^*}(B)) \cong \mathrm{P\Gamma L}(n+1, q)$. Moreover, we have $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)}$ since the actions of a 2-transitive subgroup of $\mathrm{P\Gamma L}(n+1, q)$ on the point set and the block set of $\mathrm{PG}_{n-1}(n, q)$ are permutationally isomorphic. Furthermore, if $g \in G_{(\mathcal{B})}$, then $g \in G_{(B)}$ since $\overline{\mathcal{D}^*}(B)$ is self-dual. Since this holds for every $B \in \mathcal{B}$, g fixes every vertex of Γ. Since $G \leq \mathrm{Aut}(\Gamma)$ is faithful on $V(\Gamma)$, we conclude that $g = 1$ and so G is faithful on $\mathcal{B}$. Therefore, (c) occurs.

Further, $\overline{\mathcal{D}^*}(B)$ is isomorphic to the unique 2-$(11, 5, 2)$ design if and only if $p = 5$ and $a = 2$. In this case, since the automorphism group of this symmetric 2-$(11, 5, 2)$ design is $\mathrm{PSL}(2, 11)$ (see, for example, [1, Theorem IV.7.14]), G_B^B is isomorphic to a 2-transitive subgroup of PSL(2, 11). Since $|G_B^B| \geq 11 \cdot 10$ but no proper subgroup of PSL(2, 11) has order greater than 60, we have $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(2, 11)$ and (d) occurs.

Case 2: $v = pa$ is a multiple of p, where $a \geq 2$ is an integer. In this case (2.1) becomes $ar = b(a-1)$. Thus a divides b and $a-1$ divides r. On the other hand, Fisher's inequality yields $b \leq pa$ and $r \leq p(a-1)$. So $b = at$ and $r = (a-1)t$ for some integer t between 1 and p. By (2.2), $\lambda = p(a-1)((a-1)t-1)/(at-1) = p(a-2) + p(t-1)/(at-1)$.

Subcase 2.1: $t = 1$. Then $(v, b, r, \lambda) = (pa, a, a-1, p(a-2))$, where $a \geq 3$ as $\lambda \geq 1$ by our assumption. Thus, by [12, Equation (3)], any two distinct 'blocks' of $\overline{\mathcal{D}}(B)$ intersect at $\overline{\lambda} = v - 2k + \lambda = 0$ 'points'. That is, the 'blocks' $B \setminus \Gamma(C)$ $(C \in \Gamma_{\mathcal{B}}(B))$ of $\overline{\mathcal{D}}(B)$ are pairwise disjoint and hence [12, Theorem 3.7] applies. Following [12, Section 3], define $\mathcal{P} = \bigcup_{B \in \mathcal{B}} \{B \setminus \Gamma(C) : C \in \Gamma_{\mathcal{B}}(B)\}$. Then $\mathcal{P}$ is a proper refinement of $\mathcal{B}$. Denote $\hat{B} = \{B \setminus \Gamma(C) \in \mathcal{P} : C \in \Gamma_{\mathcal{B}}(B)\}$. Then $\hat{\mathcal{B}} = \{\hat{B} : B \in \mathcal{B}\}$ is a G-invariant partition of $\mathcal{P}$. Denote by $v_{\hat{\mathcal{B}}}$, $k_{\hat{\mathcal{B}}}$, $b_{\hat{\mathcal{B}}}$, $r_{\hat{\mathcal{B}}}$ the parameters with respect to $(\Gamma_{\mathcal{P}}, \hat{\mathcal{B}})$. It can be verified (see [12, Theorem 3.7]) that $(\Gamma_{\mathcal{P}})_{\hat{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$, $v_{\hat{\mathcal{B}}} = v/p = a$, $k_{\hat{\mathcal{B}}} = v_{\hat{\mathcal{B}}} - 1 = a - 1$, $b_{\hat{\mathcal{B}}} = b = a$, $r_{\hat{\mathcal{B}}} = r = a - 1$ and $\mathcal{D}(\hat{B})$ has no repeated blocks. Thus, by [10, Theorem 1] (or [12, Theorem 3.7]), $\Gamma_{\mathcal{P}} \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ for some self-paired G-orbit Δ on the set of 3-arcs of $\Gamma_{\mathcal{B}}$. Hence (e) occurs.

Subcase 2.2: $t \geq 2$. In this case, since λ is an integer, $at-1$ is a divisor of $p(t-1)$. In particular, $at - 1 \leq p(t-1)$, which implies $a \leq p-1$ and $(p-1)/(p-a) \leq t \leq p$. Since $at-1$ does not divide $t-1$ and p is a prime, $at-1$ must be a multiple of p, say, $at - 1 = ps$, so that $p(t-1)/(at-1) = (t-1)/s$ and s divides $t-1$. Since $t = (ps+1)/a$ is an integer, a is a divisor of $ps+1$. Therefore, $(v, b, r, \lambda) = (pa, ps+1, (ps+1)(a-1)/a, p(a-2) + (ps-a+1)/(as))$. Since λ is an integer, s is a divisor of $(ps-a+1)/a$ and so s is a divisor of $a-1$. This together with $(p-1)/(p-a) \leq t = (ps+1)/a$ implies $(a-1)/(p-a) \leq s \leq a-1$. Therefore, case (f) occurs.

The rest of the proof is devoted to the case $s = 1$ in (f). In this case $\mathcal{D}^*(B)$ is a 2-$(p+1, ((p+1)(a-1))/a, p(a-2) + ((p-a+1)/a))$ design with pa 'blocks' such that each 'point' is contained in exactly $r^* = p(a-1)$ 'blocks'. Moreover, $\mathcal{D}^*(B)$ admits G_B as a group of automorphisms acting 2-transitively on the set $\Gamma_{\mathcal{B}}(B)$ of $p+1$ 'points'. All 2-transitive groups are known (see, for example, [5, Section 7.7]). First, since $p + 1 \neq q^3 + 1, q^2 + 1$ for any prime power q, $G_B^{\Gamma_{\mathcal{B}}(B)}$ cannot be a unitary, Suzuki or Ree group. Since $r < p+1$, S_{p+1} is r-transitive on $p+1$ points (in its natural action) but on the other hand $\mathcal{D}^*(B)$ has $pa < \binom{p+1}{r}$ blocks. Hence $G_B^{\Gamma_{\mathcal{B}}(B)} \ncong S_{p+1}$. Similarly, as A_{p+1} is $(p-1)$-transitive in its natural action, $G_B^{\Gamma_{\mathcal{B}}(B)} \ncong A_{p+1}$ unless $r = p-1$. In this exceptional case, $\mathcal{D}^*(B)$ has $pa = \binom{p+1}{2}$ blocks and so is isomorphic to the complementary design of the trivial design K_{p+1}. This gives the second row in Table 2.

https://doi.org/10.1017/S1446788713000669 Published online by Cambridge University Press

If $G_B^{\Gamma_{\mathcal{B}}(B)}$ is affine, then $p+1=q^n$ for some prime power q and integer $n \geq 1$, which occurs if and only if $q=2$ and $p=2^n-1$ is a prime. In this case, n must be a prime and $p=2^n-1$ is a Mersenne prime, and $G_B^{\Gamma_{\mathcal{B}}(B)}$ is isomorphic to a 2-transitive subgroup of $\mathrm{AGL}(n,2)$. Moreover, $a=2^m$ for some integer $1 \leq m \leq n-1$, and so $(v,b,r,\lambda,r^*) = (2^m(2^n-1), 2^n, 2^n-2^{n-m}, (2^m-1)(2^n-2^{n-m}-1), (2^n-1)(2^m-1))$. This gives the third row in Table 2.

If $G_B^{\Gamma_{\mathcal{B}}(B)}$ is projective, then $p+1=(q^n-1)/(q-1)$ for a prime power q and an integer $n \geq 2$. Thus $n=2$, $p=q$ and $G_B^{\Gamma_{\mathcal{B}}(B)}$ is isomorphic to a 2-transitive subgroup of $\mathrm{PGL}(2,p)$. Since G_B is transitive on the $p(p+1)(a-1)$ flags of $\mathcal{D}(B)$, $p(p+1)(a-1)$ is a divisor of $|\mathrm{PGL}(2,p)| = (p-1)p(p+1)$ and so $a-1$ is a divisor of $p-1$. Note that the second 2-transitive action of $A_5 \cong \mathrm{PSL}(2,5)$ with degree 6 is covered here, and that of $A_6 \cong \mathrm{PSL}(2,9)$ with degree 10, of A_7 with degree 15, and of $A_8 \cong \mathrm{PSL}(4,2)$ with degree 15 cannot happen since 9 and 14 are not prime. Similarly, the second 2-transitive action of $\mathrm{PSL}(2,8) \leq \mathrm{Sp}_6(2)$ of degree 28 and that of $\mathrm{PSL}(2,11) \leq M_{11}$ of degree 11 cannot happen. So we have the fourth row in Table 2.

If $G_B^{\Gamma_{\mathcal{B}}(B)}$ is symplectic, then $p+1=2^{m-1}(2^m \pm 1)$ for some $m \geq 2$. If $p+1 = 2^{m-1}(2^m+1)$, then $p=(2^{m-1}+1)(2^m-1)$, which cannot happen since p is a prime. Similarly, if $p+1=2^{m-1}(2^m-1)$, then $p=(2^{m-1}-1)(2^m+1)$, which occurs if and only if $m=2$ and $p=5$. In this exceptional case, we have $G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{Sp}_4(2)$ ($\cong S_6$), $a=2$ or 3, and hence $(v,b,r,\lambda,r^*) = (10,6,3,2,5)$ or $(15,6,4,6,10)$. The latter cannot happen since a 2-$(6,4,6)$ design does not exist [1, Table A1.1]. Thus $\mathcal{D}^*(B)$ is isomorphic to the unique 2-$(6,3,2)$ design. This gives the fifth row in Table 2.

By comparing the degree $p+1$ of $G_B^{\Gamma_{\mathcal{B}}(B)}$ with that of the ten sporadic 2-transitive groups [5, Section 7.7], one can verify that among such groups only the following may be isomorphic to $G_B^{\Gamma_{\mathcal{B}}(B)}$: M_{11} (degree $p+1=12$); M_{12} (degree $p+1=12$); M_{24} (degree $p+1=24$).

In the case of M_{24}, a is 2, 3, 4, 6, 8 or 12, and so $(v,b,r,\lambda,r^*) = (46,24,12,11,23)$, $(69,24,16,30,46)$, $(92,24,18,51,69)$, $(138,24,20,95,115)$, $(184,24,21,140,161)$ or $(276,24,22,231,253)$. It is well known [1, Ch. IV] that M_{24} is the automorphic group of the unique 5-$(24,8,1)$ design (the Witt design W_{24}), which is also a 2-$(24,8,77)$ design, and that up to isomorphism the natural action of M_{24} on the points of W_{24} is the only 2-transitive action of M_{24} with degree 24. Hence $G_B^{\Gamma_{\mathcal{B}}(B)} \not\cong M_{24}$.

In the cases of M_{11} and M_{12}, a is 2, 3, 4 or 6, and so $(v,b,r,\lambda,r^*) = (22,12,6,5,11)$, $(33,12,8,14,22)$, $(44,12,9,24,33)$ or $(66,12,10,45,55)$. Since by [4, Section II.1.3] a 2-$(12,8,14)$ or 2-$(12,9,24)$ design does not exist, the second and third possibilities can be eliminated. Thus, if $G_B^{\Gamma_{\mathcal{B}}(B)} \cong M_{11}$ or M_{12}, then $\mathcal{D}^*(B)$ is isomorphic to a 2-$(12,6,5)$ or 2-$(12,10,45)$ design. It is well known [1, Ch. IV] that M_{12} is the automorphic group of the unique 5-$(12,6,1)$ design (the Witt design W_{12}), which is also a 2-$(12,6,30)$ design. Since up to isomorphism the natural action of M_{12} on the points of W_{12} is the only 2-transitive action of M_{12} with degree 12, we have $G_B^{\Gamma_{\mathcal{B}}(B)} \not\cong M_{12}$. Further, M_{11} is the automorphic group of a 3-$(12,6,2)$ design (that is, a Hadamard 3-subdesign

https://doi.org/10.1017/S1446788713000669 Published online by Cambridge University Press

of W_{12} [1, Ch. IV]), which is also a 2-(12, 6, 5) design. Since up to isomorphism the natural action of M_{11} on the points of this design is the only 2-transitive action of M_{11} with degree 12, we conclude that if $G_B^{\Gamma_{\mathcal{B}}(B)} \cong M_{11}$ then $\mathcal{D}^*(B)$ is isomorphic to this 2-(12, 6, 5) design. This gives the last row in Table 2.

In the same fashion one can prove that, if $s = 2$ in (f), then we have the possibilities in Table 3. □

3. $p = 3, 5$

Theorem 1.1 provides a necessary condition for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive when $k = v - p$ for any prime $p \geq 3$. This condition may be sufficient for some special primes p, and in this section we prove that this is the case when $p = 3$ or 5. Moreover, when $p = 3$ we obtain more structural information about Γ (see Theorem 3.2 below). In particular, in the last case in Theorem 3.2 (which corresponds to case (f) in Theorem 1.1), Γ can be constructed from $\Gamma_{\mathcal{B}}$ by using a simple construction introduced in [12, Section 4.1]. Given a regular graph Σ with valency at least 2 and a self-paired subset Δ of the set of 3-arcs of Σ, define [12] $\Gamma_2(\Sigma, \Delta)$ to be the graph with the set of 2-*paths* (paths of length 2) of Σ as vertex set such that two distinct 'vertices' $\tau\sigma\tau'$ $(=\tau'\sigma\tau)$ and $\eta\varepsilon\eta'$ $(=\eta'\varepsilon\eta)$ are adjacent if and only if they have a common edge (that is, $\sigma \in \{\eta, \eta'\}$ and $\varepsilon \in \{\tau, \tau'\}$) and moreover the two 3-arcs (which are reverses of each other) formed by 'gluing' the common edge are in Δ. (As noted in [12], when Δ is the set of all 3-arcs of Σ, $\Gamma_2(\Sigma, \Delta)$ is exactly the path graph $P_3(\Sigma)$ introduced in [3].)

In the proof of Theorem 3.2 we will use the following lemma.

Lemma 3.1. *Let Γ be a G-symmetric graph that admits a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - i$, where $i \geq 1$. Then the multiplicity m of $\mathcal{D}(B)$ and $\overline{\mathcal{D}}(B)$ is a common divisor of r and b.*

Proof. As in [12], we may view $\mathcal{D}(B)$ and $\overline{\mathcal{D}}(B)$ as hypergraphs with vertex set B and hyperedges $\Gamma(C) \cap B$ and $B \setminus \Gamma(C)$, $C \in \Gamma_{\mathcal{B}}(B)$, respectively, with each hyperedge repeated m times. It is easy to see that as hypergraphs they have valencies $\mathrm{val}(\mathcal{D}(B)) = r = b - (ib/v)$ and $\mathrm{val}(\overline{\mathcal{D}}(B)) = b - r$, respectively. Since m is a divisor of each of these valencies, it must be a common divisor of r and b. □

Denote by $K_{n,n}$ the complete bipartite graph with n vertices in each part of its bipartition, and by $\Sigma_1 - \Sigma_2$ the graph obtained from a graph Σ_1 by deleting the edges of a spanning subgraph Σ_2 of Σ_1. Denote by $G_{B,C}$ the subgroup of G fixing B and C setwise. A few statements in the following theorem are carried over directly from Theorem 1.1, and we keep them there for the completeness of the result.

Theorem 3.2. *Let Γ be a G-symmetric graph with $V(\Gamma)$ admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 3 \geq 1$ and $\Gamma_{\mathcal{B}}$ is connected of valency $b \geq 2$, where $G \leq \mathrm{Aut}(\Gamma)$. Then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if one of the following holds:*

(a) $(v, b, r, \lambda) = (4, 4, 1, 0)$ *and* $G_B^B \cong A_4$ *or* S_4*;*

(b) $(v, b, r, \lambda) = (6, 2, 1, 0)$ *and* $\Gamma_{\mathcal{B}} \cong C_n$*, where* $n = |V(\Gamma)|/6$*;*

https://doi.org/10.1017/S1446788713000669 Published online by Cambridge University Press

(c) $(v,b,r,\lambda) = (7,7,4,2)$ *and* $G_B^B \cong \mathrm{PSL}(3,2)$*;*
(d) $(v,b,r,\lambda) = (3a,a,a-1,3a-6)$ *for some integer* $a \geq 3$*;*
(e) $(v,b,r,\lambda) = (6,4,2,1)$ *and* $G_B^{\Gamma_{\mathcal{B}}(B)} \cong A_4$ *or* S_4*.*

Moreover, in (a) we have $G_B^{\Gamma_{\mathcal{B}}(B)} \cong A_4$ *or* S_4*,* $\Gamma \cong (|V(\Gamma)|/2) \cdot K_2$*, and every connected* 4*-valent* 2*-arc transitive graph can occur as* $\Gamma_{\mathcal{B}}$ *in (a).*

In (b), we have $\Gamma \cong 3n \cdot K_2, n \cdot C_6$ *or* $n \cdot K_{3,3}$*, and* $G/G_{(\mathcal{B})} = D_{2n}$*.*

In (c), $\overline{\mathcal{D}}(B)$ *is isomorphic to the Fano plane* $\mathrm{PG}(2,2)$*,* $G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(3,2)$*,* G *is faithful on* $\mathcal{B}$*, and* $\Gamma[B,C] \cong 4 \cdot K_2$*,* $K_{4,4} - 4 \cdot K_2$ *or* $K_{4,4}$*. In the first case* Γ *is* $(G,2)$*-arc transitive, and in the last two cases* Γ *is connected of valency* 12 *and* 16 *respectively.*

In (d), the statements in (e) of Theorem 1.1 hold with $p = 3$*.*

In (e), we have $\Gamma \cong \Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$ *for a self-paired* G*-orbit* Δ *on* 3*-arcs of* $\Gamma_{\mathcal{B}}$*, and every connected* 4*-valent* $(G,2)$*-arc transitive graph can occur as* $\Gamma_{\mathcal{B}}$ *in (e).*

Proof. *Necessity.* Suppose $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive. Since $p = 3$, by Theorem 1.1, (v,b,r,λ) is one of the following:

(a) $(4,4,1,0)$; (b) $(6,2,1,0)$; (c) $(7,7,4,2)$ (for which $n = q = 2$); (d) $(3a,a,a-1,3a-6)$ (where $a \geq 3$); (e) $(6,4,2,1)$ (for which $a = 2$ and $s = 1$).

In case (a), $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)}$ is 2-transitive of degree four, and in case (b), $\Gamma[B,C] \cong 3 \cdot K_2, C_6$ or $K_{3,3}$ for adjacent $B, C \in \mathcal{B}$. The properties for cases (a), (b) and (d) follow from Theorem 1.1 immediately.

Case (c): In this case $\mathcal{D}(B)$ is the biplane of order two. In other words, $\overline{\mathcal{D}}(B)$ is isomorphic to the Fano plane $\mathrm{PG}(2,2)$. Since G_B^B induces a group of automorphisms of the self-dual $\overline{\mathcal{D}}(B)$, we have $G_B^B \leq \mathrm{Aut}(\overline{\mathcal{D}}(B)) \cong \mathrm{PSL}(3,2)$ and $G_B^{\Gamma_{\mathcal{B}}(B)} \leq \mathrm{Aut}(\overline{\mathcal{D}}(B))$. Since G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ of degree seven, we have $|G_B^{\Gamma_{\mathcal{B}}(B)}| \geq 7 \cdot 6 = 42$. Since no proper subgroup of $\mathrm{PSL}(3,2)$ has order greater than 24, it follows that $G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(3,2)$. Since the actions of an automorphism group of $\mathrm{PG}(2,2)$ on the set of points and the set of lines are permutationally isomorphic, we have $G_B^B \cong \mathrm{PSL}(3,2)$. By Theorem 1.1, G is faithful on $\mathcal{B}$.

We now prove $\Gamma[B,C] \ncong 2 \cdot C_4, C_8$, and if $\Gamma[B,C] \cong 4 \cdot K_2$ then Γ is $(G,2)$-arc transitive. Denote $A \cap \Gamma(B) = \{u_1,u_2,u_3,u_4\}$ and $B \cap \Gamma(A) = \{v_1,v_2,v_3,v_4\}$ for a fixed $A \in \Gamma_{\mathcal{B}}(B)$.

Suppose $\Gamma[A,B] \cong 2 \cdot C_4$. Without loss of generality, we may assume that each of $\{u_1,u_2,v_1,v_2\}$ and $\{u_3,u_4,v_3,v_4\}$ induces a copy of C_4 in Γ. Since $\lambda = 2$, $|B \cap \Gamma(A) \cap \Gamma(F)| = 2$ for each $F \in \Gamma_{\mathcal{B}}(B) \setminus \{A\}$. Since there are exactly six such blocks F, and since $|B \cap \Gamma(A)| = 4$ and the multiplicity of $\mathcal{D}(B)$ is one, each pair $\{v_i,v_j\}$ $(1 \leq i < j \leq 4)$ is equal to exactly one $B \cap \Gamma(A) \cap \Gamma(F)$. So there exist $C, D \in \Gamma_{\mathcal{B}}(B) \setminus \{A\}$ such that $B \cap \Gamma(A) \cap \Gamma(C) = \{v_1,v_2\}$ and $B \cap \Gamma(A) \cap \Gamma(D) = \{v_1,v_3\}$. Since $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, there exists $g \in G$ such that $(A,B,C)^g = (A,B,D)$. Hence $(B \cap \Gamma(A) \cap \Gamma(C))^g = B \cap \Gamma(A) \cap \Gamma(D)$, that is, $\{v_1,v_2\}^g = \{v_1,v_3\}$. However, since $g \in G_{A,B}$, it permutes the two cycles of $\Gamma[A,B]$ and so $\{v_1,v_2\}^g = \{v_1,v_2\}$ or $\{v_3,v_4\}$, which is a contradiction.

Suppose $\Gamma[A, B] \cong C_8$. Without loss of generality, we may assume that $\Gamma[A, B]$ is the cycle $(v_1, u_1, v_2, u_2, v_3, u_3, v_4, u_4, v_1)$. As above, there exists $C \in \Gamma_{\mathcal{B}}(B) \setminus \{A\}$ such that $B \cap \Gamma(A) \cap \Gamma(C) = \{v_1, v_2\}$. Since $r = 4$, there exist distinct $D, F \in \Gamma_{\mathcal{B}}(B) \setminus \{A, C\}$ such that $v_1 \in B \cap \Gamma(D) \cap \Gamma(F)$. Since $\lambda = 2$, either $B \cap \Gamma(A) \cap \Gamma(D)$ or $B \cap \Gamma(A) \cap \Gamma(F)$ is equal to $\{v_1, v_3\}$, say, $B \cap \Gamma(A) \cap \Gamma(D) = \{v_1, v_3\}$. Since $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, there exists $g \in G$ such that $(A, B, C)^g = (A, B, D)$. Hence $\{v_1, v_2\}^g = \{v_1, v_3\}$. However, $g \in G_{A,B}$ induces an automorphism of $\Gamma[A, B]$. On the other hand, the distances from v_1 to v_2 and v_3 in $\Gamma[A, B]$ are 2 and 4, respectively, and this is a contradiction.

So far we have proved that $\Gamma[A, B] \ncong 2 \cdot C_4, C_8$. Since $k = 4$ and $\Gamma[B, C]$ is $G_{B,C}$-edge transitive, we must have $\Gamma[B, C] \cong 4 \cdot K_2, K_{4,4} - 4 \cdot K_2$ or $K_{4,4}$. Suppose $\Gamma[B, C] \cong 4 \cdot K_2$. Then for $\alpha \in B$ the action of G_α on $\Gamma(\alpha)$ and $\Gamma_{\mathcal{B}}(\alpha)$ are permutationally isomorphic. Note that $\Gamma_{\mathcal{B}}(\alpha)$ is a block of $\mathcal{D}^*(B) \cong \mathcal{D}(B)$. Since $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(2, 7) \cong \mathrm{Aut}(\mathcal{D}^*(B))$, the setwise stabilizer of $\Gamma_{\mathcal{B}}(\alpha)$ in $G_B^{\Gamma_{\mathcal{B}}(B)}$ is isomorphic to S_4 and hence is 2-transitive on $\Gamma_{\mathcal{B}}(\alpha)$ as $|\Gamma_{\mathcal{B}}(\alpha)| = 4$. One can verify that this stabilizer is equal to G_α. Thus G_α is 2-transitive on $\Gamma_{\mathcal{B}}(\alpha)$ and so 2-transitive on $\Gamma(\alpha)$. In other words, Γ is $(G, 2)$-arc transitive when $\Gamma[B, C] \cong 4 \cdot K_2$. In the case where $\Gamma[B, C] \cong K_{4,4} - 4 \cdot K_2$ or $K_{4,4}$, since $\Gamma_{\mathcal{B}}$ is connected and $\overline{\mathcal{D}}(B) \cong \mathrm{PG}(2, 2)$, one can easily see that Γ is connected of valency 12 or 16 respectively.

Case (e): Since $(v, b, r, \lambda) = (6, 4, 2, 1)$, $\mathcal{D}^*(B)$ is the 2-(4, 2, 1) design, that is, the complete graph on four vertices. This case coincides with the case $(v, k) = (6, 3)$ in [8, Theorem 4.1(b)] and we have $G_B^{\Gamma_{\mathcal{B}}(B)} \cong A_4$ or S_4 since G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ of degree four. Since $(\lambda, r) = (1, 2)$, by [12, Theorem 4.3] we have $\Gamma \cong \Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$ for some self-paired G-orbit Δ on 3-arcs of $\Gamma_{\mathcal{B}}$. Moreover, again by [12, Theorem 4.3], for any connected 4-valent $(G, 2)$-arc transitive graph Σ and any self-paired G-orbit Δ on 3-arcs of Σ, $\Gamma = \Gamma_2(\Sigma, \Delta)$ is a G-symmetric graph admitting $\mathcal{B}_2 = \{B_2(\sigma) : \sigma \in V(\Sigma)\}$ as a G-invariant partition such that $\Gamma_{\mathcal{B}_2} \cong \Sigma$ and the corresponding parameters are $(v, b, r, \lambda) = (6, 4, 2, 1)$ and $k = v - 3 = 3$, where $B_2(\sigma)$ is the set of 2-paths of Σ with middle vertex σ. Since Σ is $(G, 2)$-arc transitive with even valency, by [10, Remark 4(c)] such a Δ exists and hence Σ can occur as $\Gamma_{\mathcal{B}}$ in (e).

Sufficiency. We now prove that each of (a)–(e) implies that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Since by Lemma 3.1 the multiplicity m of $\mathcal{D}(B)$ is a common divisor of b and r, in cases (a)–(d) we have $m = 1$. In case (e), since $b = 4$ and $\lambda \geq 1$, we have $m = 1$ as well.

In case (a), since $(v, b, r, \lambda) = (4, 4, 1, 0)$ and $k = 1$, each vertex in B has a neighbour in a unique block of $\Gamma_{\mathcal{B}}(B)$, yielding a bijection from B to $\Gamma_{\mathcal{B}}(B)$. Using this bijection, one can see that the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally isomorphic. Since $G_B^B \cong A_4$ or S_4, G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ and therefore $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive.

In case (b), since $\Gamma_{\mathcal{B}}$ is a cycle and is G-symmetric, it must be $(G, 2)$-arc transitive.

In case (c), since $(v, b, r, \lambda) = (7, 7, 4, 2)$, $\overline{\mathcal{D}}(B) \cong \mathrm{PG}(2, 2)$. Since $G_B^B \cong \mathrm{PSL}(3, 2)$ and the actions of $\mathrm{PSL}(3, 2)$ on the set of points and the set of lines of $\mathrm{PG}(2, 2)$ are permutationally isomorphic, we have $G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(3, 2)$. Since $\mathrm{PSL}(3, 2)$ is

https://doi.org/10.1017/S1446788713000669 Published online by Cambridge University Press

2-transitive on the set of lines of PG(2, 2), G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ and so $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive.

As shown in Remark 1.2(2), in case (d), $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive.

In case (e), since $G_B^{\Gamma_{\mathcal{B}}(B)} \cong A_4$ or S_4 and $b = 4$, G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ and so $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. □

The following result about the case $p = 5$ is largely a corollary of Theorem 1.1 (and Remark 1.2(2)). So we omit its proof.

Corollary 3.3. *Let Γ be a G-symmetric graph with $V(\Gamma)$ admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 5 \geq 1$ and $\Gamma_{\mathcal{B}}$ is connected of valency $b \geq 2$, where $G \leq \mathrm{Aut}(\Gamma)$. Then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if one of the following holds:*

(a) $(v, b, r, \lambda) = (6, 6, 1, 0)$ *and* $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)} \cong A_6$ *or* S_6*;*

(b) $(v, b, r, \lambda) = (10, 2, 1, 0)$*,* $\Gamma_{\mathcal{B}} \cong C_n$ *and* $G/G_{(\mathcal{B})} = D_{2n}$*, where* $n = |V(\Gamma)|/10$*;*

(c) $(v, b, r, \lambda) = (21, 21, 16, 12)$*,* $\overline{\mathcal{D}^*}(B) \cong \mathrm{PG}(2, 4)$*,* $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)}$ *is isomorphic to a 2-transitive subgroup of* $\mathrm{P\Gamma L}(3, 4)$*, and G is faithful on $\mathcal{B}$;*

(d) $(v, b, r, \lambda) = (11, 11, 6, 3)$*,* $\overline{\mathcal{D}^*}(B)$ *is isomorphic to the unique* 2-(11, 5, 2) *design and* $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(2, 11)$*;*

(e) $(v, b, r, \lambda) = (5a, a, a - 1, 5a - 10)$ *for some integer* $a \geq 3$*;*

(f) *either* (1) $(v, b, r, \lambda) = (10, 6, 3, 2)$*,* $\mathcal{D}^*(B)$ *is isomorphic to the unique* 2-(6, 3, 2) *design, and* $G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{Sp}_4(2)$ *or* $\mathrm{PSL}(2, 5)$*; or* (2) $(v, b, r, \lambda) = (15, 6, 4, 6)$*,* $\mathcal{D}^*(B)$ *is isomorphic to the complementary design of* K_6 *and* $G_B^{\Gamma_{\mathcal{B}}(B)} \cong A_6$*; or* (3) $(v, b, r, \lambda) = (20, 16, 12, 11)$*,* $\overline{\mathcal{D}^*}(B) \cong \mathrm{AG}(2, 4)$ *and* $G_B^{\Gamma_{\mathcal{B}}(B)}$ *is isomorphic to a 2-transitive subgroup of* $\mathrm{A\Gamma L}(2, 4)$.

As in Theorem 1.1, in (a) above we have $\Gamma \cong (|V(\Gamma)|/2) \cdot K_2$ and every connected 6-valent 2-arc transitive graph can occur as $\Gamma_{\mathcal{B}}$ in (a). In (b), since $\Gamma[B, C]$ is $G_{B,C}$-edge transitive, we have $\Gamma \cong 5n \cdot K_2$, $n \cdot C_{10}$, $n \cdot (K_{5,5} - C_{10})$, $n \cdot (K_{5,5} - 5 \cdot K_2)$ or $n \cdot K_{5,5}$. In (e) above, the same statements as in case (e) of Theorem 1.1 hold with $p = 5$. The three cases in (f) arise because $(a, s) = (2, 1), (3, 1), (4, 3)$ are the only pairs satisfying the conditions in (f) of Theorem 1.1. In (2) of (f), $G_B^{\Gamma_{\mathcal{B}}(B)}$ cannot be PGL(2, 5) since the latter has no transitive action of degree 15. Similarly, in (1) of (f), $G_B^{\Gamma_{\mathcal{B}}(B)} \not\cong \mathrm{PGL}(2, 5)$ because the 2-(6, 3, 2) design has ten blocks of size 3 and PGL(2, 5) is (sharply) 3-transitive of degree six. The result in (3) of (f) follows because $\overline{\mathcal{D}^*}(B)$ is a 2-(16, 4, 1) design in this case and AG(4, 2) is the unique 2-(16, 4, 1) design [4, Section 1.3].

Acknowledgements

We would like to thank Professor Cheryl E. Praeger for helpful discussions on Mathieu groups, Professor Yuqing Chen for providing the example in Remark 1.2(3), and an anonymous referee for helpful comments.

References

[1] T. Beth, D. Jungnickel and H. Lenz, *Design Theory,* 2nd edn (Cambridge University Press, Cambridge, 1999).

[2] N. L. Biggs, *Algebraic Graph Theory,* 2nd edn (Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993).

[3] H. J. Broersma and C. Hoede, 'Path graphs', *J. Graph Theory* **13** (1989), 427–444.

[4] C. J. Colbourn and J. H. Dinitz, *Handbook of Combinatorial Designs,* 2nd edn (CRC Press, Boca Raton, 2007).

[5] J. D. Dixon and B. Mortimer, *Permutation Groups* (Springer, New York, 1996).

[6] A. Gardiner and C. E. Praeger, 'A geometrical approach to imprimitive graphs', *Proc. Lond. Math. Soc.* (3) **71** (1995), 524–546.

[7] M. A. Iranmanesh, C. E. Praeger and S. Zhou, 'Finite symmetric graphs with two-arc transitive quotients', *J. Combin. Theory Ser. B* **94** (2005), 79–99.

[8] B. Jia, Z. Lu and G. X. Wang, 'A class of symmetric graphs with 2-arc transitive quotients', *J. Graph Theory* **65** (2010), 232–245.

[9] W. M. Kantor, 'Classification of 2-transitive symmetric designs', *Graphs Combin.* **1**(2) (1985), 165–166.

[10] C. H. Li, C. E. Praeger and S. Zhou, 'A class of finite symmetric graphs with 2-arc transitive quotients', *Math. Proc. Cambridge Philos. Soc.* **129** (2000), 19–34.

[11] C. H. Li, C. E. Praeger and S. Zhou, 'Imprimitive symmetric graphs with cyclic blocks', *Eur. J. Combin.* **31** (2010), 362–367.

[12] Z. Lu and S. Zhou, 'Finite symmetric graphs with 2-arc transitive quotients (II)', *J. Graph Theory* **56** (2007), 167–193.

[13] G. Xu and S. Zhou, 'Solution to a question on a family of imprimitive symmetric graphs', *Bull. Aust. Math. Soc.* **82** (2010), 79–83.

[14] S. Zhou, 'Constructing a class of symmetric graphs', *Eur. J. Combin.* **23** (2002), 741–760.

[15] S. Zhou, 'Symmetric graphs and flag graphs', *Monatsh. Math.* **139** (2003), 69–81.

[16] S. Zhou, 'Almost covers of 2-arc transitive graphs', *Combinatorica* **24** (2004), 731–745; [Erratum: **27** (2007), 745–746].

[17] S. Zhou, 'On a class of finite symmetric graphs', *Eur. J. Combin.* **29** (2008), 630–640.

GUANGJUN XU, Department of Mathematics and Statistics,
The University of Melbourne, Parkville, VIC 3010, Australia
e-mail: gx@ms.unimelb.edu.au

SANMING ZHOU, Department of Mathematics and Statistics,
The University of Melbourne, Parkville, VIC 3010, Australia
e-mail: smzhou@ms.unimelb.edu.au

https://doi.org/10.1017/S1446788713000669 Published online by Cambridge University Press