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# Quasi proximal continuity

## Panayotis Th. Lambrinos

Conditions are given, under which a quasi-proximally continuous function is quasi-uniformly continuous, or a continuous function is quasi-proximally continuous. Thus, basic results on uniform and proximal continuity are extended and some new results are obtained. Three results in the literature are shown to be false.

According to [8] and [9], a *quasi-proximity space* is a pair $(X, \delta)$, where $X$ is a non empty set and $\delta$ is a binary relation on the power set of $X$ which satisfies:

(P.1) $A\not\delta\emptyset$ , $\emptyset\not\delta A$ for every $A \subset X$ ;

(P.2) $\{x\}\delta\{x\}$ for every $x \in X$ ;

(P.3) $C\delta A \cup B$ iff $C\delta A$ or $C\delta B$ and $A \cup B\delta C$ iff $A\delta C$ or $B\delta C$ ;

(P.4) if $A\not\delta B$ then there exists a subset $E$ such that $A\not\delta E$ and $(X-E)\not\delta B$ .

The pair $(X, \delta)$ becomes a *proximity space* when:

(P.5) $A\delta B$ iff $B\delta A$ .

Pervin showed in [8] that if the closure operator $c$ is defined by $c(A) = \{x : \{x\}\delta A\}$ , then each quasi-proximity space $(X, \delta)$ gives rise to a topology $\tau(\delta)$ on $X$ and that every topological space $(X, \tau)$ is quasi-proximizable, that is, there exists a quasi-proximity $\delta$ on $X$ such that $\tau(\delta) = \tau$ .

Given a quasi-uniformity $Q$ on $X$ (for definitions see [6], [7] or [3]) the quasi-proximity $\delta$ induced by $Q$ is defined by: $A\delta B$ iff

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$(A \times B) \cap U \neq \emptyset$ for every $U$ in $Q$ ([7], p. 107). If $\delta$ is induced by $Q$, then $Q$ is said to be compatible with $\delta$.

It has been shown in [2] that for every quasi-proximity space $(X, \delta)$ there exists a unique totally bounded quasi-uniformity $Q_t$ which is compatible with $\delta$. Moreover, $Q_t$ is the coarsest quasi-uniformity which is compatible with $\delta$ and a base for $Q_t$ is the collection of all the sets of the form $\bigcup_{i=1}^{n} B_i \times A_i$, where $X = \bigcup_{i=1}^{n} B_i$, $n$ a natural number and $B_i \bar{\delta} (X-A_i)$ for every $i$, $1 \leq i \leq n$.

Clearly, $\delta$ is a proximity on $X$ iff $Q_t$ is the unique totally bounded uniformity which is compatible with $\delta$ ([2]). Therefore, since the Pervin quasi-uniformity $P$ is totally bounded ([6], p. 51), one can get, using Theorem 5 in [1], examples of quasi-proximity spaces $(X, \delta)$ (even compact spaces) such that the quasi-proximity $\delta$ is not a proximity. In fact it is sufficient for this to consider the quasi-proximity $\delta_P$ induced by $P$.

Now, as a natural extension of Definition 4.1 of [7], we have:

**DEFINITION.** Let $(X, \delta_1)$ and $(X, \delta_2)$ be two quasi-proximity spaces. A function $f : X \to Y$ is said to be quasi-proximally continuous iff $A \delta_1 B$ implies $f(A) \delta_2 f(B)$ or equivalently iff $C \bar{\delta}_2 D$ implies $f^{-1}(C) \bar{\delta}_1 f^{-1}(D)$.

Clearly if $\delta_1$, $\delta_2$ are proximities then a quasi-proximally continuous function is just a proximally continuous one (a proximity mapping in the terminology of [7]).

The next two theorems can be proved similarly to Theorems 10.8 and 4.2 of [7].

**THEOREM 1.** *Let $f : (X, Q_1) \to (Y, Q_2)$ be a quasi-uniformly continuous function; then $f : (X, \delta_1) \to (Y, \delta_2)$ is quasi-proximally continuous, where $\delta_1$, $\delta_2$ are the quasi-proximities induced respectively by the quasi-uniformities $Q_1$, $Q_2$.*

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**THEOREM 2.** *A quasi-proximally continuous function* $f : (X, \delta_1) \to (Y, \delta_2)$ *is continuous with respect to* $r(\delta_1)$ *and* $r(\delta_2)$ .

The converses of the above two theorems do not hold without special conditions even if $\delta_1, \delta_2$ are proximities (see [7], pp. 66, 20).

However, as regards the converse of Theorem 1 above, we have the following.

**LEMMA 1.** *Let* $Q^*$ *be a quasi-uniformity compatible with the quasi-proximity* $\delta^*$ *and* $Q_t$ *the unique totally bounded quasi-uniformity compatible with* $\delta$ . *If* $\delta \subset \delta^*$ , *then the supremum quasi-uniformity* $Q_t \vee Q^*$ *is compatible with* $\delta$ .

Proof. Let $\delta_1$ be the quasi-proximity induced by $Q_t \vee Q^*$ . Since $Q_t \vee Q^*$ contains $Q_t$ , it follows from Theorem 1 that $\delta_1 \subset \delta$ . Let now $A \not\delta_1 B$ ; hence there exist a basic member $W$ of $Q_t$ and a member $V$ of $Q^*$ such that $(A\times B) \cap W \cap V = \emptyset$ , $W = \bigcup_{i=1}^{m} (B_i\times A_i)$ , $X = \bigcup_{i=1}^{m} B_i$ and $B_i \not\delta (X-A_i)$ for every $i$ , $1 \le i \le m$ .

Therefore

$$\bigcup_{i=1}^{m} (A\times B) \cap (B_i\times A_i) \cap V = \bigcup_{i=1}^{m} [(A\cap B_i)\times(B\cap A_i)] \cap V = \emptyset$$

or

$$[(A\cap B_i)\times(B\cap A_i)] \cap V = \emptyset ,$$

for every $i$ , $1 \le i \le m$ .

Consequently $A \cap B_i \not\delta^* B \cap A_i$ and thus $A \cap B_i \not\delta B \cap A_i$ . Now, by axiom P.3, $B_i \not\delta (X-A_i)$ implies that $A \cap B_i \not\delta B \cap (X-A_i)$ . By axiom P.3 again, $A \cap B_i \not\delta B \cap A_i$ and $A \cap B_i \not\delta B \cap (X-A_i)$ imply $A \cap B_i \not\delta B \cap [(X-A_i)\cup A_i]$ , that is, $A \cap B_i \not\delta B$ for every $i$ , $1 \le i \le m$ . Applying again axiom P.3 we get $\bigcup_{i=1}^{m} (A\cap B_i) \not\delta B$ or $A \cap \left(\bigcup_{i=1}^{m} B_i\right) \not\delta B$ . Finally, since $X = \bigcup_{i=1}^{m} B_i$ we have

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$A \bar{\delta} B$ . We have already shown that $\delta = \delta_1$ and thus $Q_t \vee Q^*$ is compatible with $\delta$ .

THEOREM 3. *Let* $f : (X, \delta_1) \to (Y, \delta_2)$ *be a quasi-proximally continuous function and* $Q_2$ *an arbitrary quasi-uniformity compatible with* $\delta_2$ . *Then there exists a quasi-uniformity* $Q_1$ *compatible with* $\delta_1$ *such that* $f : (X, Q_1) \to (Y, Q_2)$ *is quasi-uniformly continuous. If* $Q_2$ *is totally bounded, then* $Q_1$ *may be chosen to be also totally bounded.*

*Proof.* It is well known that the quasi-uniformity $Q^*$ generated by the family $\left\{(f \times f)^{-1}(V) : V \in Q_2\right\}$ is the coarsest quasi-uniformity on $X$ such that $f : (X, Q^*) \to (Y, Q_2)$ is quasi-uniformly continuous and that if $Q_2$ is totally bounded, then so is $Q^*$ .

Let $\delta^*$ be the quasi-proximity induced by $Q^*$ . Now, one can easily show that $\delta_1 \subset \delta^*$ (see, for example, [6], p. 75). By the previous lemma, if $Q_t$ is the unique totally bounded quasi-uniformity compatible with $\delta_1$ , then $Q_t \vee Q^* = Q_1$ is also compatible with $\delta_1$ and clearly the function $f : (X, Q_1) \to (Y, Q_2)$ is quasi-uniformly continuous.

If $Q_2$ is totally bounded, then so is $Q_t \vee Q^* = Q_1$ ([5], p. 50) and thus $Q_1 = Q_t \vee Q^* = Q_t$ .

COROLLARY 1. *Let* $(X, \delta_1)$ *and* $(Y, \delta_2)$ *be quasi-proximity spaces and* $Q_{t2}$ *the unique totally bounded quasi-uniformity which is compatible with* $\delta_2$ . *Then, for every quasi-uniformity* $Q$ *compatible with* $\delta_1$ *, a function* $f : (X, \delta_1) \to (Y, \delta_2)$ *is quasi-proximally continuous iff it is quasi-uniformly continuous with respect to* $Q$ *and* $Q_{t2}$ .

COROLLARY 2. *Let* $\delta_1, \delta_2$ *be two quasi-proximities on* $X$ *and let* $Q_{ti}$ *be the unique totally bounded quasi-uniformity compatible with* $\delta_i$ *,* $i = 1, 2$ . *If* $\delta_1 \subset \delta_2$ *, then we have* $Q_{t1} \supset Q_{t2}$ .

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Proof. Use the identity function $i : (X, \delta_1) \to (X, \delta_2)$ and Corollary 1 above.

COROLLARY 3. *The* Pervin *quasi-uniformity* $P$ *is the largest totally bounded quasi-uniformity which is compatible with the topology* $r$ *of a space* $(X, r)$ .

Proof. This follows by Corollary 2 above and the fact that every quasi-proximity $\delta$ compatible with $r$ contains the Pervin quasi-proximity $\delta_p$ (see [7], Theorems 19.14, 19.7).

For an example of a totally bounded quasi-uniformity which (by Corollary 3) not only does not contain $P$ but is strictly contained in it, see [1], p. 399.

PROPOSITION 1. *For every non discrete* $T_0$ *space* $(X, r)$ , *if* $\delta$ *is a proximity compatible with the topology* $r$ *and* $\delta_p$ *is the quasi-proximity induced by the* Pervin *quasi-uniformity* $P$ *of the space, then* $\delta \not\subset \delta_p$ .

Proof. Let $\delta \subset \delta_p$ for a proximity $\delta$ compatible with $r$ . Since the Pervin quasi-uniformity $P$ is totally bounded, it follows by Corollary 1 above, that the identity function $i : (X, \delta) \to (X, \delta_p)$ is quasi-uniformly continuous with respect to every uniformity $U$ compatible with $\delta$ and to $P$ . Hence $U \supset P$ , a contradiction to Theorem 5, [1].

LEMMA 2. *Let* $Q$, $Q_t$ *be quasi-uniformities compatible with the quasi-proximity* $\delta$ *on* $X$ *and let* $Q_t$ *be totally bounded. Then, if* $Q^*$ *is a quasi-uniformity such that* $Q_t \subset Q^* \subset Q$ , *it follows that* $Q^*$ *is also compatible with* $\delta$ .

Proof. Using the identity function and Theorem 1 above we have that $Q_t \subset Q^* \subset Q$ implies $\delta \supset \delta^* \supset \delta$ , where $\delta^*$ is the quasi-proximity induced by $Q^*$ . Hence $\delta^* = \delta$ , that is, $Q^*$ is compatible with $\delta$ .

THEOREM 4. *For every quasi-uniformity* $Q$ *compatible with a given quasi-proximity* $\delta$ , *the unique totally bounded quasi-uniformity* $Q_t$ *compatible with* $\delta$ *is the largest totally bounded quasi-uniformity* *contained in* $Q$ .

Proof. Let $Q^*$ be any totally bounded quasi-uniformity contained in $Q$ ; then the supremum quasi-uniformity $Q^* \vee Q_t$ is totally bounded and $Q_t \subset Q^* \vee Q_t \subset Q$ . By the previous lemma $Q^* \vee Q_t$ is compatible with $\delta$ and since $Q_t$ is the unique totally bounded quasi-uniformity compatible with $\delta$ it follows that $Q^* \subset Q_t$ .

As regards the inverse of Theorem 2 above we have:

THEOREM 5. *Every continuous function from a compact quasi-proximity space* $(X, \delta_1)$ *into a proximity space* $(Y, \delta_2)$ *is quasi-proximally continuous.*

Proof. Let us consider a quasi-uniformity compatible with $\delta_1$ and a uniformity compatible with $\delta_2$ . Now, because of Theorem 1 above, it is sufficient to recall the fact that a continuous function from a compact quasi-uniform space into a uniform space is quasi-uniformly continuous ([3], Theorem 1).

By the above theorem, we get as a special case the well known result that every continuous function from a compact proximity space into a proximity space is proximally continuous ([7], Theorem 4.4).

Of course, a generalization of Theorem 5 above may be given by means of $r$-bounded sets by analogy to Theorem 2 in [3].

COUNTEREXAMPLE. A continuous function from a compact quasi-proximity space (even a compact proximity space) into a quasi-proximity space is not necessarily quasi-proximally continuous, as the following show:

1. Let us take the space $X = [0, 1]$ with the usual topology $r$ and let $\delta_1$ be the proximity compatible with $r$ , defined by: $A\delta_1 B$ iff $\overline{A} \cap \overline{B} \neq \emptyset$ ([7], p. 13). Let now, $\delta_2$ be the quasi-proximity compatible with $r$ defined by: $A\delta_2 B$ iff $A \cap \overline{B} \neq \emptyset$ ([8]). It is clear that $\delta_1 \not\subset \delta_2$ . Therefore, the identity function $i : (X, \delta_1) \to (X, \delta_2)$ is continuous on the compact space $(X, r)$ but is not quasi-proximally continuous. We can get the same conclusion using instead of $\delta_1$ the

proximity $\delta_1^*$ defined by: $A\delta_1^*B$ iff

$$d(A,\ B) = \inf\{|a-b| : a \in A,\ b \in B\} = 0$$

([7], p. 14).

2. Let $\delta$ be the proximity on $X = [0,\ 1]$ induced by the metric uniformity $\mathcal{U}$ on it. Hence, $A\delta B$ iff $(A\times B) \cap U_\varepsilon \neq \emptyset$, for every $U_\varepsilon = \{(x,\ y) : |x-y| < \varepsilon\}$, $\varepsilon > 0$. Let also $\delta_F$ be the quasi-proximity compatible with $r$ on $X$, induced by the quasi-uniformity $Q_F$ having as a base the family $\mathcal{B} = \{V_a : 0 < a < 1\}$,

$$V_a = \{(x,\ x) : x \in X\} \cup \{(x,\ y) : x,\ y \neq 0,\ 1 \text{ and } |x-y| < a\} \cup$$
$$\cup \{0\} \times \{y : 1-a < y < 1\} \cup \{1\} \times \{y : 0 < y < a\}$$

([1], p. 399). Now it is clear that the sets $A = \{1\}$, $B = (1/2,\ 1)$ are such that $(A\times B) \cap U_\varepsilon \neq \emptyset$ for every $\varepsilon > 0$, that is, $A\delta B$ but $A\not\delta_F B$, since for $a \leq 1/2$ we have $(A\times B) \cap V_a = \emptyset$. It follows that the identity function $i : (X,\ \delta) \to (X,\ \delta_F)$ is continuous, but is not quasi-proximally continuous, although $X$ is compact.

Another large class of counterexamples of the above type may be obtained by Proposition 1.

REMARK 1. In the light of Theorem 1, this counterexample contradicts the assertion of the Corollary, p. 56, [6], that "a continuous function from a compact uniform space into a quasi-uniform space is quasi-uniformly continuous". Hence, Theorem 4.23, p. 56, and Lemma 4.22, p. 55, in [6], are both false.

In fact, false is the underlying assertion in line 2, p. 56, that: if a subset $G$ containing the diagonal $\Delta = \{(x,\ x) : x \in X\}$ is open in the product topology $r_Q \times r_{Q^{-1}}$ (where $r_Q$, $r_{Q^{-1}}$ are the topologies induced by the quasi-uniformities $Q$ and $Q^{-1}$), then $G(x)$ is a neighbourhood of the point $x$ in the topology $r_Q$. In fact, the following hold.

THEOREM 6. *Under the above conditions* $G(x)$ *is* $r_{Q^{-1}}$*-open for every*

$x \in X$ *, but is not necessarily an* $r_Q$*-neighbourhood of the point* $x$ *.*

Proof. Let $y \in G(x)$, that is, $(x, y) \in G$. Then, there exist $V, W \in Q$ such that $(x, y) \in V(x) \times W^{-1}(y) \subset G$ and thus $G(x) \supset [V(x) \times W^{-1}(y)](x) = W^{-1}(y)$. Therefore $G(x)$ is an $r_{Q^{-1}}$-neighbourhood of every point $y \in G(x)$. It follows that $G(x)$ is $r_{Q^{-1}}$-open for every $x \in X$.

Now, we shall give a counterexample showing that $G(x)$ is not necessarily an $r_Q$-neighbourhood of $x$.

Let $Q$ be the quasi-uniformity on the set $R$ of the reals, generated by the family $U_\varepsilon = \{(x, y) : y < x+\varepsilon\}$, $\varepsilon > 0$. Then a basis for the topology $r_Q$ is the family $\{(-\infty, a) : a \in R\}$ and a basis for the topology $r_{Q^{-1}}$ is the family $\{(b, \infty) : b \in R\}$. Obviously, the set $G = \bigcup_{a \in R} [(-\infty, a+1) \times (a-1, \infty)]$ is $r_Q \times r_{Q^{-1}}$-open and contains $\Delta$, but $G(x) = \bigcup_{x<a+1} (a-1, \infty) = (x-2, \infty)$ which does not contain $r_Q$-open sets; that is, sets of the form $(-\infty, a)$.

COROLLARY 4. *Let* $M$ *be a closed set in the product topology* $r_Q \times r_{Q^{-1}}$. *Then,* $M(x)$ *is* $r_{Q^{-1}}$*-closed for every* $x \in X$ *, but not necessarily* $r_Q$*-closed.*

Proof. $M(x) = X - (X \times X - M)(x)$ for every $x \in X$.

THEOREM 7. *A function* $f : (X, r_1) \to (Y, r_2)$ *is continuous iff it is quasi-proximally continuous with respect to the* Pervin *quasi-proximities on the spaces* $\delta_{P1}, \delta_{P2}$.

Proof. Sufficiency is obvious by Theorem 2. Necessity follows by Theorem 1 and the fact that a continuous function is quasi-uniformly continuous with respect to the Pervin quasi-uniformities $P_1, P_2$ ([4], Theorem 4.1).

Note added in proof (30 June 1973). During the correction of the proofs the author found that Theorem 3 above has been announced by Metzger [5, Theorem 2, p. 131]. However, Metzger's proof of the theorem is mainly based on his Lemma 4, p. 125, the proof of which is not correct, because the claim in line 1, p. 126, that $B_1 \cup B_2 \cup \ldots \cup B_n^{\cdot} \supset B$ is not true.

EXAMPLE. Take $X = [0, 2]$ and $\tau$ the usual topology,

$$V = ([0, 1]\times[0, 1]) \cup \left((\tfrac{1}{2}, \tfrac{3}{2})\times(\tfrac{1}{2}, \tfrac{3}{2})\right) \cup \left((1, 2]\times(1, 2]\right) .$$

Then, since $V$ is open in the product space $X \times X$ and contains the diagonal, for every $A \subset X$, $A \not\delta_p X - V(A)$ where $\delta_p$ is the Pervin quasi-proximity. Let $A = [\frac{1}{4}, \frac{3}{4}]$, $B = [\frac{5}{4}, \frac{7}{4}]$, $A_1 = [\frac{3}{8}, \frac{7}{8}]$, $B_1 = [\frac{9}{8}, \frac{13}{8}]$. Clearly, $A \times B \subset (X\times X-V) \cup \left(A_1\times B_1\right)$, $A_1 \not\delta_p B_1$ but $B \not\subset B_1$.

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Department of Mathematics,
Aristotelian University of Thessaloniki,
Thessaloniki,
Greece.