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Hilbert Transformation and Representation of the $ax + b$ Group

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Abstract. In this paper we study the Hilbert transformations over $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$ from the viewpoint of symmetry. For a linear operator over $L^2(\mathbb{R})$ commutative with the $ax + b$ group, we show that the operator is of the form $\lambda I + \eta H$, where I and H are the identity operator and Hilbert transformation, respectively, and λ, η are complex numbers. In the related literature this result was proved by first invoking the boundedness result of the operator using some machinery. In our setting the boundedness is a consequence of the boundedness of the Hilbert transformation. The methodology that we use is the Gelfand–Naimark representation of the $ax + b$ group. Furthermore, we prove a similar result on the unit circle. Although there does not exist a group like the $ax + b$ group on the unit circle, we construct a semigroup that plays the same symmetry role for the Hilbert transformations over the circle $L^2(\mathbb{T})$.

1 Introduction

The Hilbert transformation given by the formula

$$\mathbf{H}f(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy, \quad x \in \mathbb{R},$$

can be first defined for functions f of finite energy and locally of the Hölder type continuity. It can then be extended to become a L^2-bounded linear operator over the whole L^2 space. In the rest of the article we will omit the prix p.v. in the case of no confusion.

The Hilbert transformation has many applications, including solving problems in aerodynamics, condensed matter physics, optics, fluids, and engineering (see, for instance, [7]). It is an especially indispensable tool in harmonic and signal analysis.

Hilbert transformations play a role in connecting harmonic with complex analysis. In general terms, a Hilbert transformation on a manifold can be defined as the mapping from the scalar part (real part) to the non-scalar part (imaginary part) of the boundary limits of complex analytic functions on one of the two regions divided by the manifold ([1]). For example, for all $u \in L^2(\mathbb{R})$, $u + i\mathbf{H}u$ belongs to the closed subspace of $L^2(\mathbb{R})$ constituted by the non-tangential boundary limits of the functions in the complex Hardy space $H^2(\mathbb{C}^+)$. The concerned closed subspace of $L^2(\mathbb{R})$ is denoted by $H^2_+(\mathbb{R})$. On the contrary, if $f \in H^2(\mathbb{C}^+)$, then there exists a function

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$u \in L^2(\mathbb{R})$ such that the non-tangential boundary limit of f, still denoted f, possesses the form $u + i\mathbf{H}u$, where u can be chosen as real-valued or complex valued. In particular, the non-tangential boundary limit of a Hardy $H^2(\mathbb{C}^+)$ function f can have the expression $(1/2)f + i(1/2)\mathbf{H}f$ on the boundary, phrased as the Plemelj formula [9,10], which further implies $\mathbf{H}f = -if$. The latter turns out to be a characterization of a function $f \in L^2(\mathbb{R})$ to be in $H^2_+(\mathbb{R})$ ([3]). The above relations can all be extended to Hardy spaces $H^p(\mathbb{C}^+)$ with $1 \le p < \infty$ ([13]). For $p = 2$, the trace operator taking $H^2(\mathbb{C}^+)$ to $H^2_+(\mathbb{R})$ is, in fact, an isometry between the Hilbert spaces. In harmonic and signal analysis there is correspondence between the real signal u and the analytic one $u + i\mathbf{H}u$ ([5,11]). In the current study we restrict ourselves to $p = 2$. For the lower half complex plane we have an analogous theory and, correspondingly, have the spaces $H^2(\mathbb{C}^-)$ and $H^2_-(\mathbb{R})$.

Denote by $H^\pm = H^2_\pm(\mathbb{R})$ the spaces consisting of, respectively, the non-tangential boundary limits of the upper and lower half Hardy spaces; the Hilbert transformation $\mathbf{H}$ can be decomposed into the sum of the two projection operators over H^+ and H^-, respectively. In fact, by the Plemelj formula,

$$L^2(\mathbb{R}) = H^+ \bigoplus H^-, \quad \mathbf{H} = (-i)(\mathbf{P}^+ - \mathbf{P}^-),$$

where P^+ and P^- denote the projection operators over H^+ and H^-, respectively. By this decomposition it is easy to check that the Hilbert transformation is a power self-inverse, and precisely,

$$(i\mathbf{H})^2 = \mathbf{I}, \quad \text{or} \quad \mathbf{H}^2 = -\mathbf{I},$$

where $\mathbf{I}$ is the identity operator [10]. We further note that

$$\mathbf{H}f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}(-i\operatorname{sgn}(\xi))f^\wedge(\xi)d\xi,$$

where $f^\wedge$ is the Fourier transform of f, and, for almost all x,

$$\mathbf{P}^\pm f(x) = \pm\frac{1}{2\pi}\int_0^{\pm\infty} e^{i\xi x}f^\wedge(\xi)d\xi = \frac{1}{2}f(x) \pm i\frac{1}{2}\mathbf{H}f(x).$$

In [15] a set of characterization conditions for an operator to be the Hilbert transformation is given, while the characterization conditions are in terms of properties of the images of the operator restricted to the exponential functions. In [5] the authors further study the aspect and give mathematical proofs of the results in [15]. In this study we give characterizations of the Hilbert transformations on the real line and on the unit circle in terms of group symmetry in the respective contexts. What is interesting is that the Hilbert transformation operator, originally defined as an analysis object, can be fully characterized through algebraic operations.

On the real line we consider the following two group operations in relation to symmetry properties of operators.

Denote by T_a the dilation operator

$$T_a f = a^{-\frac{1}{2}} f\Big(\frac{x}{a}\Big), \quad a > 0, \quad \forall f \in L_2(\mathbb{R}),$$

and by τ_b the translation operator

$$\tau_b f = f(x-b),\ b \in \mathbb{R}, \quad \forall f \in L_2(\mathbb{R}).$$

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It is evident that both T_a and τ_b are isometric mappings from $L^2(R)$ to itself. Then we have the following lemma.

Lemma 1.1 ([7]) *For $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$, both operators T_a and τ_b commute with the Hilbert transformation* **H**, *we have*

$$T_a\mathbf{H} = \mathbf{H}T_a, \tag{1.1}$$

$$\tau_b\mathbf{H} = \mathbf{H}\tau_b. \tag{1.2}$$

Lemma 1.1 reveals the physical significance of the Hilbert transformation. Equation (1.1) means that **H** is independent of scale, while (1.2) says that **H** is independent of the location of the original point. We know that translation and dilation generate the $ax + b$ group. So the Hilbert transformation is invariant under the action of the $ax + b$ group over $L^2(\mathbb{R})$. In this sense we say that the Hilbert transformation has the symmetry of the $ax + b$ group. On the other hand, this symmetry can characterize the Hilbert transformation. In fact, by using the results in [6] one first obtains that in $\mathbb{R}^n$, and in some other symmetric manifolds as well, linear operators on L^p for some $1 \le p < \infty$ commuting with translation or scaling like operations are themselves bounded operator. As a second step in [7] the author shows that a linear bounded operator on $L^p(\mathbb{R})$, $1 \le p < \infty$, commuting with the translation and scaling, is of the form $\lambda\mathbf{I} + \eta\mathbf{H}$, where **I** and **H** are the identity operator and Hilbert transformations respectively, λ, η are complex numbers.

In this paper, by using Gelfand-Naimark's irreducible representation of the $ax + b$ group and Schur theorem, we prove the characterization of the Hilbert transformation in the $L^2(\mathbb{R})$ space in terms of the $ax + b$ group. In particular, we do not assume the boundedness property of the operator commuting with the $ax + b$ group. With our approach, the boundedness is a consequence of the Gelfand–Naimark's irreducible representation theorem that avoids the big machinery established in [6].

The case of the unit circle $\mathbb{T}$ is more delicate. Denote by $L^2(\mathbb{T})$ the space of square integrable functions on $\mathbb{T}$ with the inner product $\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta$. The Hilbert transformation over $L^2(\mathbb{T})$, or *circular Hilbert transformation*, is defined as

$$\widetilde{\mathbf{H}}f(t) = \text{p.v.}\frac{1}{2\pi}\int_0^{2\pi} f(e^{is})\cot\Big(\frac{\theta - s}{2}\Big)ds,$$

where $t = e^{i\theta}$, $\tau = e^{is}$. A closely related singular integral operator is

$$\mathbf{C}f(t) = \text{p.v.}\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\tau)}{\tau - t}d\tau = \text{p.v.}\frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{is})}{e^{is} - e^{i\theta}}e^{is}ds.$$

In this article, we call **C** the *singular Cauchy transformation*. Denote by $\mathbf{H}_0$ the functional $\mathbf{H}_0 f = \frac{1}{2\pi}\int_0^{2\pi} f(e^{is})ds$ giving rise to the 0-th Fourier coefficient of the function to be expanded. It is easy to check (the Plemelj formula, see, for instance, [10] or [12]) that

$$\mathbf{C} = \frac{i}{2}\widetilde{\mathbf{H}} + \frac{1}{2}\mathbf{H}_0. \tag{1.3}$$

$\widetilde{\mathbf{H}}$ has properties analogous with $\mathbf{H}$. For instance, $L^2(\mathbb{T})$ is the direct sum of the two Hardy spaces on, respectively, the two areas of the complex plane divided by the unit circle. $\widetilde{\mathbf{H}}$, modulo a constant multiple of $\mathbf{H}_0$, is a linear combination of the projections over the two Hardy spaces, respectively, namely,

$$\mathbf{I} = \mathbf{P}^+ + \mathbf{P}^-, \qquad \mathbf{P}^\pm = \frac{1}{2}(\mathbf{I} \pm \mathbf{C}),$$
$$\widetilde{\mathbf{H}} = (-i)(\mathbf{P}^+ - \mathbf{P}^-) + i\mathbf{H}_0, \qquad \widetilde{\mathbf{H}}^2 = -\mathbf{I} + \mathbf{H}_0.$$

With the Fourier expansion

$$f(e^{it}) = \sum_{k=-\infty}^{\infty} c_k e^{ikt},$$

there, in fact, hold

$$\mathbf{P}^+ f(e^{it}) = \sum_{k=0}^{\infty} c_k e^{ikt}, \quad \mathbf{P}^- f(e^{it}) = \sum_{k=-\infty}^{-1} c_k e^{ikt},$$
$$\widetilde{\mathbf{H}} f(e^{it}) = -ic_0 + \sum_{k\neq 0} (-i\mathrm{sgn}(k)) c_k e^{ikt}.$$

It seems unnatural to study symmetry of the singular Cauchy transformation due to the non-zero curvature of the underlying manifold, viz., the circle. In this paper we deal with symmetry of the circular Hilbert transformation. At first glance, it should be the Möbius transformation group that gives rise to the characterization of $\widetilde{\mathbf{H}}$. However, there does not seem to exist a Fourier correspondence of the Möbius transformation. On the other hand, the phase translation and scale change generate the Fourier inverse of the actions of the $ax+b$ group on $L^2(\mathbb{R})$. We also wish to obtain the symmetry by the module of the $ax+b$ group, and, in order to do so, we treat (a,b) and (c,d) as identical if $ax+b \equiv cx+b \pmod{2\pi}$ for all $x \in \mathbb{R}$. Unfortunately, the equivalent classes do not form a group.

We construct a family of transformations over $\mathbb{T}$ whose natural representation is irreducible over $L^2(\mathbb{T})$. Then we obtain the characterization of $\widetilde{\mathbf{H}}$ in analogy with $\mathbf{H}$.

At the end of the paper, we add some remarks on the role of the Möbius group in relation to symmetry of the Hilbert transformation on the unit circle.

2 Induced Representations of Two Groups

In Section 1 we mentioned that translations and dilations generate a nontrivial group G, the $ax+b$ group, which is the group of all affine transformations $x \to ax+b$ of $\mathbb{R}$ with $a>0$ and $b \in \mathbb{R}$. Its underlying manifold is $(0,\infty)\times\mathbb{R}$, and the group law is defined by

$$\big((a,b)(a',b')\big)(x) = aa'x + b + ab' = (aa', b+ab')(x), \quad x \in \mathbb{R},$$

which gives

$$(a,b)(a',b') = (aa', b+ab').$$

It is easy to check the relation $(a,b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$. The measure da/a^2 is the left Haar measure, and $dadb/a$ is the right Haar measure on this group [2].

There exists a natural unitary representation π of G, of infinite dimension, over the Hilbert space $L^2(\mathbb{R})$. Denote by $\mathfrak{U}(L^2(\mathbb{R}))$ the operator group of the unitary automorphism of $L^2(\mathbb{R})$. Then the group morphism $\pi: G \to \mathfrak{U}(L^2(\mathbb{R}))$ is defined by

$$(\pi(a,b)f)(x) = \Big(\frac{1}{a}\Big)^{\frac{1}{2}} f\Big(\frac{x-b}{a}\Big), \quad x \in \mathbb{R}.$$

Equation $\pi(a,b)$ is also written as π_{ab} in this article.

It is obvious that both the Hardy spaces on the upper and lower half planes, respectively, are the invariant subspaces of the natural representation of the $ax+b$ group. So it is reducible. It is not easy to get an irreducible representation of the $ax+b$ group, since it is both noncommutative and noncompact. In 1948, Gelfand and Naimark [4] first proved the following theorem.

Theorem 2.1 (Gelfand-Naimark) *The $ax+b$ group has only two nontrivial irreducible representations, $\check{\pi}^+(a,b): (L^2(0,+\infty) \to L^2(0,+\infty))$ and $\check{\pi}^-(a,b): (L^2(-\infty,0) \to L^2(-\infty,0))$, defined as*

$$[\check{\pi}^+(a,b)f](x) = a^{\frac{1}{2}} e^{2\pi i b x} f(ax), (x > 0),$$
$$[\check{\pi}^-(a,b)f](x) = a^{\frac{1}{2}} e^{2\pi i b x} f(ax), (x < 0).$$

We define a representation of the $ax+b$ group over $L^2(\mathbb{R})$ as

$$\big(\check{\pi}(a,b)f\big)(x) = a^{\frac{1}{2}} e^{2\pi i b x} f(ax), \ a \in \mathbb{R}^+, b \in \mathbb{R}, f \in L(\mathbb{R}).$$

Then $\check{\pi}^{\pm}(a,b)$ and $\check{\pi}(a,b)$ are also written as $\check{\pi}^{\pm}_{ab}$ and $\check{\pi}_{ab}$, respectively.

It is obvious that $L^2(\mathbb{R})$ can be decomposed into the orthogonal direct sum of $L^2(-\infty,0)$ and $L^2(0,+\infty)$, *i.e.*, $L^2(\mathbb{R}) = L^2(-\infty,0) \oplus L^2(0,+\infty)$. By Theorem 2.1, the representation $\check{\pi}$ is just the sum of two irreducible representations $\check{\pi}^+$ and $\check{\pi}^-$ over $L^2(-\infty,0)$ and $L^2(0,+\infty)$, respectively.

Denote by $\mathcal{F}$ the Fourier transformation. We also denote $f^\wedge = \mathcal{F}f$. For (a,b) in the $ax+b$ group, define $\mathcal{F}(\pi)$ by

$$\big((\mathcal{F}(\pi)(a,b))f\big)(x) = \big(\mathcal{F}^{-1}(\pi_{ab}f^\wedge)\big)(x), \ f \in L^2(\mathbb{R}).$$

Then we have

$$\begin{aligned}\big((\mathcal{F}(\pi)(a,b))f\big)(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a^{-\frac{1}{2}} f^\wedge\Big(\frac{y}{a}-\frac{b}{a}\Big)e^{iyx}\,dy\\ &= a^{\frac{1}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^\wedge\Big(\frac{y-b}{a}\Big)e^{i\frac{y-b}{a}(ax)}e^{ibx}\,d\frac{y-b}{a}\\ &= a^{\frac{1}{2}}e^{ibx}f(ax) = (\check{\pi}_{a,b}f)(x).\end{aligned}$$

Thus, the Fourier transformation is just the isomorphism between the two representations π and $\check{\pi}$. Then π has two sub-representations that are equivalent by the Fourier correspondence to $\check{\pi}^+$ and $\check{\pi}^-$, respectively. Let H^+ and H^- be the the images under the Fourier transformation of $L^2(0,+\infty)$ and $L^2(-\infty,0)$, respectively. Let π^+ and π^- be the restrictions of π over H^+ and H^-, respectively.

Theorem 2.2 *For $\pi = \pi_{ab}$, we have $\pi = \pi^+ \oplus \pi^-$, where π^+ and π^- are defined by*

$$\pi^+ f = (\check{\pi}^+(a,b) f^\vee)^\wedge, \ f \in H^+,$$
$$\pi^- f = (\check{\pi}^-(a,b) f^\vee)^\wedge, \ f \in H^-.$$

Then π^+ and π^- are irreducible representations of the $ax + b$ group over H^+ and H^-, respectively.

To obtain our main results we introduce a Schur's lemma in the version of the infinite dimension. Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{A}$ be a family of transformations acting on $\mathcal{H}$ satisfying $\mathcal{H} = \bigcup_{T \in \mathcal{A}} T(\mathcal{H})$. We say that $\mathcal{A}$ acts irreducibly on $\mathcal{H}$ if there does not exist a decomposition of $\mathcal{H}$ such that $\mathcal{H}_1 \oplus \mathcal{H}_2$, where both $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant subspaces of $\mathcal{A}$.

Theorem 2.3 ([8] Dixmier's Lemma) *Let $\mathcal{H}$ be a complex separable Hilbert space. If a family of transformations $\mathcal{A}$ acts irreducibly on $\mathcal{H}$, then any linear transformation* $\mathbf{T}$ *that commutes with every $T \in \mathcal{A}$ is of the form $\mathbf{T} = \lambda \mathcal{I}$, where λ is a complex number and $\mathcal{I}$ is the identity transformation.*

Let σ be a representation of a group G over the Hilbert space $\mathcal{H}$. It is obvious that σ is irreducible if and only if $\{\sigma(g) : g \in G\}$ acts irreducibly on $\mathcal{H}$. Then by the above theorem, the following theorem is evident.

Theorem 2.4 (Schur's Lemma) *Suppose that σ is a irreducible representation of a group G over a complex separable Hilbert space $\mathcal{H}$. Then any linear transformation* $\mathbf{T}$ *that commutes with σ is of the form $\mathbf{T} = \lambda \mathcal{I}$, where λ is a complex number and $\mathcal{I}$ is the identity transformation.*

We note that $\mathbf{T}$ is not required to be a bounded operator in either of the above theorems.

The sequel will be based on the above two theorems.

3 Characterization of the Hilbert Transformation on the Line

We can now state the symmetry properties of the Hilbert transformation $\mathbf{H}$. It is obvious that the operator group $\pi(G)$ is generated by T_a and τ_b. Then by Lemma 1.1 we get that the Hilbert transformation $\mathbf{H}$ is invariant under the actions of $\pi(G)$ over $L^2(\mathbb{R})$. That is the following theorem.

Theorem 3.1 *π commutes with the Hilbert transform* $\mathbf{H}$, i.e.,

$$\pi_{ab}\mathbf{H}(f) = \mathbf{H}\pi_{ab}(f), \quad (a,b) \in ax + b, f \in L^2(\mathbb{R}).$$

Then by Theorems 2.2 and 2.4, the restrictions of π over H^+ and H^- are the scalar operators. Let us now further identify the spaces H^+ and H^-. Denote by H^2_+ and H^2_- the Hardy spaces on the upper and lower half planes, respectively.

Theorem 3.2 *$H^+ = H^2_+$ and $H^- = H^2_-$.*

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Proof It is trivial to check that both H^2_+ and H^2_- are the closed invariant subspaces of $\pi(G)$. By Theorem 2.1 there exist exactly two nontrivial irreducible representations of the $ax+b$ group. Then H^2_+ and H^2_- must be H^+ and H^-.

Given $0 \neq f \in H^+$, by the definition of H^+, its Fourier transform is $f^\wedge$ is in $L^2(0,+\infty)$. We define

$$F(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f^\wedge(t)e^{itz}\,dt = \frac{1}{2\pi}\int_0^{+\infty} f^\wedge(t)e^{itx}e^{-ty}\,dt,\ z = x+iy.$$

Obviously, for $y>0$, $F(z)$ is analytic. By the theory of the Hardy space on the upper half plane, we obtain that $f(x)$ is the non-tangential boundary limit of $F(z)$ when z tends to x from the above. That is, $f(x)$ is the boundary value of $F(z)$; *i.e., f* belongs to H^2_+. So $H^+ = H^2_+$, and then $H^- = H^2_-$. ∎

We note that from the Fourier multiplier representation of the Hilbert transformation we have

$$(\mathbf{H}f)^\wedge(\xi) = -i\operatorname{sgn}(\xi)f^\wedge(\xi),\ \xi \in \mathbb{R}.$$

The following theorem is now obvious.

Theorem 3.3 *H^+ and H^- are respectively the eigen-subspaces associated with eigenvalues $-i$ and i of the Hilbert transformation; that is,*

$$\mathbf{H}|_{H^+} = -i\mathbf{I}|_{H^+}, H|_{H^-} = i\mathbf{I}|_{H^-}.$$

Although the Hardy space theory and the related Fourier multiplier theory imply Theorem 3.3, they do not easily imply the converse of the theorem, which addresses the symmetry of the Hilbert transformation.

Theorem 3.4 *Suppose that $\mathbf{T}$ is a linear operator from $L^2(\mathbb{R})$ to itself, and $\mathbf{T}$ commutes with the natural representation of the group $ax+b$. Then there exist two complex numbers λ, η such that*

$$\mathbf{T} = \lambda\mathbf{I} + \eta\mathbf{H}. \tag{3.1}$$

Moreover, if $\mathbf{T}$ is an anti-symmetric, norm-preserving and real operator, it must be either $\mathbf{H}$ or $-\mathbf{H}$.

Proof By Theorem 2.4 both the restrictions of $\mathbf{T}$ over H^+ and H^- are the scalar operators, since $\mathbf{T}$ commutes with $\pi(G)$. Assume that $\mathbf{T} = k_1\mathbf{I}|_{H^+}$ over H^+ and $\mathbf{T} = k_2\mathbf{I}|_{H^-}$ over H^-, respectively. For $f \in L^2(\mathbb{R})$, there exist $f_1 \in H^+$ and $f_2 \in H^-$ such that $f = f_1 + f_2$. Then we have

$$\begin{aligned}\mathbf{T}f &= \mathbf{T}f_1 + \mathbf{T}f_2 = k_1f_1 + k_2f_2 = \frac{k_1+k_2}{2}(f_1+f_2) + \frac{k_1-k_2}{2i}(if_1 - if_2)\\ &= \frac{k_2+k_1}{2}\mathbf{I}f + \frac{k_2-k_1}{2i}\mathbf{H}f.\end{aligned} \tag{3.2}$$

Let $\lambda = \frac{k_2+k_1}{2}$ and $\eta = \frac{k_2-k_1}{2i}$. Equation (3.2) completes the proof of (3.1).

If **T** is a real operator, its non-real eigenvalues must appear in pairs. This implies that $k_1 = \overline{k_2}$. If **T** is also anti-symmetric, its eigenvalues must be pure complex numbers, *i.e.*, $k_1 = -k_2$. Finally, if **T** preserves the norm, we obtain that $|k_1| = |k_2| = 1$. Therefore, $\lambda = 0$ and $\eta = 1$ or -1. ■

Remark For bounded operators T, Theorem 3.3 is a known result in the literature of Hilbert transformation. Equation (3.2) is precisely [7, (4.82)]. Moreover, linear self-maps of $L^2(R)$ that commute with translations (even just with one translation) are automatically continuous [6]. Since translation operators over $L^2(\mathbb{R})$ do not have critical eigenvalue, by [6, Corollary 3.5], linear self-maps of $L^2(R)$ that commute with translations must be continuous. The relation [7, (4.82)] is proved by the multiplier theory from [14], where the assumption of boundedness of T is essential. In this paper we derive Theorem 3.3 without invoking the results from [6, 7, 14]. We use the Gelfand–Naimark representation of $ax + b$ to deal with the issue. The new methodology enables us to study Hilbert transformations on other types of manifolds with symmetry properties similar to translations and dilations. As an example, in the next section we study the Hilbert transformation on the unit circle in an analogous way.

4 Characterization of the Hilbert Transformation on the Circle

We did not find a group acting on $\mathbb{T}$ suitable for study of the Hilbert transformation over $L^2(\mathbb{T})$. Fortunately, there exists a semigroup on the unit circle that plays a similar role as the $ax + b$ group on the real line. The semigroup on the unit circle makes it possible to carry on a similar, but not identical, procedure as above for the line.

Denote by $\mathfrak{N}$ the set of pairs (n, β), $n \in \mathbb{Z}^+$, $\beta \in \mathbb{R}$. We will also use the notation $\alpha\theta+\beta$, $\alpha \in \mathbb{Q}^+$, $\beta \in \mathbb{R}$. The latter turns out to be a subgroup of the $ax+b$ group. Similar to the case of the $ax + b$ group, we would hope that $\alpha\theta + \beta$ can be considered as an affine transformation on $\mathbb{R}$ such that $(n, \beta)(\theta) = n\theta + \beta \mod (2\pi)$. Let $(n, \beta) \in \mathfrak{N}$. Define its action on $\mathbb{T}$ by $(n, \beta)t = e^{i(n\theta+\beta)}$, where $t = e^{i\theta} \in \mathbb{T}$. Unfortunately, this mechanism does not work for the non-integer α's. That is, $\alpha\theta + \beta$ is not congruent to $\alpha(\theta + 2\pi) + \beta$ modulo 2π if α is not an integer. It means that the action of $\alpha\theta + \beta$ on $\mathbb{T}$ is not well defined.

In general the element of the $\alpha\theta + \beta$ group is not an automorphism of $\mathbb{T}$. However, there still exists a natural action of the $\alpha\theta + \beta$ group over the space $L^2(\mathbb{T})$.

Define the mapping π from the $\alpha\theta + \beta$ group to the the set of bounded endomorphisms of $L_2(\mathbb{T})$ as follows: for $n \in \mathbb{Z}^+$ and $f \in L^2(\mathbb{T})$,

$$\big(\pi(n, \beta)\big) f(t) = \Big(\frac{1}{n}\Big)^{\frac{1}{2}} f(t^n e^{i\beta}), \quad t \in \mathbb{T}, \tag{4.1}$$

and for a general positive rational number α, $\alpha = q/p$, $p, q \in \mathbb{Z}^+$, $p > 1$, and $(p, q) = 1$, let

$$\Big(\pi\Big(\frac{q}{p}, \beta\Big) f\Big)(t) = \Big(\frac{p}{q}\Big)^{\frac{1}{2}} \frac{1}{p}\Big(f(e^{i(\frac{q}{p}\theta+\beta)}) + f(e^{i(\frac{q}{p}\theta+\beta)}\omega_p^1) + \cdots + f(e^{i(\frac{q}{p}\theta+\beta)}\omega_p^{p-1})\Big), \tag{4.2}$$

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where $\tau = e^{i\theta} \in \mathbb{T}$ and $\omega_p = e^{i\frac{2\pi}{p}}$.

Notice that $e^{i\frac{q2k\pi}{p}}$ is one of the p-th roots of the unity. Then the p-tuple

$$\left(e^{i\frac{q2k\pi}{p}}, e^{i\frac{q2k\pi}{p}} w_p^1, \dots, e^{i\frac{q2k\pi}{p}} w_p^{p-1} \right)$$

is just a rearrangement of $(1, \omega_p^1, \dots, \omega_p^{p-1})$. Thus, (4.2) is independent of the choice of θ. We note that (4.1) and (4.2) do not give a representation of the $\alpha\theta + \beta$ group, but we still can use it to derive the characterization of the Hilbert transformation over $L^2(\mathbb{T})$.

In the rest of the section we denote by $\pi_{\alpha\beta}$ the image of (α, β) under π. We say that π commutes with a linear operator $\mathbf{T}$ if $\pi_{\alpha\beta}\mathbf{T} = \mathbf{T}\pi_{\alpha\beta}$ for all (α, β) in the $\alpha\theta + \beta$ group.

Theorem 4.1 *π commutes with the Hilbert transformation on the unit circle.*

Proof Let $p, q \in \mathbb{Z}$ and $(p, q) = 1$. First we check that $\mathbf{H}_0$ is invariant under π. In fact, for $f \in L^2(\mathbb{T})$ we have

$$\begin{aligned}\Big(\mathbf{H}_0\Big(\pi\Big(\frac{q}{p}, \beta\Big) f\Big)\Big)(t) &= \frac{1}{2\pi}\int_0^{2\pi} \big(f(e^{i(\frac{q}{p}s+\beta)}) + \dots + f(e^{i(\frac{q}{p}s+\beta)}\omega_p^{p-1})\big) ds \\ &= \Big(\frac{p}{q}\Big)^{\frac{1}{2}} \frac{1}{2\pi}\int_0^{2\pi} f(e^{is}) ds = \Big(\pi\Big(\frac{q}{p}, \beta\Big)(\mathbf{H}_0 f)\Big)(t).\end{aligned}$$

Then by (1.3), it is sufficient to prove that π commutes with the singular Cauchy transformation $\mathbf{C}$. Notice that $\pi_{\frac{q}{p}\beta}$ is the composition of π_{q0} with $\pi_{\frac{1}{p}\beta}$, *i.e.*, $\pi_{\frac{q}{p}\beta} = \pi_{q0}\pi_{\frac{1}{p}\beta}$. We only prove the cases of $\alpha = n$ or $\frac{1}{n}$ for positive integers n.

Let $\alpha = n$ be a positive integer and $\beta \in \mathbb{R}$. It is obvious that we can assume $n > 1$. Then for $f \in L^2(\mathbb{T})$ with the Hölder type continuity, we have

$$\begin{aligned}(\mathbf{C}\pi_{n\beta})f(s) &= \frac{1}{2\pi i}\text{p.v.}\int_{\mathbb{T}} \frac{(\frac{1}{n})^{\frac{1}{2}} f((n,\beta)t)}{t-s} dt \\ &= \frac{1}{2\pi i}\text{p.v.}\int_{\mathbb{T}} \frac{(\frac{1}{n})^{\frac{1}{2}} f(t^n e^{i\beta})}{t-s} dt \\ &= \Big(\frac{1}{n}\Big)^{\frac{1}{2}} \frac{1}{2\pi}\text{p.v.}\int_0^{2\pi} \frac{f(e^{i(n\theta+\beta)})}{e^{i\theta}-s} e^{i\theta} d\theta.\end{aligned} \tag{4.3}$$

Let $\phi = n\theta$. Then by (4.3), we get

$$\begin{aligned}(\mathbf{C}\pi_{n\beta})f(s) &= \Big(\frac{1}{n}\Big)^{\frac{1}{2}} \frac{1}{2\pi}\text{p.v.}\int_0^{2n\pi} \frac{f(e^{i(\phi+\beta)})}{e^{i\frac{1}{n}\phi}-s} e^{i\frac{1}{n}\phi}\frac{d\phi}{n} \\ &= \Big(\frac{1}{n}\Big)^{\frac{1}{2}} \frac{1}{2\pi}\text{p.v.}\int_0^{2n\pi} \frac{f(e^{i(\phi+\beta)})}{e^{i\phi}-s^n} e^{i\frac{1}{n}\phi}\big(e^{i\frac{n-1}{n}\phi} + \dots + s^{n-1}\big)\frac{d\phi}{n} \\ &= \frac{1}{n}\Big(\frac{1}{n}\Big)^{\frac{1}{2}} \frac{1}{2\pi}\Big(\text{p.v.}\int_0^{2n\pi} \frac{f(e^{i(\phi+\beta)})e^{i\phi}}{e^{i\phi}-s^n} d\phi + \cdots \\ &\qquad + \text{p.v.}\int_0^{2n\pi} \frac{f(e^{i(\phi+\beta)})s^{n-1}}{e^{i\phi}-s^n} d\phi\Big)\end{aligned} \tag{4.4}$$

We divide the right-hand side of (4.4) into n parts. Then

$$
\begin{aligned}
(4.5)\qquad & \frac{1}{n}\Big(\frac{1}{n}\Big)^{\frac{1}{2}}\frac{1}{2\pi}\text{p.v.}\int_0^{2n\pi}\frac{f(e^{i(\phi+\beta)})e^{i\phi}}{e^{i\phi}-s^n}d\phi \\
&= \Big(\frac{1}{n}\Big)^{\frac{1}{2}}\frac{1}{2\pi}\text{p.v.}\int_0^{2\pi}\frac{f(e^{i(\phi+\beta)})e^{i\phi}}{e^{i\phi}-s^n}d\theta \\
&= \Big(\frac{1}{n}\Big)^{\frac{1}{2}}\frac{1}{2\pi i}\text{p.v.}\int_0^{2\pi}\frac{f(e^{i\psi})de^{i\psi}}{e^{i\psi}-s^ne^{i\beta}} \\
&= \Big(\frac{1}{n}\Big)^{\frac{1}{2}}\frac{1}{2\pi i}\text{p.v.}\int_{\mathbb{T}}\frac{f(\tau)d\tau}{\tau-s^ne^{i\beta}} = \pi_{n\beta}(\mathbf{C}f)(s).
\end{aligned}
$$

Similarly, for $1 < k \leq n-1$, we have

$$
\begin{aligned}
(4.6)\qquad & \text{p.v.}\int_0^{2n\pi}\frac{f(e^{i(\phi+\beta)})e^{i\frac{n-1-k}{n}\theta}s^k}{e^{i\phi}-e^{i\beta}}d\theta \\
&= \sum_{l=1}^{n}\text{p.v.}\int_{2(l-1)}^{2l\pi}\frac{f(e^{i(\phi+\beta)})e^{i\frac{n-1-k}{n}\theta}s^k}{e^{i\phi}-s^n}d\theta \\
&= \text{p.v.}\int_0^{2\pi}\frac{f(e^{i(\phi+\beta)})s^k}{e^{i\phi}-e^{i\beta}}\Big(\sum_{l=1}^{n}e^{i\frac{n-1-k}{n}(\theta+2l\pi)}\Big)d\theta \\
&= \text{p.v.}\int_0^{2\pi}\frac{f(e^{i(\phi+\beta)})s^k}{e^{i\phi}-e^{i\beta}}e^{i\frac{n-1-k}{n}\theta}\Big(\sum_{l=1}^{n}e^{i\frac{n-1-k}{n}(2l\pi)}\Big)d\theta.
\end{aligned}
$$

Notice that $n > k > 1$ and $e^{i\frac{n-1-k}{n}(2l\pi)}$, $l = 1, 2, \dots, n$, are just all of the n-roots of unity, which means that

$$
(4.7)\qquad \sum_{l=1}^{n}e^{i\frac{n-1-k}{n}(2l\pi)} = 0.
$$

Now by (4.4), (4.5), (4.6), and (4.7), the proposition is proved for $\alpha \in \mathbb{Z}^+$.

For $\alpha = \frac{1}{n}$, we have

(4.8)

$$
\begin{aligned}
(\mathbf{C}\pi_{\alpha\beta})f(s) &= \frac{1}{2\pi i}\text{p.v.}\int_{\mathbb{T}}\frac{(\pi_{\alpha\beta}f)(t)}{t-s}dt \\
&= \frac{1}{2\pi i}\text{p.v.}\int_0^{2\pi}n^{\frac{1}{2}}\frac{1}{n}\Big(f(e^{i\frac{1}{n}\theta}e^{i\beta})+\cdots+f(e^{i\frac{1}{n}\theta}\omega_n^{n-1}e^{i\beta})\Big)\frac{1}{e^{i\theta}-s}de^{i\theta}.
\end{aligned}
$$

For $1 \leq k \leq n$, let $\phi = \frac{1}{n}\theta$ in the k-th term of the sum of the above integrand. Then we have

$$
\begin{aligned}
(4.9)\quad & \frac{1}{2\pi i}\text{p.v.}\int_0^{2\pi}n^{\frac{1}{2}}\frac{1}{n}(f(e^{i\frac{1}{n}\theta}\omega_n^{k-1}e^{i\beta})\frac{de^{i\theta}}{e^{i\theta}-s} \\
&= \frac{1}{2\pi}n^{\frac{1}{2}}\text{p.v.}\int_0^{\frac{2}{n}\pi}\frac{f(e^{i\phi}\omega_n^{k-1}e^{i\beta})e^{in\phi}}{e^{in\phi}-s}d\phi \\
&= \frac{1}{2\pi}n^{\frac{1}{2}}\text{p.v.}\int_0^{\frac{2}{n}\pi}\frac{f(e^{i\phi}\omega_n^{k-1}e^{i\beta})e^{i(n-1)\phi}e^{i\phi}}{(e^{i\phi}-e^{i\frac{1}{n}\psi})(e^{i\phi}-e^{i\frac{1}{n}\psi}w_n)\cdots(e^{i\phi}-e^{i\frac{1}{n}\psi}w_n^{n-1})}d\phi,
\end{aligned}
$$

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where $s = e^{i\psi} \in \mathbb{T}$ and $w_n = \frac{1}{\omega_n} = e^{-i\frac{2\pi}{n}}$. Notice that w_n is still one of the n-th roots of the unity. Then we obtain that $(1-w_n)\cdots(1-w_n^{n-1}) = \lim_{z\to 1}\frac{z^n-1}{z-1} = n$ and

$$
\begin{aligned}
\frac{e^{i(n-1)\phi}}{(e^{i\phi}-e^{i\frac{1}{n}\psi})(e^{i\phi}-e^{i\frac{1}{n}\psi}w_n)\cdots(e^{i\phi}-e^{i\frac{1}{n}\psi}w_n^{n-1})}
&= \frac{1}{(1-w_n)\cdots(1-w_n^{n-1})}\Big(\frac{1}{e^{i\phi}-e^{i\frac{1}{n}\psi}}+\cdots+\frac{1}{e^{i\phi}-e^{i\frac{1}{n}\psi}w_n^{n-1}}\Big)\\
&= \frac{1}{n}\Big(\frac{1}{e^{i\phi}-e^{i\frac{1}{n}\psi}}+\cdots+\frac{1}{e^{i\phi}-e^{i\frac{1}{n}\psi}w_n^{n-1}}\Big).
\end{aligned}
\tag{4.10}
$$

Denote by

$$
A_{kj} = \frac{1}{2\pi}\frac{n^{\frac{1}{2}}}{n}\text{p.v.}\int_0^{\frac{2}{n}\pi}\frac{f(e^{i\phi}\omega_n^{k-1}e^{i\beta})e^{i\phi}}{e^{i\phi}-e^{i\frac{1}{n}\psi}w_n^{j-1}}d\phi,\ \ j,k=1,\dots,n-1.
\tag{4.11}
$$

From (4.8)–(4.11), we have

$$
(\mathbf{C}\pi_{\alpha\beta})f(s) = \sum_{k=1}^{n}\sum_{j=1}^{n}A_{kj} = \sum_{m=0}^{n-1}\Big(\sum_{\substack{1\le k,j\le n\\ j-k\equiv m(\operatorname{mod} n)}}A_{k,j}\Big).
\tag{4.12}
$$

It is obvious that $\sum_{1\le k,j\le n,\,j-k\equiv 0(\operatorname{mod}\,n)}A_{k,j} = \sum_{j=1}^{n}A_{j,j}$. But by (4.11), we obtain

$$
\begin{aligned}
\sum_{j=1}^{n}A_{j,j} &= \frac{1}{2\pi}\frac{n^{\frac{1}{2}}}{n}\text{p.v.}\Big(\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}e^{i\beta})e^{i\phi}}{e^{i\phi}-e^{i\frac{1}{n}\psi}}d\phi+\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n e^{i\beta})e^{i\phi}}{e^{i\phi}-e^{i\frac{1}{n}\psi}w_n}d\phi\\
&\qquad +\cdots\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^{n-1}e^{i\beta})e^{i\phi}}{e^{i\phi}-e^{i\frac{1}{n}\psi}w_n^{n-1}}d\phi\Big)\\
&= \frac{1}{2\pi i}\frac{n^{\frac{1}{2}}}{n}\text{p.v.}\Big(\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}e^{i\beta})}{e^{i\phi}-e^{i\frac{1}{n}\psi}}de^{i\phi}+\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}e^{i\frac{2\pi}{n}}e^{i\beta})e^{i\frac{2\pi}{n}}}{e^{i\phi}e^{i\frac{2\pi}{n}}-e^{i\frac{1}{n}\psi}}de^{i\phi}\\
&\qquad +\cdots\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}e^{i\frac{2(n-1)\pi}{n}}e^{i\beta})e^{i\frac{2(n-1)\pi}{n}}}{e^{i\phi}e^{i\frac{2(n-1)\pi}{n}}-e^{i\frac{1}{n}\psi}}de^{i\phi}\Big)\\
&= \frac{1}{2\pi i}\frac{n^{\frac{1}{2}}}{n}\text{p.v.}\Big(\int_0^{\frac{2\pi i}{n}}+\cdots+\int_{\frac{2(n-1)\pi}{n}}^{2\pi}\Big)\frac{f(e^{i\phi}e^{i\beta})de^{i\phi}}{e^{i\phi}-e^{i\frac{1}{n}\psi}}\\
&= \frac{1}{2\pi i}\frac{n^{\frac{1}{2}}}{n}\text{p.v.}\int_0^{2\pi}\frac{f(e^{i\phi}e^{i\beta})de^{i\phi}}{e^{i\phi}-e^{i\frac{1}{n}\psi}} = \frac{1}{2\pi i}\frac{n^{\frac{1}{2}}}{n}\text{p.v.}\int_0^{2\pi}\frac{f(e^{i\phi})de^{i\phi}}{e^{i\phi}-e^{i\frac{1}{n}\psi}e^{i\beta}}
\end{aligned}
$$

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Similarly, for $m \geq 1$, recalling that $\omega_n^k, \omega_n^{k+1}, \dots, \omega_n^{k+n-1}$ still run through the n-th roots of the unity, we get

(4.13)
$$\begin{aligned}
&\sum_{\substack{1\le k,j\le n\\ j-k\equiv m(\mathrm{mod}\ n)}} A_{k,j}\\
&= \frac{1}{2\pi}\frac{n^{\frac12}}{n}\mathrm{p.v.}\Big(\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^m e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}}de^{i\phi}+\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^{m+1}e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}w_n}de^{i\phi}\\
&\quad+\cdots+\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^{n-1}e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}w_n^{n-m-1}}de^{i\phi}+\int_0^{\frac{2\pi i}{n}}\frac{f(e^{i\phi}e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}w_n^{n-m}}de^{i\phi}\\
&\quad+\cdots+\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^{m-1}e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}w_n^{n-1}}de^{i\phi}\Big)\\
&= \frac{1}{2\pi i}\frac{n^{\frac12}}{n}\mathrm{p.v.}\Big(\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^m e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}}de^{i\phi}+\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^{m+1}e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}w_n}de^{i\phi}\\
&\quad+\cdots+\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^{n-1}e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}w_n^{n-m-1}}de^{i\phi}+\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^{n}e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}w_n^{n-m}}de^{i\phi}\\
&\quad+\cdots+\int_0^{\frac{2\pi}{n}}\frac{f(e^{i\phi}\omega_n^{n+m-1}e^{i\beta})}{e^{i\phi}-e^{i\frac1n\psi}w_n^{n-1}}de^{i\phi}\Big)\\
&= \frac{1}{2\pi i}\frac{n^{\frac12}}{n}\mathrm{p.v.}\int_{\mathbb{T}}\frac{f(\tau)}{\tau-e^{i\frac1n\psi}\omega_n^m e^{i\beta}}d\tau,\qquad m=1,2,\dots,n-1.
\end{aligned}$$

From (4.12) and (4.13), we have

(4.14)
$$(\mathbf{C}\pi_{\frac1n,\beta})f(s) = n^{\frac12}\frac1n\Big(\sum_{m=0}^{n-1}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(\tau)}{e^{i\frac1n\psi}\omega_n^m e^{i\beta}-\tau}d\tau\Big) = \pi_{\frac1n,\beta}(\mathbf{C}f)s.$$

Equation (4.14) is valid for $f \in L^2(\mathbb{T})$, because $\mathbf{C}$ is a bounded operator over $L^2(\mathbb{T})$, and the class of functions of the Hölder continuity is dense in $L^2(\mathbb{T})$. This completes the proof of the theorem. ∎

5 Decomposition of the Concerned Operators and Spaces on the Circle

In the rest of the paper we discuss the irreducibility of π. Then we characterize the circular Hilbert transformation $\widetilde{\mathbf{H}}$.

In the previous section, we pointed out that $G = \{(\alpha,\beta), \alpha \text{ or } \frac{1}{\alpha} \in \mathbb{Z}^+, ; b \in \mathbb{R}\}$ is not a group. But we can still prove that the family of $\pi_{(\alpha,\beta)}$, $(\alpha,\beta) \in G$, acts irreducibly on some subspaces of $L^2(\mathbb{T})$.

Given a family of functions $\mathfrak{M}$ in $L^2(\mathbb{T})$. Denote by

(5.1)
$$Z(\mathfrak{M}) = \bigcap_{f\in\mathfrak{M}}\{n \in \mathbb{Z}, f_n = 0\},$$

where f_n are the Fourier coefficients of f such that $f(t) = \sum_{n=-\infty}^{\infty} f_n t^n$. By (5.1), $Z(\mathfrak{M}) = \varnothing$ means that for every $n \in \mathbb{Z}$ there exists at least one $f^{(n)} \in \mathfrak{M}$ such that $f_n^{(n)} \neq 0$.

Theorem 5.1 *Assume that $Z(\mathfrak{M})$ is empty. Suppose that ϕ belongs to $L^2(\mathbb{T})$ and satisfies $f * \phi \equiv 0$, for all $f \in \mathfrak{M}$. Then $\phi \equiv 0$.*

Proof Assume that

$$\phi(t) = \sum_{n=-\infty}^{\infty} \phi_n t^n = \sum_{n=-\infty}^{\infty} \phi_n e^{in\theta}, \quad t = e^{i\theta},$$
$$f(t) = \sum_{n=-\infty}^{\infty} f_n t^n = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}, \quad t = e^{i\theta}$$

for $f \in \mathfrak{M}$.

Then we obtain

$$(f * \phi)(e^{i\theta}) = \mathcal{F}^{-1}(\widehat{f\phi}) = \sum_{n=-\infty}^{\infty} f_n \phi_n e^{in\theta},$$

which gives $f_n \phi_n = 0$, for all $f \in \mathfrak{M}$, especially $f_n^{(n)} \phi_n = 0$. Thus, $\phi_n = 0, \forall n \in \mathbb{Z}$, which means that $\phi \equiv 0$. ■

Denote by $H^+(\mathbb{T})$ the Hardy space on the unit disc, and by $H^-(\mathbb{T})$ the Hardy space on the complement of the unit disc in the whole complex plane. Denote by H^0 the subspace of constant functions, and by $\widetilde{H}^+(\mathbb{T})$ its orthogonal complement in $H^+(\mathbb{T})$. Then we obtain that $L^2(\mathbb{T}) = \widetilde{H}^+ \oplus H^0 \oplus H^-$. It is obvious that H^0 is invariant to π. It is easy to check that there do not exist $f \in \widetilde{H}^+(\mathbb{T})$ and $\pi_{(\alpha,\beta)}$ such that $\pi_{(\alpha,\beta)} f$ is a constant function. Thus both H_0 and $\widetilde{H}^+(\mathbb{T})$ are invariant spaces of $\pi_{(\alpha,\beta)}$, $(\alpha,\beta) \in G$.

The following theorem plays the same role on the circle as the Gelfand–Naimark representation for the $ax + b$ group on the real axis.

Theorem 5.2 *The family of $\pi_{(\alpha,\beta)}$, $(\alpha,\beta) \in G$, acts irreducibly over $\widetilde{H}^+(\mathbb{T})$ and $H^-(\mathbb{T})$, respectively.*

Proof We only prove the first part. The proof of the other part is similar.

If the family $\pi_{(\alpha,\beta)}$ were not irreducible over $\widetilde{H}^+(\mathbb{T})$, there would exist two proper subspaces $H_1, H_2 \subset \widetilde{H}^+(\mathbb{T})$ such that $\widetilde{H}^+(\mathbb{T}) = H_1 \oplus H_2$, and both H_1 and H_2 are the invariant spaces of $\{\pi_{(\alpha,\beta)}, (\alpha,\beta) \in G\}$.

Claim: $Z(H_1) \cap \mathbb{Z}^+$ cannot be empty.

Otherwise, assume that $Z(H_1) \cap \mathbb{Z}^+ = \varnothing$. Since $Z(H^-(\mathbb{T})) = \mathbb{Z}^+ \cup \{0\}$, $Z(H^0) = \mathbb{Z} \setminus \{0\}$, we obtain that $Z(H^-(\mathbb{T})) \cap Z(H^0) = \mathbb{Z}^+$ and

$$Z\big(H_1 \cup H^0 \cup H^-(\mathbb{T})\big) = Z(H_1) \cap Z(H^0) \cap Z\big(H^-(\mathbb{T})\big) = \varnothing.$$

Let $h \in H_2$. Since all H^0, H_1, and H_2 are invariant spaces of the transforms $\pi_{(1,\theta)}$ for $\theta \in \mathbb{R}$, we have

$$\langle \pi_{(1,\theta)} f, h \rangle = 0, \quad \forall f \in H_1 \cup H^0 \cup H^-(\mathbb{T}), \quad \theta \in \mathbb{R}.$$

Recalling that $\pi_{(1,\theta)} f(e^{is}) = f(e^{i(s-\theta)})$, we obtain that

$$f * \overline{h}(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i(s-\theta)})\overline{h(e^{is})}ds = 0, \quad \forall f \in H_1 \cup H^0 \cup H^-(\mathbb{T}).$$

Now by Theorem 5.1, we obtain $h \equiv 0$, which contradicts with H_2 being proper. So $Z(H_1) \cap \mathbb{Z}^+$ cannot be empty.

According to the just proved Claim there exists at least one positive integer m such that $m \in Z(H_1)$. Since $g(t) = t^m \in \widetilde{H}^+(\mathbb{T})$, there exist $f'(t) \in H_1$ and $f''(t) \in H_2$ such that $t^m = f'(t) + f''(t)$. Recall that $m \in Z(H_1)$ means $f'_m = 0$. Then we have

$$0 = \langle f', f''\rangle = \langle f', t^m - f'\rangle = \langle f', t^m\rangle - \langle f'.f'\rangle = f'_m - \|f'\|^2 = -\|f'\|^2,$$

which gives $f'(t) = 0$. So $t^m = f''(t) \in H_2$. Now by (4.2), we have $t = \pi_{(\frac{1}{m},0)} t^m \in H_2$. As consequence, $t^n = \pi_{(n,0)} t$ are all in H_2 for $n \in \mathbb{Z}^+$. Hence,

$$\widetilde{H}^+(\mathbb{T}) = \overline{\operatorname{span}(t, t^2, \dots,)} \subseteq H_2 \subseteq \widetilde{H}^+(\mathbb{T}),$$

which contradicts with the assumption that H_2 is proper. ■

By the Plemelj formula [10], it is easy to prove the following theorem.

Theorem 5.3 *Let $\widetilde{\mathbf{H}}$ be the Hilbert transformation on $L^2(\mathbb{T})$. Then*

$$\widetilde{\mathbf{H}}|_{\widetilde{H}^+} = -i\mathbf{I}|_{\widetilde{H}^+}, \quad \mathbf{C}|_{H^0} = 0, \quad \mathbf{C}|_{H^-} = i\mathbf{I}|_{H^-}.$$

We end this paper with the inverse of Theorem 5.3.

Theorem 5.4 *Let T be a bounded operator from $L^2(\mathbb{T})$ to itself. Assume that $\widetilde{\mathbf{T}}$ commutes with $\pi_{(\alpha,\beta)}$, $(\alpha,\beta) \in G$. Then there exist three complex numbers λ, η, ω such that*

$$\widetilde{\mathbf{T}}|_{\widetilde{H}^+} = \lambda\mathbf{I}|_{\widetilde{H}^+}, \quad \widetilde{\mathbf{T}}|_{H^0} = \eta\mathbf{I}_{H_0}, \quad \widetilde{\mathbf{T}}|_{H^-} = \omega\mathbf{I}|_{H^-}.$$

Proof It is obvious that the family of $\pi_{(\alpha,\beta)}$, $(\alpha,\beta) \in G)$ acts irreducibly on H^0. By Theorems 5.2 and 2.3, T must be scalar operators on, respectively, H^+, H^0, and H^-. ■

Remark (i) Theorem 5.4, as far as we are aware, is a new result. It is also not easy to prove by the methods from [7, 14], because of the lack of the general dilation on the unit circle.

(ii) It is also possible to characterize the circular Hilbert transformation by the symmetry of the generalized Möbius group containing both the rotations and Möbius transforms. Define $\tau_\theta z = e^{i\theta} z$, $\theta \in \mathbb{R}$ and $\varphi_a(z) = \frac{z-a}{1-az}$, $a \in (0,1)$. Then the Möbius group, $\mathcal{M}$, is generated by τ_θ and φ_a. There exists a natural representation of $\mathcal{M}$ over $L^2(\mathbb{T})$ as follows: for $\varphi \in \mathcal{M}$ with the expression $\varphi(z) = e^{i\theta}\frac{z-a}{1-az}$, we define

$$\big((\pi_{\mathcal{M}}\varphi)(f)\big)(t) = \frac{\sqrt{1-a^2}}{1-at} f\big(\varphi^{-1}(t)\big), \quad f \in L^2(\mathbb{T}).$$

It is easy to check that $\pi_{\mathcal{M}}$ is a unitary representation and commutes with $\mathbf{C}$. We can also prove that its restrictions to H^+ and H^- are respectively irreducible. Then we obtain Theorem 5.3.

But the Fourier series expansions of the Möbius transforms are complicated. So we cannot obtain the precise structures of **C** in the phase space as discussed in the real line case.

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