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CONGRUENCE COHERENT DISTRIBUTIVE DOUBLE p-ALGEBRAS

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For distributive double p-algebras, a close connection is established between being congruence coherent and congruence regular. Every congruence regular distributive double p-algebra is congruence coherent. Even though every congruence coherent distributive double p-algebra that has either a non-empty core or finite range is congruence regular, an example is given that is congruence coherent but not congruence regular.

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1. Introduction

An algebra A is said to be *congruence coherent* providing, for each congruence Θ and subalgebra B, whenever B contains some Θ congruence class, then B is a union of the Θ congruence classes contained in it. A variety is *coherent* if every one of its members is coherent. This notion was introduced in Geiger [7] who showed that coherent varieties are definable by a Mal'cev condition and are congruence regular, but not conversely. For an arbitrary algebra A, if $A \times A$ is congruence coherent, than, as shown by Duda [6], A is congruence regular and, as subsequently shown by Clark and Fleisher [4], congruence permutable.

Some specific varieties of algebras have also been investigated. In [3], congruence coherent de Morgan algebras were characterized by showing that a de Morgan algebra is congruence coherent if and only if it is a Boolean algebra or isomorphic to one of three other finite de Morgan algebras. Furthermore, it was shown that a distributive p-algebra or a Heyting algebra is congruence coherent if and only if it is Boolean.

In this paper, we shall see that the situation for distributive double p-algebras is considerably more complicated. Although, ultimately, we are unable to characterize the congruence coherent distributive double p-algebras, we shall show that they are closely linked with those that are congruence regular.

It will be shown that every congruence regular distributive double p-algebra is congruence coherent (Theorem 3.3). Although (as will be seen in Section 5) not every congruence coherent distributive double p-algebra is congruence regular, it is if it has either a non-empty core or finite range (Theorem 3.4). In Section 4, a necessary condition is given for a distributive double p-algebra to be congruence coherent: for any two distinct prime ideals which are neither minimal nor maximal, there exists a minimal or a maximal prime ideal which is comparable with precisely one of them (Theorem 4.1). In Section 5, we give an example of a congruence coherent distributive double p-algebra which is not congruence regular. In fact, with [4] in mind, we give an example that is not congruence permutable which, since the algebra in question is a distributive double p-algebra, automatically yields that it is not congruence regular. Obviously, our example has an empty core and infinite range.

The variety of distributive double p-algebra is dually equivalent to a category of ordered topological spaces. On balance, the results given here are better presented in the dual category. As such we will work strictly in the dual category rather than crossing backwards and forwards between the two. All necessary background will be given in Section 2.

2. Preliminaries

An algebra $(L; \vee, \wedge, {}^*, {}^+, 0, 1)$ of type $(2,2,1,1,0,0)$ is a *distributive double p-algebra* if $(L; \vee, \wedge, 0, 1)$ is a distributive $(0,1)$-lattice, * is the unary operation of pseudocomplementation (that is, for a and $b \in L$, $a \wedge b = 0$ if and only if $b \leqq a^*$), and $+$ is dual pseudocomplementation (that is, for a and $b \in L$, $a \vee b = 1$ if and only if $b \geqq a^+$).

For a distributive double p-algebra L, $a \in L$ is *dense* providing $a^* = 0$ and *dually dense* providing $a^+ = 1$. If $D(L)$ and $D^d(L)$ denote the sets of dense and dually dense elements of L, respectively, then the *core* of L is given by $K(L) = D(L) \cap D^d(L)$. Standard computation (see, for example, [2] or [8]) shows that, for $a \in L$, $a^{+*} \leqq a$. Thus, for $n \geqq 0$, $a^{n(+*)} \geqq a^{n+1(+*)}$ where $a^{0(+*)}$ denotes a and $a^{k+1(+*)} = a^{k(+*)+*}$ for any $k \geqq 0$. If $a^{n(+*)} = a^{n+1(+*)}$ for some $n \geqq 0$, then a is said to have *finite range*; L has *finite range* in the event that every element does.

An algebra A is *congruence regular* providing $\Theta = \Psi$ whenever congruences Θ and Ψ on A have a congruence class in common and *congruence permutable* providing, for any congruences Θ and Ψ, $\Theta \circ \Psi = \Psi \circ \Theta$ where $\circ$ denotes relational product. As shown by Varlet [11], a distributive double p-algebra L is congruence regular if and only if there is no 3-element chain in its poset of prime ideals and, as shown in [1], it is congruence permutable if and only if there is no 4-element chain in its poset of prime ideals. Of particular interest for distributive double p-algebra is the *determination congruence* Φ where, for a and $b \in L$, Φ is defined by $a \equiv b(\Phi)$ if and only if $a^* = b^*$ and $a^+ = b^+$. As shown by Varlet [11], L is congruence regular if and only if $\Phi = \omega$. (In fact, the congruence regular distributive double p-algebras form a variety.)

In [9], Priestley showed that the variety of distributive $(0,1)$-lattices is dually equivalent to a suitable category of topological spaces. Since every distributive double p-algebra is a $(0,1)$-distributive lattice, there is a corresponding subcategory to which the variety of distributive double p-algebras is dually equivalent. A brief outline sufficient for our basic requirements follows (for further details see, for example, the text [5] or the survey paper [10]).

Let $(P; \leqq)$ be a poset. For $X \subseteq P$, let $(X] = \{y \in P: y \leqq x \text{ for some } x \in X\}$ and $[X) = \{y \in P: y \geqq x \text{ for some } x \in X\}$ denote, respectively, the *order-ideal* and *order-filter* generated by X (denoted $(x]$ and $[x)$ in the event $X = \{x\}$). Sometimes we may write

$(X]_P$ or $[X)_P$ to emphasise that the order-ideal or order-filter is in the poset P. Then X is *convex* providing $X=(X]\cap[X)$. Let $Min(P)$ and $Max(P)$ denote, respectively, the set of minimal elements and the set of maximal elements of P and $Ext(P)=Min(P)\cup Max(P)$. For $x\in P$, let $Min(x)=Min(P)\cap(x]$ and $Max(x)=Max(P)\cap[x)$. For a poset $(Q;\leqq)$, a mapping $\phi:P\to Q$ is *order-preserving* providing $\phi(x)\leqq\phi(y)$ whenever $x\leqq y$.

Given a topology τ defined on a poset P, $(P;\leqq,\tau)$ is called an *ordered topological space*. It is *totally order-disconnected* if, for any x and $y\in P$, $x\ngeqq y$ implies that there exists a clopen order-ideal $U\subseteq P$ such that $x\in U$ and $y\notin U$. If $(P;\leqq,\tau)$ is a compact totally order-disconnected space, then it is called a *Priestley space*. For a Priestley space $(Q;\leqq,\rho)$, a map $\phi:P\to Q$ is a *Priestley map* if it is continuous and order-preserving. Let $\mathbf{P}$ denote the category of all Priestley spaces together with all Priestley maps. As shown in **[9]**, the variety $\mathbf{D}$ of distributive $(0,1)$-lattices is dually equivalent to $\mathbf{P}$. Under this dual equivalence, for a distributive $(0,1)$-lattice L associated with a Priestley space P, L is isomorphic to the distributive $(0,1)$-lattice $(D(P);\cup,\cap,\emptyset,P)$ where $D(P)$ denotes the set of all clopen order-ideals of P, $(P;\leqq)$ is order-isomorphic to the post $(S(L);\subseteq)$ where $S(L)$ denotes the poset of all prime ideals of L ordered by inclusion, and τ has a sub-basis the family of sets $(\{I\in S(L):a\notin I\}:a\in L)$ together with their complements. Further, for $K\in\mathbf{D}$ associated with $Q\in\mathbf{P}$, if $f:K\to L$ is associated with the continuous order-preserving map $\phi:P\to Q$, then $f(a)=b$ if and only if $\phi^{-1}(A)=B$ where A and B are clopen order-ideals that represent a and b. Moreover, f is one-to-one or onto if and only if ϕ is onto or a one-to-one order-isomorphism, respectively. Since congruences correspond to onto $(0,1)$-lattice homomorphisms, it follows that the congruence of L are in one-to-one correspondence with the closed subsets of P. Thus, for a congruence Θ on L associated with a closed subset Y of P, $a\equiv b(\Theta)$ if and only if $A\cap Y=B\cap Y$. We shall adopt the convention that $u, v, w, x,\ldots$ denote points of P and that $a, b, c, d,\ldots$ denote elements of L. Whenever we wish to implicitly refer to the correspondence between elements of an algebra and clopen order-ideals of the associated Priestley space, we will let elements $a, b, c,\ldots$ correspond to clopen order-ideals $A, B, C,\ldots$.

A Priestley space $(P;\leqq,\tau)$ is a *dp-space* if, for every clopen order-ideal or clopen order-filter $U\subseteq P$, both $[U)$ and $(U]$ are clopen. For dp-spaces P and Q, a continuous order-preserving map $\phi:P\to Q$ is a *dp-map* if, for every $x\in P$, $\phi(Min(x))=Min(\phi(x))$ and $\phi(Max(x))=Max(\phi(x))$. We shall need the fact that the variety of distributive double p-algebras $\mathbf{D}_{dp}$ is dually equivalent to the category $\mathbf{P}_{dp}$ whose objects are all dp-spaces and morphisms are all dp-maps. For a distributive double p-algebra L associated with a dp-space, P, if $a\in L$, then $A^*=P\backslash[A)$ and $A^+=(P\backslash A]$. A closed subset Y of P is a *c-subset* providing $Ext(Y)\subseteq Y$. As can be seen from above, it follows that congruences on L are in one-to-one correspondence with c-subsets of P. In particular, the determination congruence Φ corresponds to the closed subset $Ext(P)$.

3. The relationship to congruence regularity

Let K be a subalgebra of a regular distributive double p-algebra L and Θ a non-trivial congruence of L. Let $(P;\leqq,\tau)$ and $(Q;\leqq,\rho)$ be the dp-spaces corresponding to L and K, respectively. Since the identity is an embedding of K into L, there is an associated onto dp-map $\phi: P \to Q$. Further, there is a c-subset Y of P such that, for a and $b \in L$, $a \equiv b(\Theta)$ if and only if $A \cap Y = B \cap Y$. Let $a \in L$ be such that $[a]\Theta \subseteq K$.

Lemma 3.1. *If, for distinct x and $y \in P$, $\phi(x) = \phi(y)$, then $x \in Y$.*

Proof. Since L is congruence regular, P has no 3-element chain (see Section 2) and, hence, Y is an order-component of P. Suppose, contrary to hypothesis, that $x \notin Y$.

Consider first when $x \| y$. Suppose $y \in A$. Then, $z \ngeq x$ for every $z \in Y$. Because P is a dp-space and Y is a closed subset, it follows that there is a clopen order-ideal U such that $x \notin U$ and $Y \subseteq U$. By hypothesis, there exists a clopen order-ideal V such that $x \notin V$ and $y \in V$. Thus, for the clopen order-filter $C = P \backslash (U \cup V)$, $x \in C$, $y \notin C$, and $Y \cap C = \emptyset$. Set $B = A \backslash C$. Consequently, for the clopen order-ideal B, $x \notin B$ and $y \in B$ which, since $\phi(x) = \phi(y)$, implies that $B \neq \phi^{-1}(D)$ for any clopen order-ideal $D \subseteq Q$ and, hence, $b \notin K$. However, by choice, $A \cap Y = B \cap Y$ and, so, $a \equiv b(\Theta)$, a contradiction. Alternatively, suppose $y \notin A$. Then there exist a clopen order-filter U such that $x \notin U$ and $Y \subseteq U$ and a clopen order filter V such that $x \notin V$ and $y \in V$. Thus, for $C = P \backslash (U \cup V)$, $x \in C$, $y \notin C$, and $Y \cap C = \emptyset$. Set $B = A \cup C$. It follows that, for the clopen order-ideal B, $x \in B$, $y \notin B$, and $A \cap Y = B \cap Y$. Thus, once more, $b \notin K$ and $a \equiv b(\Theta)$, a contradiction.

Without loss of generality, it remains to consider $x < y$. In which case there exist a clopen order-filter U such that $x \notin U$, $y \in U$, and $U \cap Y = \emptyset$ and a clopen order-ideal V such that $x \in V$, $y \notin V$, and $Y \cap V = \emptyset$. For $y \in A$, set $B = A \cap (P \backslash U)$ and, for $y \notin A$, set $B = A \cup V$. Either way, for the clopen order-ideal B, $b \notin K$ and $a \equiv b(\Theta)$, a contradiction. □

Lemma 3.2. *If $a \equiv b(\Theta)$, then $y = x$ whenever $\phi(y) = \phi(x)$ and $x \in (A \backslash B) \cup (B \backslash A)$.*

Proof. Suppose $x \in (A \backslash B) \cup (B \backslash A)$ and $\phi(x) = \phi(y)$. If $x \neq y$, then, by Lemma 3.1, $x \in Y$ which is impossible since $A \cap Y = B \cap Y$. □

Theorem 3.3. *If a distributive double p-algebra is congruence regular, then it is congruence coherent.*

Proof. It is to be shown that $b \in K$ whenever $a \equiv b(\Theta)$.

Since $a \in K$, there exists a clopen order-ideal $C \subseteq Q$ such that $A = \phi^{-1}(C)$. It is to be shown that if $B \subseteq P$ is a clopen order-ideal such that $A \cap Y = B \cap Y$, then $B = \phi^{-1}(D)$ for some clopen order-ideal $D \subseteq Q$.

Set

$$D = (C \cup \phi(B \backslash A)) \backslash \phi(A \backslash B).$$

We establish first that $B = \phi^{-1}(D)$. Consider $x \in B$. Then $x \in B \backslash A$ or $\phi^{-1}(C)$ depending on whether or not $x \in A$. Consequently, $\phi(x) \in C \cup \phi(B \backslash A)$. If $\phi(x) \in \phi(A \backslash B)$, then $\phi(x) = \phi(y)$ for some $y \in A \backslash B$. Since $x \in B$, $x \neq y$ which violates Lemma 3.2. Thus,

$\phi(x) \in (C \cup \phi(B\backslash A))\backslash\phi(A\backslash B) = D$ and $B \subseteqq \phi^{-1}(D)$. Alternatively, consider $x \in \phi^{-1}(D)$. Since $\phi(x) \in C \cup \phi(B\backslash A)$, $\phi(x) \in C$ or $\phi(x) = \phi(y)$ for some $y \in B\backslash A$. Thus, either $x \in A$ or, by Lemma 3.2, $x \in B\backslash A$. If $x \in A$, then, since $\phi(x) \notin \phi(A\backslash B)$, $x \notin A\backslash B$ and, hence, $x \in B$. Either way, $x \in B$. Thus, $\phi^{-1}(D) \subseteqq B$ and, as required, $B = \phi^{-1}(D)$.

It remains to show that D is a clopen order-ideal. Since ϕ is onto, the image of any clopen set under ϕ is clopen and, hence, D is clopen. Suppose then that $y \prec x$ for some $x \in D$. It is to be shown that $y \in D$. If $y \in \phi(A\backslash B)$, then $y = \phi(u)$ for some $u \in A\backslash B$. Since $x \succ y$ and K is congruence regular, $x \in Max(Q)$. It follows, as ϕ is a dp-map, that $x = \phi(v)$ for some $v > u$. Thus, $v \in B$ and, hence, $u \in B$, a contradiction. Thus $y \notin \phi(A\backslash B)$ and it remains to show that either $y \in C$ or $y \in \phi(B\backslash A)$. Since $x \in D$, either $x \in C$ or $x \in \phi(B\backslash A)$. If $x \in C$, then $y \in C$. If $x \in \phi(B\backslash A)$, then $x = \phi(v)$ for some $v \in B\backslash A$. Since $x \succ y$ and K is congruence regular, $y \in Min(Q)$. Thus, $y = \phi(u)$ for some $u < v$. If $u \notin A$, then $y \in \phi(B\backslash A)$. If $u \in A$, then $y \in C$. Either way, $y \in D$. □

As we shall see in Section 5, a congruence coherent distributive double p-algebra is not necessarily congruence regular. However, as Theorem 3.4 shows, the converse does hold when some natural extra hypothesis is added.

For the remainder of this section, L is a congruence coherent distributive double p-algebra with dp-space $(P; \leqq, \tau)$.

Theorem 3.4. *If a distributive double p-algebra is congruence coherent and has either non-empty core or finite range, then it is congruence regular.*

Proof. Suppose L is not congruence regular. Then there exists a chain $u < w < v$ in P where u and $v \in Ext(P)$. Since both $Min(P)$ and $Max(P)$ are closed subsets of P, $Ext_s(P) = Min(P) \cap Max(P)$ is too.

For any $c \in L$, $c \in K(L)$ if and only if $Min(P) \subseteqq C$ and $Max(P) \cap C = \emptyset$. Consequently, if $K(L) \neq \emptyset$, then $Ext_s(P) = \emptyset$. If $Ext_s(P) = \emptyset$, then set $C = \emptyset$ and observe that $C^{n(+*)} = \emptyset$ for every $n \geqq 0$.

Alternatively, $K(L) = \emptyset$ and, since $u \notin Ext_s(P)$ by hypothesis, there exists a clopen order-ideal C such that $Ext_s(P) \subseteqq C$ but $u \notin C$. Since $C^{n+1(+*)} \subseteqq C^{n(+*)}$, $u \notin C^{n(+*)}$ for any $n \geqq 0$. Further, if $x \in Ext_s(P) \cap C^{n(+*)}$, then $x \notin C^{n(+*)+}$ and, hence, $x \in C^{n(+*)+*}$. In other words, $Ext_s(P) \subseteqq C^{n(+*)}$ for any $n \geqq 0$. By choice, $Ext_s(P) \neq \emptyset$ and, by hypothesis, L has finite range. Thus, $C^{n(+*)} = C^{n+1(+*)}$ for some $n \geqq 0$.

Either way, choose n such that $C^{n(+*)} = C^{n+1(+*)}$ and observe that $C^{n(+*)}$ is an order-filter. Were this not the case then, for some x and $y \in P$, $x > y$, $x \notin C^{n(+*)}$, and $y \in C^{n(+*)}$. Since $x \notin C^{n(+*)}$, $x \in C^{n(+*)+}$. Thus, $y \in C^{n(+*)+}$ and, so, $y \notin C^{n(+*)+*}$ which is absurd.

Since $P\backslash C^{n(+*)}$ is clopen and $Min(P) \cap (P\backslash C^{n(+*)})$ and $Max(P) \cap (P\backslash C^{n(+*)})$ are disjoint closed sets, there exists a clopen order-ideal A such that $w \in A$, $(Min(P) \cap (P\backslash C^{n(+*)})) \subseteqq A$, $Max(P) \cap A = \emptyset$, and $C^{n(+*)} \cap A = \emptyset$.

Choose $w' \in C^{n(+*)}$ (if possible) and consider the dp-space $(Q; \leqq_Q, \tau_Q)$ where $Q = \{u, w', v\}$, $\leqq_Q$ is the order relation on Q inherited from P, and τ_Q is the subspace topology on Q (note that, since Q is finite, τ_Q is discrete).

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Consider $\phi: P \to Q$ given by

$$\phi(x) = \begin{cases} v, & \text{for } x \in P \backslash (A \cup C^{n(+*)}); \\ w', & \text{for } x \in C^{n(+*)}; \\ u, & \text{otherwise.} \end{cases}$$

It is not hard to see that ϕ is an onto *dp*-map. Let K denote the corresponding subalgebra of L.

Consider the determination congruence Φ on L. Since $[1]\Phi = \{1\} \subseteq K$, K is the union of the Φ congruence classes contained in it. For $A = \phi^{-1}(\{u\})$, $a \in K$ and, hence $[a]\Phi \subseteq K$. Arguing as before, there exists a clopen order-ideal B such that $w \notin B$ but, as for A, $(Min(P) \cap (P \backslash C^{n(+*)})) \subseteq B$, $Max(P) \cap B = \emptyset$, and $C^{n(+*)} \cap B = \emptyset$. In particular, $A \cap Ext(P) = B \cap Ext(P)$ and, so $a \equiv b(\Phi)$. However, $B \neq \phi^{-1}(D)$ for any order-ideal $D \subseteq Q$ and, hence, $b \notin K$ a contradiction. □

The following corollary is immediate, since if L has a chain K as a subalgebra with $\geqq 4$ elements, then it is neither congruence regular (see Section 2) nor does it have an empty core (since $|K(L)| \geqq |K| - 2$).

Corollary 3.5. *If a distributive double p-algebra is congruence coherent, then no chain having $\geqq 4$ elements can be a subalgebra.*

4. A necessary condition for congruence coherence

The next theorem is better stated in terms of the duality.

Theorem 4.1. *For a distributive double p-algebra L associated with a dp-space P, if L is congruence coherent, then, for distinct u and $v \in P \backslash Ext(P)$, $Ext(u) \neq Ext(v)$.*

Proof. Suppose, contrary to hypothesis, that there exist distinct u and $v \in P \backslash Ext(P)$ such that $Ext(u) = Ext(v)$.

Let X denote the smallest closed convex set containing u and v. Then either $X = \{u, v\}$ or, say, $u < v$ and $X = [u, v]$. Let $Q = (P \backslash X) \cup \{w\}$ for some new element $w \notin P$.

Define an order relation $\prec$ on Q by, for x and $y \in Q$, $x \prec y$ if and only if one of the following holds:

$x < y$,	for x and $y \in P \backslash X$;
$x < r$ and $s < y$,	for x and $y \in P \backslash X$ and some r and $s \in X$;
$x < r$ and $y = w$,	for $x \notin X$ and some $r \in X$;
$x = w$ and $s < y$,	for $y \notin X$ and some $s \in X$.

To see that $(Q; \preceq)$ really is a poset, it must be shown that, for every x, y and $z \in Q$, $x \prec x$ is false and that $x \prec z$ whenever $x \prec y$ and $y \prec z$. For $x \in Q$, the only possibility for $x \prec x$ to hold is that $x < r$ and $s < x$ for some r and $s \in X$ with $x \notin X$. Since X is convex, this is impossible. Suppose then that, for x, y and $z \in Q$, $x \prec y$ and $y \prec z$. Consider first when x, y and $z \in P \backslash X$. If either $x<y$ or $y<z$, say $x<y$, then either $y<z$ or, for some r and $s \in X$, $y<r$ and $s<z$. If $y<z$ then $x<z$ and, otherwise, $x<r$. Either way, $x \prec z$. If neither $x<y$ nor $y<z$, then, for some p, q, r and $s \in X$, $x<r$, $s<y$, $y<p$ and $q<z$ and, so, $y \in X$ which is absurd. Suppose now that at least one of x, y or z is w. If $y=w$, then, for some r and $s \in X$, $x<r$ and $s<z$ which implies that $x \prec z$. Finally, it remains to consider when x or z is w, say $x=w$. Thus, $s<y$ for some $s \in X$. Either $y<z$, or, for some r and $t \in X$, $y<r$ and $t<z$, or, for some $r \in X$, $y<r$ and $z=w$. If $y<z$, then $s<z$ and, so, $x \prec z$. Otherwise, $y \in X$ which is impossible.

Let ρ denote the quotient topology on Q obtained from the map $\phi: P \to Q$ where

$$\phi(x)=\begin{cases} x, & \text{for } x \notin X; \\ w, & \text{otherwise.} \end{cases}$$

Certainly, $(Q; \precsim, \rho)$ is a compact ordered topological space. Suppose that, for x and $y \in Q$, $x \not\succsim y$. If x or $y=w$, say $x=w$, then $y \not\leqq r$ for any $r \in X$. Hence, there exists a clopen order-ideal $B \subseteqq P$ such that $x \in B$, $y \notin B$, and $X \subseteqq B$. Let $A=(B \backslash X) \cup \{w\} \subseteqq Q$. Since $\phi^{-1}(A)=B$, A is a clopen subset of Q. To see that A is an order-ideal, suppose $p \succ q$ for some $p \in A$. If $q \neq w$, then either $p>q$ or $q<r$ for some $r \in X$. Either way, $q \in B \backslash X$. If neither x nor $y=w$, then $x \not\geqq y$ and either $x \not\geqq r$ for every $r \in X$ or $y \not\leqq s$ for every $s \in X$, say $x \not\geqq r$ for every $r \in X$. Then there exists a clopen order-ideal $A \subseteqq P$ such that $x \in A$, $y \notin A$, and $A \cap X=\emptyset$. Since $A \subseteqq Q$ and $\phi^{-1}(A)=A$, A is a clopen subset of Q. Suppose $p \succ q$ for some $p \in A$. Either $p>q$ and $q \in A$ or $p>r$ for some $r \in X$ which is impossible. Hence, $(Q; \precsim, \rho)$ is a Priestley space.

To see that $(Q; \precsim, \rho)$ is a dp-space, let $A \subseteqq Q$ be a clopen order-ideal. It must be shown that $[A)$ and $(P \backslash A]$ are clopen. Since Q is a Priestley space and $w \notin Min(Q)$, there exists a clopen order-ideal $B \subseteqq Q$ such that $w \notin B$ and $Min(B)=Min(A)$. Hence, $[B)=[A)$ and, so, without loss of generality, we may assume that $w \notin A$. Then, by hypothesis, $A=\phi^{-1}(A)$ is a clopen subset of P. Suppose, for x and $y \in P$, $x \in A$ and $x>y$. Since $x \not\succsim w$, $x \not\geqq r$ for any $r \in X$ and, so, $y \notin X$. Thus, $x \succ y$ and $y \in A$. In other words, A is an order-ideal of P and, so, $[A)_P$ is a clopen subset of P. We must show that $\phi^{-1}([A)_Q)=[A)_P$. Suppose $x \in \phi^{-1}([A)_Q)$. Then $\phi(x) \in [A)_Q$ and, so, $\phi(x) \succsim y$ for some $y \in A \cap Min(Q)$. Then $x \geqq y$ or $x \geqq r$ and $y \leqq s$ for some r and $s \in X$. Since $y \in Min(r)= Min(s)$, in either case $x \geqq y$. Hence, $x \in [A)_P$ and $\phi^{-1}([A)_Q) \subseteqq [A)_P$. Conversely, suppose $x \in [A)_P$. Then $x \geqq y$ for some $y \in A \cap Min(P)$. Either way, $\phi(x) \succsim y \in A \cap Min(Q)$. Thus, $[A)_P \subseteqq \phi^{-1}([A)_Q)$ and, as required, $\phi^{-1}([A)_Q)=[A)_P$. An analogous argument shows $(P \backslash A]$ is clopen.

To see that ϕ is a dp-map, it must be shown that $Min(\phi(x))=\phi(Min(x))$ and $Max(\phi(x))=\phi(Max(x))$. Suppose $y \in Min(\phi(x))$. Thus, $y \precsim \phi(x)$. If $x \not\geqq y$, then $\phi(x) \succsim w$, $x \geqq r$ and $y \leqq s$ for some r and $s \in X$, and, since $y \in Min(r)=Min(s)$, $x \geqq y$. Either way, $x \geqq y$ and, so, $y \in \phi(Min(x))$. Conversely, if $y \in \phi(Min(x))$, then $y \leqq x$ and, so, $y \precsim \phi(x)$. Since $y \in Min(Q)$, $y \in Min(\phi(x))$ and, as required, $Min(\phi(x))=\phi(Min(x))$. An analogous argument shows that $Max\ (\phi(x))=\phi(Max(x))$.

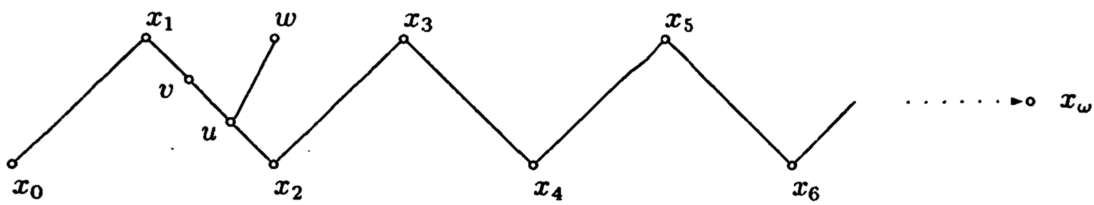

FIGURE 1

To summarize, $(Q;\ \leqq, \rho)$ is a dp-space and $\phi: P \to Q$ is an onto dp-map. Thus Q corresponds to a subalgebra K of L. If Φ is the determination congruence on L, then $[1]\Phi = \{1\} \subseteq K$. Hence, to complete the proof of Theorem 4.1, it is sufficient to show that K is not a union of Φ congruence classes.

Choose a clopen order-ideal $C \subseteq Q$ such that $w \in C$ and $C \cap Max(Q) = \emptyset$. For $A = \phi^{-1}(C)$, $A \cap Max(P) = \emptyset$. Say, without loss of generality, $u \not\geqq v$. Then there exists a clopen order-ideal $D \subseteq P$ such that $u \in D$, $v \notin D$, and $Min(P) \subseteq D$. Thus $B = A \cap D$ is a clopen order-ideal such that $B \cap Ext(P) = A \cap Ext(P)$ and, so, $a \equiv b(\Phi)$. However, $B \neq \phi^{-1}(E)$ for any clopen order-ideal $E \subseteq Q$ and, so, $b \notin K$. Since $a \in K$ by choice, we conclude L *is not congruence coherent.* ☐

5. An example

This section is devoted to giving an example of a congruence coherent distributive double p-algebra which is not congruence regular. In fact, the example is not congruence permutable (and, *de facto*, not congruence regular).

Let $(P; \tau)$ be the one point compactification of the set $\{u, v, w\} \cup \{x_i : i < \omega\}$ by the point x_ω. Define an order relation $\leqq$ on P to be the transitive closure of

$$\{(u, w), (x_2, u), (u, v), (v, x_1)\} \cup \{(x_{2i}, x_{2i+1}),\ (x_{2(i+1)}, x_{2i+1}) : 0 \leqq i < \omega\}.$$

Clearly, $(P;\ \leqq, \tau)$ (diagrammed in Figure 1) is a dp-space.

If L denotes the associated distributive double p-algebra, then, since $(P; \leqq)$ has a 4-element chain, L is not congruence permutable. We will show that L is congruence coherent. Notice that, as required by Theorem 3.4, L has an empty core but does not have finite range. Note also that, since $x_2 < u < v < x_1$ is a 4-element chain in $(P; \leqq)$, as required by Theorem 4.1, $Ext(u) \neq Ext(v)$.

Let K be a subalgebra of L with dp-space $(Q;\ \leqq, \rho)$. Further, let $\phi: P \to Q$ be the onto dp-map associated with the identity mapping from K to L.

Consider when K is one of the two trivial subalgebras of L. Certainly, if $K = L$, then it is the union of all the Θ congruence classes of L for any congruence Θ on L. Suppose alternatively that $K = \{0, 1\}$. Then, since L has more than 3 elements, if K contains a Θ congruence class it must be the case that either $[0]\Theta = \{0\}$ or $[1]\Theta = \{1\}$. However, $[0]\Theta = \{0\}$ if and only if $\Theta \subseteq \Phi$ if and only if $[1]\Theta = \{1\}$. In other words, in this case too, if K contains a Θ congruence class, then it is a union of Θ congruence classes.

Hence, in order to show that L is congruence coherent, it is sufficient to establish the following:

Lemma 5.1. *The only subalgebras of L are trivial.*

Proof. Observe that K is trivial if and only if either ϕ is an order-isomorphism or $|Q|=1$. If ϕ is one-to-one, then, since $P\backslash Ext(P)=\{u,v\}$ and $u<v$, that ϕ is an order-isomorphism is an immediate consequence of the fact that it is a dp-map. Thus, it is sufficient to show that $|Q|=1$ whenever ϕ is not one-to-one. With no further ado, suppose ϕ is not one-to-one and consider the various possibilities:

Were $\phi(x_{2i})=\phi(x_{2j+1})$ for any i and $j<\omega$, then, since ϕ is a dp-map, $\phi(x_{2i})\in Ext_s(P)=Min(P)\cap Max(P)$. Thus, $\phi(x)=\phi(y)$ for all x and $y\in P\backslash\{x_\omega\}$ which, as ϕ is a continuous map, yields $|Q|=1$. Henceforth, we assume that, for any i and $j<\omega$, $\phi(x_{2i})\neq\phi(x_{2j+1})$.

Suppose $\phi(x_0)=\phi(x_2)$. Since ϕ is a dp-map, it follows that $|Max(\phi(x_0))|=1$ and $|Min(\phi(x_1))|=1$. Assume that $\phi(x_{2i})=\phi(x_0)$ and $\phi(x_{2i+1})=\phi(x_1)$ for some $i<\omega$. Then, from $\phi(x_{2i+1})=\phi(x_1)$, it follows that $\phi(x_{2(i+1)})=\phi(x_0)$ and, so, $\phi(x_{2(i+1)+1})=\phi(x_1)$. In other words, $\phi(x_{2i})=\phi(x_0)$ and $\phi(x_{2i+1})=\phi(x_1)$ for every $i<\omega$. Since ϕ is continuous, $\phi(x_\omega)=\phi(x_0)=\phi(x_1)$, contrary to assumption. Henceforth, suppose $\phi(x_0)\neq\phi(x_2)$.

Suppose $\phi(w)=\phi(x_1)$. Since $|Min(w)|=1$, $|Min(\phi(x_1))|=1$ and, so, $\phi(x_0)=\phi(x_2)$ which contradicts our hypothesis. Henceforth, assume $\phi(w)\neq\phi(x_1)$.

Suppose $\phi(x_3)=\phi(x_1)$. We claim that, for $i\geqq 0$, $\phi(x_{3+6i})=\phi(x_1)$ and $\phi(x_{4+6i})=\phi(x_0)$. Since $\phi(x_3)=\phi(x_1)$, $\phi(x_4)=\phi(x_0)$ and the claim holds for $i=0$. Suppose that it is valid for some $i<\omega$. Thus, $\phi(x_{5+6i})=\phi(x_1)$ and, so, $\phi(x_{6+6i})=\phi(x_2)$. Then $\phi(x_{7+6i})=\phi(w)$ and, hence, $\phi(x_{8+6i})=\phi(x_2)$. It follows that $\phi(x_{3+6(i+1)})=\phi(x_1)$ and, so, $\phi(x_{4+6(i+1)})=\phi(x_0)$, as required. Since ϕ is continuous, $\phi(x_\omega)=\phi(x_0)=\phi(x_1)$, contrary to hypothesis. Henceforth, assume $\phi(x_3)\neq\phi(x_1)$.

Suppose $\phi(x_3)=\phi(w)$. We claim that, for $i\geqq 0$, $\phi(x_{4+6i})=\phi(x_2)$ and $\phi(x_{5+6i})=\phi(x_1)$. Since $\phi(x_3)=\phi(w)$, $\phi(x_4)=\phi(x_2)$ and, so, $\phi(x_5)=\phi(x_1)$. Suppose then that the claim holds for some $i<\omega$. Since $\phi(x_{4+6i})=\phi(x_2)$ and $\phi(x_{5+6i})=\phi(x_1)$, $\phi(x_{6+6i})=\phi(x_0)$ and, hence, $\phi(x_{7+6i})=\phi(x_1)$. It follows that $\phi(x_{8+6i})=\phi(x_2)$ and, so, $\phi(x_{9+6i})=\phi(w)$. Thus, as required, $\phi(x_{4+6(i+1)})=\phi(x_2)$ and, so, $\phi(x_{5+6(i+1)})=\phi(x_1)$. Once more, as ϕ is continuous, $\phi(x_\omega)=\phi(x_1)=\phi(x_2)$, contrary to hypothesis and, henceforth, we may assume $\phi(x_3)\neq\phi(w)$.

To summarize at this point, for distinct i and $j\leqq 3$, $\phi(x_i)$, $\phi(x_j)$, and $\phi(w)$ are distinct.

We claim that, for $i\geqq 1$, ϕ is one-to-one on $\{w\}\cup\{x_j: j\leqq 2i+1\}$. Say, ϕ is one-to-one on $\{w\}\cup\{x_j: j\leqq 2i+1\}$, but not on $\{w\}\cup\{x_j: j\leqq 2(i+1)\}$. Then, since ϕ is a dp-map, $\phi(x_{2(i+1)})=\phi(x_{2i})$ and, arguing inductively, $\phi(x_{2i+1+j})=\phi(x_{2i+1-j})$ for $j\leqq 2i-1$. However, for $j=2i-1$, $\phi(x_{4i})=\phi(x_2)$ which, since $|Max(x_{4i})|=2$ and $|Max(\phi(x_2))|=3$, is absurd. Thus, ϕ is one-to-one on $Ext(P)\backslash\{x_\omega\}$. Since $\phi(x_\omega)=\phi(y)$ for some $y\in\phi(Ext(P)\backslash\{x_\omega\})$ implies $|Q|=1$, we conclude that ϕ is one-to-one on $Ext(P)$.

Three possibilities remain: either $\phi(v)=\phi(x_1)$, or $\phi(u)=\phi(v)$, or $\phi(u)=\phi(x_2)$. Since this implies that either $\phi(x_0)=\phi(x_1)$, or $\phi(w)=\phi(x_1)$, or $\phi(x_3)=\phi(x_1)$ or $\phi(w)$, respectively, each violates the working hypothesis. □

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