15

States of lowest energy: statics

Quantizing the Abraham model results in the Pauli–Fierz Hamiltonian which is a self-adjoint operator under rather general conditions. Thus the dynamics is well defined and we can start to investigate some of its properties. The most basic item is the states of lowest energy. They really come in two varieties: (i) If the electron is bound by a strong external electrostatic potential, like the Coulomb potential of a nailed-down nucleus, then the lowest energy state is the ground state, where the electron is at rest modulo quantum fluctuations. (ii) If there are no external potentials, then the total momentum is conserved and the state of lowest energy must be determined for every fixed total momentum, which then describes the electron together with its surrounding photon cloud traveling at constant velocity. Physically the most important information is the energy–momentum relation which gives the lowest energy E at given total momentum P. Both item (i) and item (ii) are discussed in this chapter. In case (i) one expects to have always a ground state provided the external potential is binding. In case (ii) the infrared divergence of the Pauli–Fierz model becomes visible. As will be explained in more detail in section 19.1, for total momentum $P \neq 0$ the state of lowest energy is not in Fock space. An electron traveling at nonzero velocity binds an infinite number of photons. To avoid such a subtlety, for item (ii) we proceed as if the photon had a tiny mass.

The external fields manufactured with macroscopic devices under laboratory conditions are weak and have a slow variation when measured in units of the effective size of the charge, roughly given through the inverse size of the form factor $\widehat{\varphi}$. Such external fields thus constitute a small perturbation in item (ii) and, as for the Abraham model, an important dynamical issue is to understand the motion of the charge in terms of an effective one-particle Hamiltonian. The energy–momentum relation must play an important role, but there will be additional pieces accounting for the spin precession. Our discussion of this topic is postponed to section 16, to keep the lengths of the chapters in reasonable proportion.

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15.1 Bound charge

The hydrogen atom has a stable ground state and thus makes the size of atoms of the order of a few ångströms. The problem under discussion is whether this ground state persists as the quantized transverse modes of the Maxwell field are taken into consideration. Since the electron now has the opportunity to bind photons, one would expect it to have effectively a larger mass. This intuition is confirmed through the path integral of chapter 14, which suggests that the fluctuations in the stochastic trajectories are reduced due to the additional interaction energy W from the integration over the Maxwell field. Thus the coupling to the photons should enhance binding.

To put such reasoning on more solid grounds, we recall that for a Schrödinger operator $H_{\mathrm{S}} = -(1/2m)\Delta + V$ with a Coulomb-like potential, i.e. a potential V such that $\lim_{|x|\to\infty} V(x) = 0$, it is rather straightforward to ensure a stable ground state. Let us assume that V is infinitesimally bounded with respect to $-\Delta$. Then the bottom of the continuous spectrum, denoted by E_{c}, satisfies $E_{\mathrm{c}} = 0$ and one only has to make sure that the energy is lowered when the electron is moved from infinity to the potential region. This means that one has to find a trial wave function such that $\langle \psi , H_{\mathrm{S}}\psi \rangle < 0$. By the Kato–Rellich theorem H_{S} is bounded from below. Thus H_{S} must have an eigenvalue at the bottom of its spectrum. The ground state wave function ψ_{g} is nodeless, since $\mathrm{e}^{-tH_{\mathrm{S}}}$ is positivity improving; compare with section 14.3(i). Hence the ground state is unique. To adapt such reasoning to the Pauli–Fierz Hamiltonian

$$H = \frac{1}{2m}(p - eA_{\varphi}(x))^2 + H_{\mathrm{f}} + V(x) = H^0 + V, \tag{15.1}$$

one faces the difficulty that there are photon excitations of arbitrarily small energies. Thus H has no spectral gap and a variational bound will not do. The convential approach is to first assume an infrared cutoff in the form factor $\widehat{\varphi}$ by setting $\widehat{\varphi}(k) = 0$ for $|k| \le \sigma$ and to adopt the construction explained in property (vi) of section 15.2.1. This yields the existence of a ground state $\psi_{\mathrm{g},\sigma}$ for the cutoff Hamiltonian H_σ. One is then left to show that as $\sigma \to 0$ the sequence of ground states $\psi_{\mathrm{g},\sigma}$ has a limit ψ_{g} which is the desired ground state for H. The difficulty is that as $\sigma \to 0$ the number of bound photons could increase without limit resulting in the physical ground state lying outside of Fock space. This is one aspect of the infrared problem to be discussed in more detail in section 19.1. Thus one has to establish a bound on the number of low-energy (soft) photons in the ground state. We explain some parts of the argument which allow us to illustrate the pull-through formula that will also be handy later on.

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Theorem 15.1 (Soft photon bound). *Let ψ_g be a ground state of the Pauli–Fierz Hamiltonian H of (15.1), $H\psi_\mathrm{g} = E\psi_\mathrm{g}$. Then the average number of photons is bounded as*

$$\langle \psi_\mathrm{g}, N_\mathrm{f}\psi_\mathrm{g}\rangle \le c_0\langle \psi_\mathrm{g}, x^2\psi_\mathrm{g}\rangle. \tag{15.2}$$

Proof: Clearly

$$\langle \psi_\mathrm{g}, N_\mathrm{f}\psi_\mathrm{g}\rangle = \sum_{\lambda=1,2}\int \mathrm{d}^3k \|a(k,\lambda)\psi_\mathrm{g}\|^2. \tag{15.3}$$

Through a virial-type argument we plan to make use of the fact that ψ_g is an eigenfunction, and start with the pull-through formula

$$[H, a(k,\lambda)] = -\omega(k)a(k,\lambda) + e\widehat{\varphi}\frac{1}{\sqrt{2\omega}}\mathrm{e}^{-\mathrm{i}k\cdot x}\frac{1}{m}e_\lambda(k)\cdot(p - eA_\varphi(x)). \tag{15.4}$$

Note that

$$\frac{1}{m}(p - eA_\varphi(x)) = \mathrm{i}[H, x]. \tag{15.5}$$

Therefore

$$\begin{aligned}(H+\omega)a(k,\lambda) - a(k,\lambda)H = e\widehat{\varphi}\frac{1}{\sqrt{2\omega}}\big(&\mathrm{i}[H, \mathrm{e}^{-\mathrm{i}k\cdot x}e_\lambda(k)\cdot x] \\ &-\mathrm{i}[H, \mathrm{e}^{-\mathrm{i}k\cdot x}]e_\lambda(k)\cdot x\big).\end{aligned} \tag{15.6}$$

The commutator with $\mathrm{e}^{-\mathrm{i}k\cdot x}$ is

$$[H, \mathrm{e}^{-\mathrm{i}k\cdot x}] = -\frac{1}{m}k\cdot(p - eA_\varphi(x))\mathrm{e}^{-\mathrm{i}k\cdot x} - \frac{1}{2m}k^2\mathrm{e}^{-\mathrm{i}k\cdot x} \tag{15.7}$$

and applied to ψ_g,

$$\begin{aligned}a(k,\lambda)\psi_\mathrm{g} &= \mathrm{i}e\widehat{\varphi}\frac{1}{\sqrt{2\omega}}(H - E + \omega)^{-1} \\ &\Big((H-E) + \frac{1}{2m}k^2 + \frac{1}{m}k\cdot(p - eA_\varphi(x))\Big)\mathrm{e}^{-\mathrm{i}k\cdot x}e_\lambda(k)\cdot x\psi_\mathrm{g} \\ &= \mathrm{i}e\widehat{\varphi}\frac{1}{\sqrt{2\omega}}(\phi_1 + \phi_2 + \phi_3).\end{aligned} \tag{15.8}$$

Thus, choosing $e > 0$ for notational convenience,

$$\|a(k,\lambda)\psi_\mathrm{g}\| \le e|\widehat{\varphi}|\frac{1}{\sqrt{2\omega}}(\|\phi_1\| + \|\phi_2\| + \|\phi_3\|) \tag{15.9}$$

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and

$$\|\phi_1\| \le \|x\psi_g\|, \quad \|\phi_2\| \le \frac{1}{2m} k^2 \frac{1}{\omega} \|x\psi_g\|,$$
$$\|\phi_3\| \le |k| \|(H - E + \omega)^{-1} \frac{1}{m} \widehat{k} \cdot (p - eA_\varphi(x)) e^{-ik\cdot x} e_\lambda(k) \cdot x\psi_g\|. \tag{15.10}$$

To estimate the norm of ϕ_3 we use

$$\|(H - E + \omega)^{-1} \widehat{k} \cdot (p - eA_\varphi(x))\| = \|\widehat{k} \cdot (p - eA_\varphi(x))(H - E + \omega)^{-1}\| \tag{15.11}$$

and

$$\begin{aligned} &\langle \psi, (H - E + \omega)^{-1} (\widehat{k} \cdot (p - eA_\varphi(x)))^2 (H - E + \omega)^{-1} \psi \rangle \\ &\quad \le \langle \psi, (H - E + \omega)^{-1} (p - eA_\varphi(x))^2 (H - E + \omega)^{-1} \psi \rangle \\ &\quad \le \langle \psi, (H - E + \omega)^{-1} (c_1 H + c_2)(H - E + \omega)^{-1} \psi \rangle \\ &\quad \le \langle \psi, \psi \rangle \Big[\sup_{\lambda \ge 0} \big(c_1(\lambda + E) + c_2 \big)(\lambda + \omega)^{-2} \Big], \end{aligned} \tag{15.12}$$

provided V_- is H bounded. Inserting in (15.10)

$$\|\phi_3\| \le |k| \Big(c_1 \frac{1}{\omega} + c_2 \sqrt{\omega} \Big) \|x\psi_g\| \tag{15.13}$$

is obtained. With these estimates we return to (15.3) to get

$$\langle \psi_g, N_f \psi_g \rangle \le c_0 \int d^3k |\widehat{\varphi}(k)|^2 \frac{1}{\omega} (1 + \omega^{-2} k^4 + \omega^{-2} k^2 + \omega k^2) \|x\psi_g\|^2, \tag{15.14}$$

which proves (15.2). □

Bounds on $\|x\psi_g\|$ are available from the diamagnetic inequality combined with functional integration, see section 14.3(i), and from yet another pull-through-type argument, see section 20.1.

Note that in (15.14) we can still afford the two extra powers ω^{-2} close to $k = 0$. This is consistent with a decay as $|t|^{-4}$ in the effective action given at the end of section 14.2.

The modern variant for the existence of a ground state relies on having an energy gain when the electron is moved from infinity to the potential region. Thereby, as discussed at length in section 20.1, the existence of a ground state for atoms and molecules is also ensured. To be complete we now state

Theorem 15.2 (Unique ground state). *Let $V = V_+ - V_-$ be the decomposition of the external potential V into positive and negative parts. It is assumed that V_- is*

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infinitesimally bounded relative to p^2*, i.e.* $|\langle\psi, V_-\psi\rangle| \le \varepsilon\langle\psi, p^2\psi\rangle + b(\varepsilon)\langle\psi,\psi\rangle$ *for every* $\varepsilon > 0$*, and that* $\frac{1}{2m}p^2 + V$ *has a ground state with isolated ground state energy. Then the Hamiltonian* H *of (15.1) has a unique ground state* $\psi_g \in \mathcal{H}$*, i.e.* $H\psi_g = E\psi_g$ *and* E *is the lowest energy.*

Proof: The existence is proved by Griesemer, Lieb and Loss (2001). The uniqueness relies on the fact that the semigroup e^{-tH} is positivity improving in a suitable basis, see Hiroshima (2000a) and section 14.3. □

Note that in Theorem 15.2 there is no restriction on the magnitude of the charge.

15.2 Energy–momentum relation, effective mass

For the Abraham model the motion of the charge subject to slowly varying external potentials is determined by the energy–momentum relation $E(P)$. There is good reason to expect the same scenario quantum mechanically, which poses two problems. First of all one has to study $E(P)$, which makes a two-line computation classically but turns out to be much harder in quantum theory. Secondly, given $E(P)$, we have to explain how it governs the effective one-particle theory. This topic is deferred to chapter 16.

Since there are no external forces acting on the electron, the Pauli–Fierz Hamiltonian reads

$$H = \frac{1}{2m}(p - eA_\varphi(x))^2 + H_f. \tag{15.15}$$

As shown already, the total momentum

$$P = p + \sum_{\lambda=1,2}\int d^3k\, k a^*(k,\lambda)a(k,\lambda) = p + P_f \tag{15.16}$$

is conserved, $[H, P] = 0$. Therefore H can be decomposed according to the subspaces of constant P. This is achieved through the unitary transformation

$$U = e^{ix\cdot P_f}, \tag{15.17}$$

which more explicitly is given by

$$(U\psi)_n(k, k_1, \lambda_1, \ldots, k_n, \lambda_n) = \psi_n(k - \sum_{j=1}^{n} k_j, k_1, \lambda_1, \ldots, k_n, \lambda_n), \tag{15.18}$$

using the momentum representation $p = k$, $x = i\nabla_k$. Then

$$UHU^{-1} = \frac{1}{2m}(P - P_f - eA_\varphi)^2 + H_f \tag{15.19}$$

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with the shorthand

$$A_\varphi = A_\varphi(0). \tag{15.20}$$

Not to overload notation we return to p instead of P, remembering that p is still canonically conjugate to x but now stands for the total momentum. The Hamiltonian under study is then

$$H_p = \frac{1}{2m}(p - P_{\mathrm{f}} - eA_\varphi)^2 + H_{\mathrm{f}}. \tag{15.21}$$

For each fixed p, H_p acts on Fock space $\mathcal{F}$. Thus we may think of the unitary U as a map from $L^2(\mathbb{R}^3) \otimes \mathcal{F}$ to the direct integral $\int^\oplus \mathrm{d}^3 p \mathcal{F}_p$ such that $UHU^{-1} = \int^\oplus \mathrm{d}^3 p H_p$. For the remainder of this section we will regard p simply as a parameter. The scalar product $\langle \cdot, \cdot \rangle$ is in Fock space throughout.

Definition 15.3 *The energy–momentum relation, $E(p)$, of the Pauli–Fierz Hamiltonian is given by*

$$E(p) = \inf_{\psi, ||\psi||=1} \langle \psi, H_p \psi \rangle. \tag{15.22}$$

The effective mass m_{eff} is the inverse curvature of $E(p)$ at $p = 0$. Since $E(p)$ is rotation-invariant,

$$(m_{\mathrm{eff}})^{-1} \delta_{\alpha\beta} = \partial_{p_\alpha} \partial_{p_\beta} E(p)|_{p=0}. \tag{15.23}$$

There is no simple scheme to compute $E(p)$ and m_{eff}, but we will establish some qualitative properties of $E(p)$ which point in the right direction. In order not to lose sight of the goal we state

Claim 15.4 (Energy–momentum relation). *Let $\omega(k) = \sqrt{m_{\mathrm{ph}}^2 + k^2}$ with $m_{\mathrm{ph}} >$ 0. There exists a threshold value, p_{c}, of the total momentum such that for all $|p| < p_{\mathrm{c}}$, H_p has a unique ground state $\psi_p \in \mathcal{F}$,*

$$H_p \psi_p = E(p) \psi_p. \tag{15.24}$$

$E(p)$ is separated by a gap from the continuous spectrum, i.e. if $E_{\mathrm{c}}(p)$ denotes the bottom of the continuous spectrum, then

$$E_{\mathrm{c}}(p) - E(p) = \Delta(p) > 0. \tag{15.25}$$

In Claim 15.4 we assumed a small photon mass m_{ph}. Thus at $p = 0$ excitations require at least an energy m_{ph}. For physical photons $m_{\mathrm{ph}} = 0$, however. Arbitrarily small-energy excitations are possible and the spectral gap closes, which is one part of the infrared behavior of the Pauli–Fierz model. The assumption $m_{\mathrm{ph}} > 0$ introduces a spectral gap, so to speak, by hand. An alternative scheme to separate

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the ground state band from the continuum is to decouple all modes with $|k| \leq \sigma$ by replacing the true $\widehat{\varphi}$ by $\widehat{\varphi}_\sigma$, where $\widehat{\varphi}_\sigma = \widehat{\varphi}$ for $|k| \geq \sigma$ and $\widehat{\varphi}_\sigma = 0$ for $|k| < \sigma$.

We made the proviso that the ground state band ceases to exist beyond the threshold p_c, where we allow for $p_c = \infty$. If $p_c < \infty$, then the electron cannot be accelerated beyond the maximal momentum p_c. For $|p| > p_c$, H_p has no ground state. States with $|p| > p_c$ decay into lower-momentum states through the emission of Čerenkov radiation. In fact the same phenomenon occurs classically if in the given medium the speed of light propagation is less than the maximal speed of the charge.

To investigate $E(p)$, let us first have a look at the uncoupled system, $e = 0$. Then the eigenstate in (15.24) is the Fock vacuum Ω with eigenvalue $p^2/2m$. The energies in the one-photon subspace are $\omega(k) + (p-k)^2/2m$, which is already part of the continuous spectrum. The energy in the n-photon subspace is $(2m)^{-1}(p - \sum_{j=1}^n k_j)^2 + \sum_{j=1}^n \omega(k_j) \geq (2m)^{-1}(p - \sum_{j=1}^n k_j)^2 + \omega(\sum_{j=1}^n k_j)$ and for low energies it suffices to take the one-photon part of the continuous spectrum into account. If p is small, $|p| < m$ $(= mc)$, the lowest energy is $p^2/2m$ separated by a gap of order $\omega(0) = m_{\mathrm{ph}}$ from the continuum. On the other hand, for $|p| > m$, the eigenvalue $p^2/2m$ is embedded in the continuum and expected to turn into a resonance, once e is different from zero. In some model systems it is found that $p_c < \infty$ for $e = 0$, but $p_c = \infty$ at any $e \neq 0$. Whether $p_c = \infty$ depends also on the form of the kinetic energy of the electron. If instead of $\frac{1}{2m}p^2$ as kinetic energy one repeats the argument just given for the relativistic cousin $\sqrt{p^2 + m^2}$, then $p_c = \infty$ at $e = 0$ and it remains so for $e > 0$. For the Pauli–Fierz model (in three dimensions) the accepted opinion is that the electron cannot be accelerated beyond $p_c \cong \mathcal{O}(mc)$.

Perturbation theory assures us that the isolated ground state energy band for $|p| < p_c$ at $e = 0$ will persist for small nonzero e. The range of validity of perturbation theory is set by $\omega(0) = m_{\mathrm{ph}}$ and is therefore very narrow. To improve and to be able to let $m_{\mathrm{ph}} \to 0$ we have to employ nonperturbative techniques, for which we follow Fröhlich (1974). Only the core of each argument is explained; the shorter ones are given immediately in the text and the longer ones are shifted to an appendix. Here is our list.

Property (i): *$E(p)$ is rotation invariant.*

According to section 13.5 there is a unitary operator U_R such that $U_R^* H_p U_R = H_{Rp}$ with R an arbitrary rotation. Therefore $E(p) = E(Rp)$.

Property (ii): *The bound*

$$E(0) \leq E(p) \tag{15.26}$$

holds.

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From the functional integral representation, compare with chapter 14 and the further explanations in the appendix, it will become clear that

$$|\langle F, U\mathrm{e}^{\mathrm{i}\pi N_\mathrm{f}/2}\mathrm{e}^{-tH_p}\mathrm{e}^{-\mathrm{i}\pi N_\mathrm{f}/2}U^{-1}F\rangle| \leq \langle |F|, U\mathrm{e}^{-tH_0}U^{-1}|F|\rangle \quad (15.27)$$

for $t \geq 0$. We choose $\mathrm{e}^{-\mathrm{i}\pi N_\mathrm{f}/2}U^{-1}F = \psi_p$, or else an approximate ground state if ψ_p does not exist. Let $\mu(\mathrm{d}\lambda)$ be the spectral measure for $U^{-1}|F|$ under H_0 and $\lambda_{\min}$ be the left edge of its support. Taking the limit $t \to \infty$ in (15.27), we obtain

$$E(p) \geq \lambda_{\min} \geq E(0). \quad (15.28)$$

One would expect $E(p)$ to be increasing in $|p|$, but no conclusive argument seems to be available.

Property (iii): *As a bound we have*

$$E(p) - E(0) \leq \frac{1}{2m}p^2. \quad (15.29)$$

The inequality (15.29) follows from a variational argument. One has

$$\begin{aligned} E(p) \leq \langle\psi_0, H_p\psi_0\rangle &= \langle\psi_0, \Big(\frac{1}{2m}p^2 + H_0 - \frac{1}{m}p\cdot(P_\mathrm{f} + eA_\varphi)\Big)\psi_0\rangle \\ &= E(0) + \frac{1}{2m}p^2 - \frac{1}{m}p\cdot\langle\psi_0, (P_\mathrm{f} + eA_\varphi)\psi_0\rangle \\ &= E(0) + \frac{1}{2m}p^2, \end{aligned} \quad (15.30)$$

since $H_0\psi_0 = E(0)\psi_0$ and $\frac{1}{m}\langle\psi_0, (P_\mathrm{f} + eA_\varphi)\psi_0\rangle = \nabla E(0) = 0$ by rotation invariance.

Property (iv): *As a bound we have*

$$E(p) \leq E(p-k) + \omega(k). \quad (15.31)$$

In particular, $E(p) - E(0) \leq \omega(p)$.

The proof is given in the appendix. There is also a corresponding lower bound.

Property (v): *There are constants* $c_1 > 0$, c_2 *such that* $E(p) \geq c_1|p| + c_2$. The proof is given in the appendix.

The next property expresses the stability against one-photon excitations. Define

$$\Delta(p) = \inf_k\{E(p-k) - E(p) + \omega(k)\}. \quad (15.32)$$

Then by property (iv) $\Delta(p) \geq 0$.

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Property (vi): *For the bottom of the continuous spectrum we have*

$$E_{\mathrm{c}}(p) = E(p) + \Delta(p). \tag{15.33}$$

If $\Delta(p) > 0$, *then* H_p *has a ground state at* $E(p)$.

The proof is given in the appendix. We want to infer from the bounds on $E(p)$ that $\Delta(p) > 0$, at least for small $|p|$. As a substitute for the missing proof of the monotonicity of $E(p)$, note that from second-order perturbation in p

$$\partial_{p_\alpha}\partial_{p_\beta}E(p) = \frac{1}{m}\delta_{\alpha\beta} - 2\langle\psi_p, \big(m^{-1}(p - P_{\mathrm{f}} - eA_\varphi) - \nabla E(p)\big)_\alpha (H_p - E(p))^{-1} \times \big(m^{-1}(p - P_{\mathrm{f}} - eA_\varphi) - \nabla E(p)\big)_\beta \psi_p\rangle. \tag{15.34}$$

This leads to

Property (vii): $E(p) = \frac{1}{2m}p^2 + t(p)$. t *is convex down.*

From property (ii) we conclude that $t(p) - t(0) \geq -\frac{1}{2m}p^2$, which means that $t(p) - t(0)$ cannot bend down too fast. This allows us to establish

Property (viii): *If* $|p| \leq (\sqrt{3} - 1)m$, *then* $\Delta(p) > 0$ *and* H_p *has a ground state separated by the gap* $\Delta(p)$ *from the continuum.*

Finally, the uniqueness follows from the overlap with the Fock vacuum.

Property (ix): *If* $|p| < p_{\mathrm{c}}$ *and if*

$$\frac{2e^2}{m}\int \mathrm{d}^3k|\widehat{\varphi}|^2\omega^{-1}E(p)(E(p-k) - E(p) + \omega)^{-2} < \frac{1}{2}, \tag{15.35}$$

then H_p *has a unique ground state.*

Again the proof is given in the appendix. If $|p| < (\sqrt{3} - 1)m \leq p_{\mathrm{c}}$ and (15.35) holds, then $E(p, e)$ is analytic jointly in p and e as a standard consequence of perturbation theory.

In summary, properties (i)–(ix) lend support to the qualitative behavior of the energy–momentum relation as schematically presented in figure 15.1. The bold line indicates the ground state. $E(0)$ increases with the coupling. The gap of size m_{ph} is not shown. As $m_{\mathrm{ph}} \to 0$ the gap closes. To understand what really happens in this limit, one has to study the infrared scaling of the Pauli–Fierz Hamiltonian with care. Explicit expressions for $E(p)$ do not seem to be available. Computationally only perturbation in e is accessible. To second order one obtains

$$E(p) = \frac{e^2}{2m}\frac{2}{3}\int \mathrm{d}^3k|\widehat{\varphi}|^2(2\omega)^{-1} + \frac{1}{2m}p^2\Big(1 - \frac{e^2}{m}\frac{1}{3}\int \mathrm{d}^3k|\widehat{\varphi}|^2\Big(2\omega(\omega + \frac{1}{2m}k^2)\Big)^{-1}\Big) + \mathcal{O}(e^4), \tag{15.36}$$

which can be trusted only for sufficiently small p. $E(0)$ increases in e and in the

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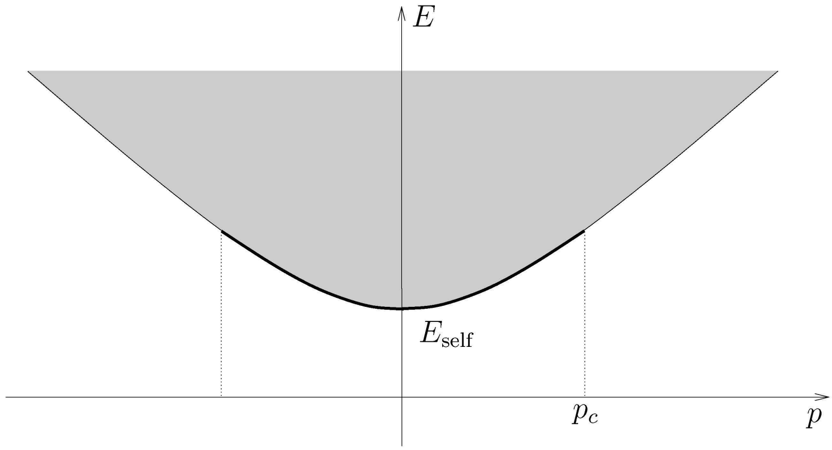

Figure 15.1: The energy–momentum relation for the Pauli–Fierz Hamiltonian.

ultraviolet cutoff, as does m_{eff}. Equation (15.36) confirms the physical intuition that the coupling to the Maxwell field effectively increases the mass of the electron. Note that already to order e^2 the effective mass differs from that obtained in the dipole approximation, compare with (14.58), and thus from the effective mass of the classical cousin, the Abraham model.

The nature of the excited states, even close to the ground state band, is left untouched by the present considerations. Physically one expects, as we have indeed established for the Abraham model, a dynamically transient stage when by radiating photons the electron adjusts to the long-time freely propagating state of the form $\mathrm{e}^{-\mathrm{i}tE(p)} f(p)\psi_{\mathrm{g}}(p)$. Here the amplitudes $f(p)$ vanish for $|p| > p_{\mathrm{c}}$ and are determined through the initial conditions. In spectral terms, this implies that H_p has a purely absolutely continuous spectrum except for the possible eigenvalue at $E(p)$. The only powerful technique available for establishing such a property is the method of positive commutators and, as its sisters, Mourre estimates and complex dilations, cf. chapter 17. Let us see how this method applies to the Pauli–Fierz Hamiltonian H_p.

In the abstract setting one starts from a self-adjoint operator H on some Hilbert space $\mathcal{H}$ and searches for another self-adjoint operator, the conjugate operator D, such that

$$[H, \mathrm{i}D] \geq c_0 > 0. \tag{15.37}$$

Then H has a purely absolutely continuous spectrum. The example to keep in mind here is $H = x$ and $\mathrm{i}D = -\partial_x$. In our context, clearly, (15.37) is too strong. The appropriate modification reads

$$[H, \mathrm{i}D] \geq c_0 - R \tag{15.38}$$

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with R a positive trace class operator. This form allows one to count eigenvalues. If $H\psi_n = E_n\psi_n$, $\|\psi_n\| = 1$, then $\langle\psi_n, [H, \mathrm{i}D]\psi_n\rangle_{\mathcal{H}} = 0 \geq c_0 - \langle\psi_n, R\psi_n\rangle$ and, by summing over n, $\mathrm{tr}P_{\mathrm{pp}} \leq c_0^{-1}\mathrm{tr}R$, where P_{pp} is the projection onto the linear span of all eigenfunctions. The Mourre estimate (15.38) ensures that H restricted to $(1 - P_{\mathrm{pp}})\mathcal{H}$ has a purely absolutely continuous spectrum. Inequality (15.38) could be still too strong and is weakened by projecting onto an appropriate energy interval Δ as

$$E_\Delta[H, \mathrm{i}D]E_\Delta \geq c_0 E_\Delta - E_\Delta R E_\Delta, \tag{15.39}$$

where E_Δ is the spectral projection of H for the interval $\Delta \subset \mathbb{R}$.

For the Pauli–Fierz operator the natural candidate for the conjugate operator is the generator D_1 of dilations in photon space, i.e. $(\mathrm{e}^{-\mathrm{i}D_1 t} f)(k) = t^{3/2} f(tk)$. Then

$$\mathrm{i}D_1 = -\frac{1}{2}\big(\widehat{k} \cdot \partial_k + \partial_k \cdot \widehat{k}\big) \tag{15.40}$$

as operator on $L^2(\mathbb{R}^3, \mathrm{d}^3k)$. We denote the second quantization of D_1 by

$$D = \sum_{\lambda=1,2} \int \mathrm{d}^3k a^*(k, \lambda) D_1 a(k, \lambda). \tag{15.41}$$

With these preparations

$$[H_p, \mathrm{i}D] = N_{\mathrm{f}} - \frac{1}{m}\mathrm{d}\Gamma(\widehat{k}) \cdot (p - P_{\mathrm{f}} - eA_\varphi) + \frac{1}{m}eA_{\varphi_1} \cdot (p - P_{\mathrm{f}} - eA_\varphi), \tag{15.42}$$

where $\mathrm{d}\Gamma(\widehat{k}) = \sum_{\lambda=1,2} \int \mathrm{d}^3k \widehat{k} a^*(k, \lambda) a(k, \lambda)$ and $\widehat{\varphi}_1 = \sqrt{\omega}\mathrm{i}D_1 \frac{1}{\sqrt{\omega}}\widehat{\varphi}$.

Let us abbreviate $B = p - P_{\mathrm{f}} - eA_\varphi$. By the Kato–Rellich theorem

$$\begin{aligned}\frac{e}{m}(A_{\varphi_1} \cdot B) &\leq \frac{e}{2m}\big((A_{\varphi_1})^2 + B^2\big)\\ &\leq \frac{e}{2m}(c_1 H_p + c_2) + eH_p \leq e(c_1 H_p + c_2)\end{aligned} \tag{15.43}$$

with coefficients c_1, c_2 independent of p and e and whose value may change from line to line. Similarly, using the fact that $[N_{\mathrm{f}}, B]$ is H_p-bounded and $\mathcal{O}(e)$,

$$\begin{aligned}\frac{1}{m}\mathrm{d}\Gamma(\widehat{k}) \cdot B &= \frac{1}{m} B N_{\mathrm{f}}^{1/2} \cdot N_{\mathrm{f}}^{-1/2}\mathrm{d}\Gamma(\widehat{k})\\ &\leq \frac{1}{m}\frac{1}{2m}\big(BN_{\mathrm{f}}^{1/2}\big)^2 + \frac{1}{2}N_{\mathrm{f}}\\ &\leq \frac{1}{m}N_{\mathrm{f}}^{1/2} H_p N_{\mathrm{f}}^{1/2} + \frac{1}{2}N_{\mathrm{f}} + e(c_1 H_p + c_2).\end{aligned} \tag{15.44}$$

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Let E_Σ be the spectral projection of H_p onto the interval $(-\infty, \Sigma]$. Combining (15.42), (15.43), and (15.44) and using the property that $N_\mathrm{f} \geq 1 - P_\Omega$, the final result reads

$$E_\Sigma[H_p, \mathrm{i}D]E_\Sigma \geq E_\Sigma(1 - P_\Omega)E_\Sigma\Big(\frac{1}{2} - \frac{1}{m}\Sigma\Big) - e(c_1\Sigma + c_2)E_\Sigma. \quad (15.45)$$

Inequality (15.45) has the structure anticipated in (15.39) with $\Delta = (-\infty, \Sigma]$ and R the one-dimensional projection P_Ω. Thus we count the number of eigenvalues in $(-\infty, \Sigma]$ as

$$\mathrm{tr}[P_{\mathrm{pp}}E_\Sigma] \leq \Big(1 - e(c_1\Sigma + c_2)\Big(\frac{1}{2} - \frac{1}{m}\Sigma\Big)^{-1}\Big)^{-1} \quad (15.46)$$

which can be made strictly less than 2 by adjusting e. We have not tried to optimize the constants. But the net result is that, upon fixing e_0, p_c sufficiently small and $\Sigma = p_\mathrm{c}^2/4m$, say, in the interval $(-\infty, \Sigma]$ the operator H_p has a purely absolutely continuous spectrum and a single, nondegenerate eigenvalue located at $E(p)$, provided $|e| < e_0$ and $|p| \leq p_\mathrm{c}$. To study the high-energy/high-momentum part of the spectrum other methods will have to be developed.

15.2.1 Appendix: Properties of $E(p)$

We prove properties (iv), (v), (vi), (viii), and (ix).

Property (vi): Fix p and choose the momentum lattice $(\delta\mathbb{Z})^3$ with lattice spacing $\delta > 0$. The 3-axis of the lattice is parallel to p. Correspondingly, $\mathbb{R}^3$ is partitioned into cubes $\mathcal{C}_\delta(n) = \{k | (n_\alpha - \frac{1}{2})\delta \leq k_\alpha < (n_\alpha + \frac{1}{2})\delta, \alpha = 1, 2, 3\}$ with integer n_α. The one-particle space $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 = \mathfrak{h}$ is decomposed into a discrete and a fluctuating part,

$$\mathfrak{h} = \mathfrak{h}_\mathrm{d} \oplus \mathfrak{h}_\mathrm{f}. \quad (15.47)$$

$\psi \in \mathfrak{h}_\mathrm{d}$ is constant over each cube and $\psi \in \mathfrak{h}_\mathrm{f}$ satisfies $\int_{\mathcal{C}_\delta(n)} \mathrm{d}^3k\psi(k, \lambda) = 0$ for all $n \in \mathbb{Z}^3$. Such an orthogonal decomposition of the one-particle space factorizes the Fock space as

$$\mathcal{F} = \mathcal{F}_\mathrm{d} \otimes \mathcal{F}_\mathrm{f}. \quad (15.48)$$

If Ω_f is the Fock vacuum of $\mathcal{F}_\mathrm{f}$, we set $\mathcal{F}_\delta = \mathcal{F}_\mathrm{d} \otimes \Omega_\mathrm{f}$ and $\mathcal{F} = \mathcal{F}_\delta \oplus \mathcal{F}_\delta^\perp$.

We want H_p to respect the factorization (15.48). This is achieved by replacing $k, \widehat{\varphi}/\sqrt{2\omega}$, and ω by their lattice approximation k_δ, $(\widehat{\varphi}/\sqrt{2\omega})_\delta$, and ω_δ, where we set $f_\delta(k) = \delta^{-3}\int_{\mathcal{C}_\delta(n)} \mathrm{d}^3kf(k)$ for $k \in \mathcal{C}_\delta(n)$. Then H_p is approximated

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by $H_p(\delta) = \frac{1}{2m}(p - P_{\mathrm{f}}(\delta) - eA_\varphi(\delta))^2 + H_{\mathrm{f}}(\delta)$, which factorizes according to (15.48) as

$$H_p(\delta) = \frac{1}{2m}\big(p - P_{\mathrm{f,d}} \otimes 1 - 1 \otimes P_{\mathrm{f,f}} - eA_{\varphi,\mathrm{d}} \otimes 1\big)^2 + H_{\mathrm{f,d}} \otimes 1 + 1 \otimes H_{\mathrm{f,f}}. \tag{15.49}$$

The fluctuating part of $A_\varphi(\delta)$ vanishes, since $\int \mathrm{d}^3k(\widehat{\varphi}/\sqrt{2\omega})_\delta \psi = 0$ for each $\psi \in \mathfrak{h}_{\mathrm{f}}$. Note that $[H_p(\delta), 1 \otimes P_{\Omega_{\mathrm{f}}}] = 0$, with $P_{\Omega_{\mathrm{f}}}$ the projection onto Ω_{f}, and therefore $H_p(\delta)$ is reduced by the subspaces $\mathcal{F}_\delta, \mathcal{F}_\delta^\perp$. The bottom of the spectrum of $H_p(\delta)$ is denoted by $E(p, \delta)$.

We want to establish a lower bound on $H_p(\delta) \restriction \mathcal{F}_\delta^\perp$. We choose $\psi \in \mathcal{F}_{\mathrm{d}}$ and $\theta \in \mathcal{F}_{\mathrm{f}}$ with fixed n, i.e. $\theta(\underline{k}, \underline{\lambda}) = \theta(k_1, \lambda_1, \ldots, k_n, \lambda_n)$, $n \geq 1$. Then, with $\varphi = \psi \otimes \theta$,

$$\begin{aligned}
\langle \varphi, H_p(\delta)\varphi\rangle_{\mathcal{F}} &= \langle \psi \otimes \theta, H_p(\delta)\psi \otimes \theta\rangle_{\mathcal{F}} \\
&= \sum_{\underline{\lambda}} \int \mathrm{d}^{3n}\underline{k} |\theta(\underline{k}, \underline{\lambda})|^2 \langle \psi, \frac{1}{2m}\Big(p - P_{\mathrm{f,d}} - \sum_{j=1}^n k_{j\delta} - eA_{\varphi,\mathrm{d}}\Big)^2 \psi\rangle_{\mathcal{F}_{\mathrm{d}}} \\
&\quad + \langle \psi, H_{\mathrm{f,d}}\psi\rangle_{\mathcal{F}_{\mathrm{d}}}\langle \theta, \theta\rangle_{\mathcal{F}_{\mathrm{f}}} + \langle \psi, \psi\rangle_{\mathcal{F}_{\mathrm{d}}}\langle \theta, H_{\mathrm{f,f}}\theta\rangle_{\mathcal{F}_{\mathrm{f}}} \\
&= \sum_{\underline{\lambda}} \int \mathrm{d}^{3n}\underline{k} |\theta(\underline{k}, \underline{\lambda})|^2 \langle \psi, H_{p - \sum_{j=1}^n k_{j\delta}, \mathrm{d}}\psi\rangle_{\mathcal{F}_{\mathrm{d}}} \\
&\quad + \langle \psi, \psi\rangle_{\mathcal{F}_{\mathrm{d}}}\langle \theta, H_{\mathrm{f,f}}\theta\rangle_{\mathcal{F}_{\mathrm{f}}} \\
&\geq \inf_{\underline{k}}\{E(p - \sum_{j=1}^n k_{j\delta}, \delta) + \sum_{j=1}^n \omega_\delta(k_j)\}\langle \psi, \psi\rangle_{\mathcal{F}_{\mathrm{d}}}\langle \theta, \theta\rangle_{\mathcal{F}_{\mathrm{f}}} \\
&\geq \inf_k\{E(p - k, \delta) + \omega_\delta(k)\}\langle \varphi, \varphi\rangle_{\mathcal{F}}.
\end{aligned} \tag{15.50}$$

By finite linear combinations this bound extends to a dense set: if $\varphi = \psi_1 \otimes \theta_1 + \psi_2 \otimes \theta_2$ with both θ_1 and θ_2 in the n-photon subspace, one only has to repeat the computation in (15.50). If they belong to different photon numbers, we use $\theta_1 \perp \theta_2$. If $E^\perp(p, \delta)$ denotes the bottom of the spectrum of $H_p(\delta) \restriction \mathcal{F}_\delta^\perp$, we conclude that

$$E^\perp(p, \delta) \geq \inf_k\{E(p - k, \delta) + \omega_\delta(k)\}. \tag{15.51}$$

$H_p(\delta) \restriction \mathcal{F}_\delta$ consists of a large, but finite number of oscillators with strictly positive frequencies. Therefore $H_p(\delta) \restriction \mathcal{F}_\delta$ has a discrete spectrum. Let

$$\Delta(p, \delta) = \inf_k\{E(p - k, \delta) - E(p, \delta) + \omega_\delta(k)\}. \tag{15.52}$$

If $\Delta(p, \delta) \geq \Delta_0 > 0$ independently of δ, then $E^\perp(p, \delta) - E(p, \delta) \geq \Delta(p, \delta) \geq \Delta_0$ by (15.51) and the ground state of $H_p(\delta)$ is in $\mathcal{F}_\delta$. The spectral projection $\chi_{[E(p,\delta), E(p,\delta)+\Delta_0]}(H_p(\delta))$ is a nonzero compact operator.

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The next step is to show that $H_p(\delta)$ converges to H_p as $\delta \to 0$. Technically one proves that for the difference of resolvents the limit

$$\lim_{\delta\to 0} \|(H_p(\delta) - z)^{-1} - (H_p - z)^{-1}\| = 0 \tag{15.53}$$

holds provided z is sufficiently negative. The argument uses the first-order expansion for the resolvent and Kato–Rellich bounds of the type used in the proof of Theorem 13.3. The norm resolvent convergence (15.53) ensures that $\chi_{[E(p,\delta),E(p,\delta)+\Delta_0]}(H_p(\delta))$ converges in norm to $\chi_{[E(p),E(p)+\Delta_0]}(H_p)$ and that this operator is compact as a norm limit of compact operators. Since the limit operator is nonzero by construction, H_p has a ground state at $E(p)$.

To confirm that $E_c(p) = E(p) + \Delta(p)$ with $\Delta(p) = \inf_k\{E(p-k) - E(p) + \omega(k)\}$ the first part of (15.50) is repeated with a one-photon wave function $\theta(k_1, \lambda_1)$ well concentrated at k_0 with k_0 such that $\Delta(p) = E(p-k_0) - E(p) + \omega(k_0)$. There is an infinite number of orthogonal states, which by construction have an energy arbitrarily close to $E(p) + \Delta(p)$. This proves (vi).

Property (iv): From the pull-through formula for a^* we obtain

$$H_p a^*(k,\lambda) = a^*(k,\lambda)(H_{p-k} + \omega(k)) - \frac{e}{m\sqrt{2\omega(k)}}\widehat{\varphi}(k)e_\lambda \cdot (p - P_f - eA_\varphi). \tag{15.54}$$

Let $\psi_{p-k,\delta}$ be an approximate ground state for H_{p-k} with energies in the interval $[E(p-k), E(p-k)+\delta]$ (or let ψ_{p-k} be equal to the ground state if it exists), and let us consider the one-photon excitation $\varphi_\delta = a^*(f_\delta)\psi_{p-k,\delta}$ with f_δ sharply centered at k. From (15.54) one infers

$$\begin{aligned}
E(p)\langle\varphi_\delta,\varphi_\delta\rangle &\le \langle\varphi_\delta, H_p\varphi_\delta\rangle \\
&= \langle\varphi_\delta, H_p a^*(f_\delta)\psi_{p-k,\delta}\rangle \\
&= \omega(k)\langle\varphi_\delta,\varphi_\delta\rangle + \langle\varphi_\delta, a^*(f_\delta)H_{p-k}\psi_{p-k,\delta}\rangle \\
&\quad - \sum_{\lambda'}\int d^3k' \frac{e}{m\sqrt{2\omega(k')}}\widehat{\varphi}(k')f_\delta(k',\lambda') \\
&\quad \times \langle\varphi_\delta, e_{\lambda'}(k')\cdot(p - P_f - eA_\varphi)\psi_{p-k,\delta}\rangle \\
&\le \langle\varphi_\delta,\varphi_\delta\rangle\big(\omega(k) + E(p-k) + \mathcal{O}(\delta)\big) \\
&\quad + \frac{1}{\sqrt{m}}\langle f_\delta, \frac{1}{\sqrt{\omega}}\widehat{\varphi}\rangle_{\mathfrak{h}}\langle\varphi_\delta,\varphi_\delta\rangle^{1/2}\langle\psi_{p-k,\delta}, H_p\psi_{p-k,\delta}\rangle^{1/2}.
\end{aligned} \tag{15.55}$$

We can now choose f_δ such that the last term multiplied by $\langle\varphi_\delta,\varphi_\delta\rangle^{-1}$ vanishes in the limit $\delta \to 0$. Thereby the bound of property (iv) results.

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Property (v): We have for $0 < a_1 < 1, a_2 > 0, (1 - a_1)(1 + a_2) = 1$,

$$\begin{aligned} 2m\langle\psi, H_p\psi\rangle &\geq a_1\langle\psi, (p - P_{\mathrm{f}})^2\psi\rangle - a_2\langle\psi, e^2 A_\varphi^2\psi\rangle + 2m\langle\psi, H_{\mathrm{f}}\psi\rangle \\ &\geq a_1\langle\psi, (p - P_{\mathrm{f}})^2\psi\rangle + (2m - a_2 a)\langle\psi, H_{\mathrm{f}}\psi\rangle - a_2 b\langle\psi, \psi\rangle, \end{aligned} \tag{15.56}$$

where the relative bound $\langle\psi, e^2 A_\varphi^2\psi\rangle \leq a\langle\psi, H_{\mathrm{f}}\psi\rangle + b\langle\psi, \psi\rangle$ is used. We choose a_2 such that $2m - a_2 a > 0$. Since for $\alpha > 0$

$$\frac{1}{2}\alpha(p - P_{\mathrm{f}})^2 + H_{\mathrm{f}} \geq |p| - \frac{1}{2}\alpha^{-1}, \tag{15.57}$$

the constants in (15.57) and (15.56) can be adjusted so as to give the desired bound.

Property (viii): By rotational invariance it suffices to consider (15.32) along a line passing through the origin. We will denote these functions by the same symbol as before. Using properties (ii) and (iv) we obtain

$$\begin{aligned} E(p-k) - E(p) + \omega(k) &= E(p-k) - E(0) - E(p) + E(0) + \omega(k) \\ &\geq -\omega(p) + \omega(k) \end{aligned} \tag{15.58}$$

and it suffices to take the minimum over the interval $|k| \leq |p|$. By reflection symmetry, one may pick $p \geq 0$. We use the decomposition of E from property (vii) and will show that

$$\Delta(p) = \min_{|k| \leq p} \left\{ \frac{1}{2m}(p-k)^2 - \frac{1}{2m}p^2 + t(p-k) - t(p) + \omega(k) \right\} > 0 \tag{15.59}$$

provided $p < m/2$. This will come about by

Lemma: *Let $f : \mathbb{R} \to \mathbb{R}$ be convex, even, with $f(0) = 0$ and $f(x) \leq \frac{1}{2m}x^2$. Then the bounds*

$$-1 + \frac{1}{m}x \leq f'(x) \leq 1 + \frac{1}{m}x \tag{15.60}$$

hold for $|x| \leq m$.

Proof: If (15.60) holds for $x \geq 0$, by reflection symmetry it also holds for $x \leq 0$. So let us take $x \geq 0$. $f'(0) = 0$ and f' is increasing. Therefore we only have to check the upper bound. Let x_0 be the smallest x such that $f'(x_0) = 1 + \frac{1}{m}x_0$. Then,

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since f' is increasing,

$$
\begin{aligned}
0 &\le -f(x) + \frac{1}{2m}x^2 \\
&= \frac{1}{2m}x_0^2 - f(x_0) - \int_{x_0}^{x} \mathrm{d}y \Big(f'(y) - \frac{1}{m}y \Big) \\
&\le \frac{1}{2m}x_0^2 - \int_{x_0}^{x} \mathrm{d}y \Big(f'(x_0) - \frac{1}{m}y \Big) \\
&= \frac{1}{2m}x^2 + \Big(1 + \frac{1}{m}x_0\Big)(x_0 - x)
\end{aligned} \tag{15.61}
$$

for all $x \ge x_0$, which can be satisfied only if $x_0 \ge m$. □

The lemma is used in (15.59) with $f(k) = -t(k) + t(0)$, which by properties (ii) and (vii) satisfies the assumptions, and we set

$$
t(p-k) - t(p) = \int_p^{p-k} \mathrm{d}x t'(x). \tag{15.62}
$$

If $k > 0$, the lower bound in (15.60) is applicable provided $p < m$, $0 \le k \le 2p$. If $k < 0$, the upper bound in (15.60) is applicable provided $p - k \le m$ and thus $p \le m/2$. The bounds put together yield

$$
\frac{1}{2m}(p-k)^2 - \frac{1}{2m}p^2 + t(p-k) - t(p) + \omega(k) \ge -|k| + \omega(k) > 0 \tag{15.63}
$$

for $p \le m/2$ and $|k| \le p$. Refining the last step of the argument the bound can be improved to $p \le (\sqrt{3} - 1)m$, which implies $p_\mathrm{c} \ge (\sqrt{3} - 1)m$.

Property (ix): As in the second part of the proof of Theorem 15.5 below, one estimates the overlap of the ground state vector with Ω by using the analog of the pull-through formula (15.76). (15.69) is replaced then by (15.35).

Finally we have to show (15.27), for which purpose we Trotterize H_p as the sum of $\frac{1}{2m}(p - P_\mathrm{f} - eA_\varphi)^2$ and H_f in the function space representation. We have

$$
|U\mathrm{e}^{\mathrm{i}\pi N_\mathrm{f}/2}\mathrm{e}^{-tH_\mathrm{f}}\mathrm{e}^{-\mathrm{i}\pi N_\mathrm{f}/2}U^{-1}F| \le U\mathrm{e}^{\mathrm{i}\pi N_\mathrm{f}/2}\mathrm{e}^{-tH_\mathrm{f}}\mathrm{e}^{-\mathrm{i}\pi N_\mathrm{f}/2}U^{-1}|F|, \tag{15.64}
$$

since $[H_\mathrm{f}, N_\mathrm{f}] = 0$ and $\mathrm{e}^{-tH_\mathrm{f}}$ has a positive kernel in function space. Recall the transformation (14.67). Linearizing the square with the Gaussian measure μ_G of

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mean zero and variance t/m, one obtains

$$
\begin{aligned}
&(U\mathrm{e}^{\mathrm{i}\pi N_\mathrm{f}/2}\mathrm{e}^{-t(p-P_\mathrm{f}-eA_\varphi)^2/2m}\mathrm{e}^{-\mathrm{i}\pi N_\mathrm{f}/2}U^{-1}F)(A(\cdot))\\
&\quad = (U\mathrm{e}^{-t(p-P_\mathrm{f}-eE_{\perp\tilde{\varphi}})^2/2m}U^{-1}F)(A(\cdot))\\
&\quad = \int \mu_G(\mathrm{d}\lambda)\mathrm{e}^{\mathrm{i}\lambda\cdot p}U\mathrm{e}^{-\mathrm{i}\lambda\cdot(P_\mathrm{f}+eE_{\perp\tilde{\varphi}})}U^{-1}F(A(\cdot))\\
&\quad = \int \mu_G(\mathrm{d}\lambda)\mathrm{e}^{\mathrm{i}\lambda\cdot p}F(A(\cdot+\lambda)+\lambda e\tilde{\varphi}(\cdot)),
\end{aligned}
\tag{15.65}
$$

since P_f shifts and $E_{\perp\tilde{\varphi}}$ translates the field. In fact the components of $(p - P_\mathrm{f} - eE_{\perp\tilde{\varphi}})$ do not commute and in (15.65) there are errors of order t^2 which vanish as the Trotter spacing tends to zero. Taking absolute values on both sides of (15.65) yields

$$
|\cdot| \le \int \mu_G(\mathrm{d}\lambda)|F(A(x+\lambda)+\lambda e\tilde{\varphi}(x))| \tag{15.66}
$$

and similarly for functionals of a finite number of fields. Therefore

$$
\begin{aligned}
&|U\mathrm{e}^{\mathrm{i}\pi N_\mathrm{f}/2}\mathrm{e}^{-t(p-P_\mathrm{f}-eA_\varphi)^2/2m}\mathrm{e}^{-\mathrm{i}\pi N_\mathrm{f}/2}U^{-1}F|\\
&\quad \le U\mathrm{e}^{\mathrm{i}\pi N_\mathrm{f}/2}\mathrm{e}^{-t(P_\mathrm{f}+eA_\varphi)^2/2m}\mathrm{e}^{-\mathrm{i}\pi N_\mathrm{f}/2}U^{-1}|F|.
\end{aligned}
\tag{15.67}
$$

Iterating the bounds (15.64) and (15.67) results in (15.27).

15.3 Two-fold degeneracy in the case of spin

For the effective spin dynamics a crucial input is the two-fold degeneracy of the ground state of the Pauli–Fierz operator with spin, which will be established here for sufficiently small e. The restriction on e is presumably an artifact of the method.

The Hamiltonian under consideration is

$$
H_p = \frac{1}{2m}(p - P_\mathrm{f} - eA_\varphi)^2 - \frac{e}{2m}\sigma\cdot B_\varphi + H_\mathrm{f} \tag{15.68}
$$

acting on $\mathbb{C}^2 \otimes \mathcal{F}$, where $A_\varphi = A_\varphi(0)$, $B_\varphi = B_\varphi(0)$; compare with Eq. (15.21). We require $m_\mathrm{ph} > 0$. Let P_g be the projection onto the ground state subspace and P_0 be the projection onto the subspace spanned by $\chi\otimes\Omega$, $\chi\in\mathbb{C}^2$, $\mathrm{tr}P_0 = 2$. We assume $|p| < p_\mathrm{c}$. Then $\mathrm{tr}P_\mathrm{g} \ge 1$ by the arguments for the proof of property (vi).

Theorem 15.5 (Two-fold degeneracy of the ground state band). *If* $\Delta(p) > 0$ *and whenever*

$$
\frac{2e^2}{m}\int \mathrm{d}^3k|\widehat{\varphi}|^2\omega^{-1}\Big(E(p)+\frac{1}{2m}k^2\Big)\big(E(p-k)-E(p)+\omega\big)^{-2} < \frac{1}{3}, \tag{15.69}
$$

then $\mathrm{tr}P_\mathrm{g} = 2$.

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For the Pauli–Fierz model with spin a proof of property (ii) is missing. If, very reasonably, it is assumed, then $\Delta(p) > 0$ for $|p| \le (\sqrt{3} - 1)m$.

Proof: We assume $\widehat{\varphi}$ to be real which can always be achieved through a suitable canonical transformation.

Let z be real and sufficiently negative. We claim that

$$P_0(z - H_p)^{-1} P_0 = a(z) P_0 \tag{15.70}$$

with real coefficient $a(z)$.

In (15.68) we set $H_0 = \frac{1}{2m}(p - P_\mathrm{f})^2 + H_\mathrm{f}$ and $H_p = H_0 + H_1$. H_0 does not depend on spin and when restricted to the n-photon subspace it is multiplication by a real function. By the Kato–Rellich theorem the resolvent expansion

$$\langle \chi \otimes \Omega, (z - H_p)^{-1} \chi \otimes \Omega \rangle = \sum_{n=0}^{\infty} \langle \chi \otimes \Omega, (z - H_0)^{-1} \big(H_1 (z - H_0)^{-1} \big)^n \chi \otimes \Omega \rangle \tag{15.71}$$

is convergent. Expanding the product yields as generic term

$$\prod_{j=1}^{m} (a_j + \mathrm{i} b_j \cdot \sigma) \tag{15.72}$$

with real coefficients a_j, b_j, depending on $k_1, \lambda_1, \dots, k_m, \lambda_m$. Using the equality

$$(a_1 + \mathrm{i} b_1 \cdot \sigma)(a_2 + \mathrm{i} b_2 \cdot \sigma) = a_1 a_2 - b_1 \cdot b_2 + \mathrm{i} \sigma \cdot (a_1 b_2 + a_2 b_1 - b_1 \times b_2) \tag{15.73}$$

it follows that

$$\langle \chi \otimes \Omega, (z - H_p)^{-1} \chi \otimes \Omega \rangle = a(z) \langle \chi, \chi \rangle + \mathrm{i} b(z) \cdot \langle \chi, \sigma \chi \rangle \tag{15.74}$$

with real coefficients $a(z), b(z)$. Since the left-hand side is real, $b(z) = 0$ which proves (15.70).

Equation (15.70) holds on the negative real axis and therefore extends by analyticity to the full resolvent set. In particular, one can integrate (15.70) over a small contour encircling $E(p)$, the ground state energy of H_p. Then

$$P_0 P_\mathrm{g} P_0 = c_1 P_0. \tag{15.75}$$

By the pull-through argument

$$\begin{aligned} [a(k,\lambda), H_p] &= (H_{p-k} - H_p + \omega(k)) a(k,\lambda) \\ &\quad - \frac{e}{m} \frac{\widehat{\varphi}}{\sqrt{2\omega}} \big(e_\lambda \cdot (p - P_\mathrm{f} - e A_\varphi) - \frac{1}{2} (e_\lambda \times \mathrm{i} k) \cdot \sigma \big). \end{aligned} \tag{15.76}$$

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Let now $\psi \in P_{\mathrm{g}}\mathcal{H}$. Then

$$\begin{aligned}\langle\psi, N_{\mathrm{f}}\psi\rangle &= \sum_{\lambda=1,2}\int \mathrm{d}^3k\|a(k,\lambda)\psi\|^2 \\ &\leq \frac{2e^2}{m}\int \mathrm{d}^3k|\widehat{\varphi}|^2\omega^{-1}\Big(E(p)+\frac{1}{2m}k^2\Big) \\ &\quad\times\big(E(p-k)-E(p)+\omega\big)^{-2} = c_0. \end{aligned} \tag{15.77}$$

Since $\mathrm{tr}[P_{\mathrm{g}}(1-P_0)] \leq \mathrm{tr}[P_{\mathrm{g}}N_{\mathrm{f}}] \leq c_0 \mathrm{tr}P_{\mathrm{g}}$, one concludes

$$(1-c_0)\mathrm{tr}P_{\mathrm{g}} \leq \mathrm{tr}[P_{\mathrm{g}}P_0] \leq 2. \tag{15.78}$$

If $c_0 < 1$, then $c_1 > 0$, with c_1 the constant in (15.75). Suppose $\mathrm{tr}P_{\mathrm{g}} = 1$. Then P_{g} projects along ψ and $P_0P_{\mathrm{g}}P_0$ along $P_0\psi$ which contradicts (15.75). Thus $\mathrm{tr}P_{\mathrm{g}} \geq 2$. On the other hand if $c_0 < \frac{1}{3}$, then $\mathrm{tr}P_{\mathrm{g}} < 3$. In conjunction, $\mathrm{tr}P_{\mathrm{g}} = 2$ as was to be shown. □

An alternative approach would be to use the positive commutator technique as explained at the end of section 15.2. It says that, provided $|e| < e_0$, $|p| < p_{\mathrm{c}}$, the ground state of H_p is exactly two-fold degenerate and that in a band above the ground state energy there is only an absolutely continuous spectrum.

Notes and references

Section 15.1

Our discussion of the soft photon bound is taken from Bach (private lecture notes) and Bach, Fröhlich and Sigal (1998a). If the potential V is attractive, but so weak that H_{at} has no ground state, then a sufficiently strong coupling to the radiation field will generate a ground state, since the mass of the particle is effectively increased (Hiroshima and Spohn 2001; Hainzl 2002; Hainzl *et al.* 2003; Chen *et al.* 2003). The property of e^{-tH} to be positivity improving is not known to hold under additional terms, for instance including an external vector potential or spin. As explained to us by V. Bach, a soft photon bound as in Theorem 15.1 automatically estimates the overlap with the Fock vacuum. If $|e|$ is sufficiently small, this overlap is larger than $1/2$ and uniqueness is guaranteed.

With the Maxwell field replaced by a scalar field, compare with section 19.2, ground state properties are investigated in Gérard (2000) and Betz *et al.* (2002), where references to earlier work are given.

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Section 15.2

The key properties of the energy–momentum relation are established in Fröhlich (1974), where also the missing points of rigor are supplied. In fact Fröhlich discusses Nelson's model of a particle coupled to a scalar field, compare with section 19.2. In that case, as for example explained in Spohn (1988), $e^{-tH(p)}$, $t > 0$, is positivity improving in Fock space, within an exponentially small error e^{-t}. From this property uniqueness of the ground state ψ_p is deduced by the argument explained in section 14.3. The overlap argument of property (ix) is a substitute which works only for small e. In his recent PhD thesis Chen (2001) establishes that $E(p)$ has a limit as $m_{\rm ph} \to 0$. The limit function $E(p)$ is twice continuously differentiable for $|p|$ sufficiently small. Thus the effective mass of the electron remains well defined even in the physical case $m_{\rm ph} = 0$, under the restriction of small e and, of course, an ultraviolet cutoff. An example where $p_{\rm c} = 1$ for $e = 0$ and $p_{\rm c} = \infty$ for $e > 0$ is the Fröhlich polaron in two dimensions (Spohn 1988). Positive commutator methods at fixed total momentum are developed in the highly recommended paper by Fröhlich, Griesemer and Schlein (2003), where the complete proof for the Nelson model, see section 19.2 for its definition, can be found. Positive-commutator methods and the related Mourre estimates are most useful also in cases where the electron is confined by an external potential. We refer to Skibsted (1998), Bach, Fröhlich and Sigal (1998b), Dereziński and Gérard (1999), Bach, Fröhlich, Sigal and Soffer (1999), and Georgescu, Gérard and Møller (2004). A precusor is Hübner and Spohn (1995b).

Section 15.3

The material is taken from Hiroshima and Spohn (2002).

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