A NOTE ON CONTINUED FRACTIONS

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1. Introduction. Any real number y leads to a continued fraction of the type

$$(1) \qquad y \sim b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \dots,$$

where a_i, b_i are integers which satisfy the inequalities

$$(2) \qquad 1 \leqslant a_i \leqslant b_i \qquad (i = 1, 2, \dots,),$$

by means of the algorithm

$$(3) \qquad \begin{aligned} y = y_0 &= [y_0] + \frac{a_1}{y_1} = b_0 + \frac{a_1}{y_1}, y_1 \geqslant a_1, \\ y_1 &= [y_1] + \frac{a_2}{y_2} = b_1 + \frac{a_2}{y_2}, y_2 \geqslant a_2, \end{aligned}$$

the a's being assigned positive integers. The process terminates for rational y; the last denominator b_k satisfying $b_k \geqslant a_k + 1$. For irrational y, the process does not terminate. For a preassigned set of numerators $a_i \geqslant 1$, this C.F. development of y is unique; its value being y.

Bankier and Leighton **(1)** call such fractions (1), which satisfy (2), proper continued fractions. Among other questions, they studied the problem of expanding quadratic surds in periodic continued fractions. They state that "it is well-known that not only does every periodic regular continued fraction represent a quadratic irrational, but the regular continued fraction expansion of a quadratic irrational is periodic. Such a result would not be expected to hold in general for proper continued fraction representations of quadratic irrationals" **(1**, p. 662**)**.

In point of fact, as I prove in this note, *every* quadratic irrational admits of infinitely many periodic proper continued fraction representations. Indeed, only one term is needed in the periodic part and at most three terms in the non-periodic part:

$$(4) \qquad b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \left(\frac{a_3}{b_3}\right)_\infty .$$

Moreover, in infinitely many representations $a_3 = b_3 = 2c$ or, again, with $a_3 = c$, $b_3 = 2c$. For the class of quadratic irrationals whose regular continued fraction expansion has a period with an *odd* number of terms, it is possible to have (in infinitely many ways) $a_3 = 1$, $b_3 = 2c$.

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It may be noted that Bankier and Leighton obtained periodic proper continued fraction expansions in infinitely many ways for the class of quadratic irrationals whose regular continued fraction expansion (i) is purely periodic and (ii) has an odd number of terms in the period. My results are stated explicitly in the following three theorems.

THEOREM 1. *Any real quadratic irrational can be expressed as a proper periodic continued fraction of the form* (4), *in which the period consists of one term only and the non-periodic part* (*which may be empty*) *contains at most three terms.*

The expansion is possible, in infinitely many ways, with $b_3 = 2c$, *an even integer.*

THEOREM 2. *For a given quadratic irrational, there are infinitely many of the expansions in Theorem* 1 *satisfying*

$$a_3 = b_3 = 2c.$$

This is also true with $a_3 = c,\ b_3 = 2c$.

THEOREM 3. *Let* θ *be any quadratic irrational and write*

$$\theta = b_0 + \zeta_0, \qquad b_0 = [\theta], \qquad 0 < \zeta_0 < 1,$$

where

$$\zeta_0 = \frac{P_0 \pm \sqrt{N_0}}{R_0}.$$

Let s_0 *be the least positive integer satisfying*

$$R_0 | s_0(P_0^2 - N_0).$$

Then, if the regular continued fraction for $s_0\sqrt{N_0}$ *has a period with an odd number of elements, infinitely many of the expansions* (4) *for* θ, *have* $a_3 = 1$.

For the proofs we require five lemmas and these are stated and proved in §§ 3, 4.

2. A conjecture. Some time ago I made a conjecture (which I cannot prove) about these representations, when the integers a_i are assigned in a special way. Let $y > 1$ be an irrational number. Assign $a_1 = b_0$. Determine b_1 and assign $a_2 = b_1$. Determine b_2 and assign $a_3 = b_2$, and so on. In this way, we determine a *unique* expansion

$$y = b_0 + \frac{b_0}{b_1 +}\, \frac{b_1}{b_2 +}\, \frac{b_2}{b_3 + \dots}, \tag{5}$$

in which the integers b_i satisfy the inequalities

$$1 \leqslant b_0 \leqslant b_1 \leqslant b_2 \leqslant \dots . \tag{6}$$

Plainly, if $b_i = b$ from some point on, y must be a quadratic surd.

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CONJECTURE. *If $y > 1$ is a quadratic irrational, the expansion* (5) *is ultimately periodic, that is, from some point on, the b_i have a fixed value.*

Examples:

$$\frac{2\sqrt{2}+1}{3} = 1 + \frac{1}{3}+\frac{3}{4}+\left(\frac{4}{4}\right)_\infty,$$

$$\frac{b}{a}(a^2+1)^{\frac{1}{2}} = b + \frac{b}{2a^2}+\frac{2a^2}{4a}+\left(\frac{4a^2}{4a^2}\right)_\infty, \qquad (b = 1, 2, \dots, 2a^2),$$

$$\frac{b}{a}(a^2+2a)^{\frac{1}{2}} = b + \frac{b}{a}+\frac{a}{2a}+\left(\frac{2a}{2a}\right)_\infty, \qquad (b = 1, 2, \dots, a),$$

$$\frac{b}{3a}(9a^2+3)^{\frac{1}{2}} = b + \frac{b}{6a^2}+\frac{6a^2}{12a^2}+\left(\frac{12a^2}{12a^2}\right)_\infty, \qquad (b = 1, 2, \dots, 6a^2).$$

3. In what follows, N is a positive non-square integer,

(7) $$c = [N^{\frac{1}{2}}], \qquad N = c^2 + a, \qquad 1 \leqslant a \leqslant 2c.$$

We express each of $N^{\frac{1}{2}}$, $-N^{\frac{1}{2}}$ as proper continued fractions with a single term in the periodic part and then apply the results to the general quadratic irrational.

LEMMA 1. *Let*

$$\xi = \frac{a}{2c}+\frac{a}{2c}+\frac{a}{2c}+\dots.$$

Then

$$\xi = -c + N^{\frac{1}{2}}.$$

Proof. Since $\xi > 0$, $\xi(2c+\xi) = a$, we have

$$\xi = -c + (c^2+a)^{\frac{1}{2}}.$$

LEMMA 2. *Let*

$$\eta = \frac{2c+1-a}{2c+1}+\frac{a}{2c}+\frac{a}{2c}+\dots.$$

Then

$$\eta = c + 1 - N^{\frac{1}{2}}.$$

Proof. Note that

$$\eta = \frac{2c+1-a}{2c+1+\xi} = \frac{(c+1)^2 - N}{c+1+N^{\frac{1}{2}}} = c + 1 - N^{\frac{1}{2}}.$$

4. We next consider quadratic irrationals of the following three types:

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$$\text{I.}\quad \zeta = \frac{P+N^{\frac{1}{2}}}{R}\,; 0 < \zeta < 1, R \geqslant 1, -N^{\frac{1}{2}} < P < N^{\frac{1}{2}}, R|(N-P^2),$$

$$\text{II.}\quad \zeta = \frac{P+N^{\frac{1}{2}}}{R}\,; 0 < \zeta < 1, R \geqslant 1, P > N^{\frac{1}{2}}, R|(P^2-N),$$

$$\text{III.}\quad \zeta = \frac{P-N^{\frac{1}{2}}}{R}\,; 0 < \zeta < 1, R \geqslant 1, R|(P^2-N).$$

LEMMA 3. *For surds of type I, define integers a_1, b_1, by the conditions*

$$a_1R = N - P^2, \qquad b_1 = [N^{\frac{1}{2}} - P] = c - P.$$

Then

$$\zeta = \frac{a_1}{b_1+}\,\frac{a}{2c+}\,\frac{a}{2c+}\cdots = \frac{a_1}{b_1+\xi}.$$

Proof. Note that

$$\frac{P+N^{\frac{1}{2}}}{R} = \frac{a_1}{-P+N^{\frac{1}{2}}},$$

where

$$0 < \frac{a_1}{N^{\frac{1}{2}}-P} < 1, N^{\frac{1}{2}} - P > 0.$$

Hence $1 \leqslant a_1 < N^{\frac{1}{2}} - P$ or $a_1 \leqslant c - P = b_1$. Thus

$$\frac{a_1}{b_1+\xi} = \frac{a_1}{b_1 - c + N^{\frac{1}{2}}} = \frac{a_1}{N^{\frac{1}{2}} - P} = \zeta,$$

and the result follows from Lemma 1.

LEMMA 4. *For surds of type* II *define integers a_2, b_2 by the conditions*

$$a_2R = P^2 - N, \qquad b_2 = [P - N^{\frac{1}{2}}] = P - c - 1.$$

Then

$$\zeta = \frac{a_2}{b_2+\eta} = \frac{a_2}{b_2+}\,\frac{2c+1-a}{2c+1}\,\frac{a}{+2c+}\,\frac{a}{2c+}\cdots.$$

Proof. Observe that

$$\zeta = \frac{P+N^{\frac{1}{2}}}{R} = \frac{a_2}{P-N^{\frac{1}{2}}},$$

where

$$0 < \frac{a_2}{P-N^{\frac{1}{2}}} < 1, P - N^{\frac{1}{2}} > 0.$$

Hence

$$1 \leqslant a_2 < P - N^{\frac{1}{2}}, \qquad 1 \leqslant a_2 \leqslant P - c - 1 = b_2.$$

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Thus our continued fraction for ζ is proper and its value is clearly

$$\frac{a_2}{b_2+\eta} = \frac{a_2}{b_2+c+1-N^{\frac{1}{2}}} = \frac{a_2}{P-N^{\frac{1}{2}}}.$$

LEMMA 5. *For surds of type* III *define an integer* a_2 *by the condition* $a_2R = P^2 - N$. *Then*

$$\zeta = \frac{a_2}{(P+c)+\xi} = \frac{a_2}{P+c+}\,\frac{a}{2c+}\,\frac{a}{2c+}\dots.$$

Proof. Since

$$\zeta = \frac{a_2}{P+N^{\frac{1}{2}}} = \frac{a_2}{(P+c)+\xi},$$

where $0 < \zeta < 1$, $1 \leqslant a_2 < P + N^{\frac{1}{2}}$, $1 \leqslant a_2 < P + c$, the result follows from Lemma 1.

5. Proofs of theorems 1, 2, and 3. Let the quadratic irrational θ, say, be expressed in the form $\theta = b_0 + \zeta_0$, where $b_0 = [\theta]$, $0 < \zeta_0 < 1$ and

$$\zeta_0 = \frac{P_0 \pm N_0^{\frac{1}{2}}}{R_0}.$$

Since we can also express ζ_0 in the form $(P \pm N^{\frac{1}{2}})/R$, where

$$R = sR_0, \qquad P = sP_0, \qquad N = s^2N_0 \qquad (s \geqslant 1)$$

it belongs to one of the types I, II, or III, provided that the integer s is chosen so that $R|(P^2 - N)$. It is sufficient, then, if $R_0|s(P_0^2 - N)$, and plainly s can be so chosen in infinitely many ways ($s = tR_0/g$, where g is the greatest common divisor of R_0 and $P_0^2 - N_0$ and t is any integer $\geqslant 1$). Thus Theorem 1 follows immediately from Lemmas 3, 4, and 5.

Observe that $s_0 = R_0/(R_0, P_0^2 - N_0)$ is the least positive integer such that $R_0|s_0(P_0^2 - N_0)$. Then we may write

$$\zeta_0 = \frac{P_0 \pm N_0^{\frac{1}{2}}}{R_0} = \frac{P \pm N^{\frac{1}{2}}}{R}$$

where $N = t^2s_0^2N_0$ $(t \geqslant 1)$. Recall that

$$c^2 + a = N = t^2s_0^2N_0$$

where

$$c = [N^{\frac{1}{2}}], \qquad 1 \leqslant a \leqslant 2c,$$

by (7). To obtain $a = 2c$, it is enough to solve the Pellian equation

$$(c+1)^2 - t^2s_0^2N_0 = 1$$

for c and t. Since $s_0^2N_0$ is not a square, there are infinitely many such pairs. Similarly, for $a = c$ we require the solutions of the Pellian equation

$$(2c+1)^2 - 4t^2s_0^2N_0 = 1,$$

which also gives infinitely many pairs. This proves Theorem 2.

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For Theorem 3 we require $a = 1$ and then it is enough to solve the Pellian equation

$$c^2 - s_0^2 N_0 t^2 = -1.$$

Now, it is well known that if the regular continued fraction for $s_0 N_0^{\frac{1}{2}}$ has a period with an *odd* number of elements, then this has infinitely many solutions in c and t. This proves Theorem 3.

6. Examples for the well-known quadratic irrational $\frac{1}{2}(1 + \sqrt{5})$ are listed.

$$\text{(i)} \quad \tfrac{1}{2}(1+\sqrt{5}) = 1 + \frac{2t}{c+t+}\left(\frac{2c}{2c}\right)_\infty,$$

where c and t satisfy $(c+1)^2 - 5t^2 = 1$, so that

$$(c+1) + t\sqrt{5} = (2+\sqrt{5})^{2n}, \quad (n = 1, 2, \ldots,),$$

for example,

$$(c, t) = (8, 4), (160, 72), \ldots$$

$$\text{(ii)} \quad \tfrac{1}{2}(1+\sqrt{5}) = 1 + \frac{2t}{c+t+}\left(\frac{c}{2c}\right)_\infty,$$

where c and t satisfy $(2c+1)^2 - 5(2t)^2 = 1$, so that

$$(2c+1) + 2t\sqrt{5} = (2+\sqrt{5})^{2n}, \qquad (n = 1, 2, \ldots,)$$

for example,

$$(c, t) = (4, 2), (80, 30), \ldots,$$

$$\text{(iii)} \quad \tfrac{1}{2}(1+\sqrt{5}) = 1 + \frac{2t}{c+t+}\left(\frac{1}{2c}\right)_\infty,$$

where c and t satisfy $c^2 - 5t^2 = -1$ and so

$$c + t\sqrt{5} = (2+\sqrt{5})^{2n-1} \quad (n = 1, 2, \ldots,)$$

for example,

$$(c, t) = (2, 1), (38, 17) \ldots.$$

References

1. J. D. Bankier and W. Leighton, *Numerical continued fractions*, Amer. J. Math., *64* (1942), 653–668.

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