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BEST POSSIBLE NETS IN A NORMED LINEAR SPACE[1]

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1. In this note we examine the question of the existence of a best possible N-net for a bounded set in a normed linear space. A sufficient condition for existence is given which leads to easy proofs of some of the standard results. The pertinent reference here is the paper by Garkavi [1].

2. Let E be a normed linear space and let M be a bounded set in E. Any system of N points in E will be called an N-net. For a given M and the net $S_N=\{y_1, y_2, \ldots y_N\}$ define

$$R(M, S_N) = \sup_{x\in M}\min_{1\le k\le N} \|x-y_k\|$$

and

$$R_N(M) = \inf_{S_N} R(M, S_N)$$

the infimum being taken over all N-nets in E. Since we assume M is bounded, $R_N(M)$ always exists. A net S_N^* such that $R(M, S_N^*)=R_N(M)$ is called a best N-net for M in E. If $N=1$, a best N-net is called a Čebyčev center for M in E. The existence question we consider is the determination of those spaces with the property that every bounded set has a best N-net, and in particular, those spaces where every bounded set has a Čebyčev center. We note that if a space E has a best N-net for all bounded M and if $N^*<N$, then E has a best N^*-net for all bounded M. (By an argument similar to that used in the last part of the proof of theorem I in [1].)

3. There is a particular topology for E that is related to the existence of a best N-net.

Definition. For a given bounded set $M=\{x_\alpha : \alpha\in A\}$ in E, define the $M(N)$ topology on E by taking as a subbase E and the complements of norm closed balls $B(x, r)$ where $x\in M$ and $r>0$ is such that there is a net $\bar{S}=\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N\}$ in E such that $\bar{S}\cap B(x_\alpha, r)\neq\varnothing$, for each $\alpha\in A$, the index set for M. Further require that $r\le R_N(M)+1$.

Definition. If for a fixed N, a space E is compact in every $M(N)$ topology, call E an $M(N)$ space.

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THEOREM 1. *If E is an $M(N)$ space, then every bounded set in E has a best N-net.*

Proof. Let $M^*=\{x_\alpha:\alpha\in A\}$ be a bounded set in an $M(N)$ space E. Then E is compact in the $M^*(N)$ topology. Let

$$B_\alpha^j=\left\{y\in E:\ \|y-x_\alpha\|\le R_N(M)+\frac{1}{j}\right\}$$

for $\alpha\in A$ and j a positive integer. Let

$$G_j=\left\{(y_1,y_2,\ldots,y_N)\in E^N:\ \bigcup_{i=1}^{N}\{y_i\}\cap B_\alpha^j\neq\phi,\qquad \forall\alpha\in A\right\}$$

for j a positive integer. For each j, G_j is clearly non-empty. We endow E^N with the product topology corresponding to the $M^*(N)$ topology on E. We now show that in this topology, G_j is closed. Let $\bar{x}=(\bar{y}_1,\bar{y}_2,\ldots,\bar{y}_N)$ be a point in E^N to which the net $\{x_p:p\in D\}$ converges, where $x_p\in G_j$ for all p and $x_p=\{y_1^p,y_2^p,\ldots,y_N^p\}$. We must show that for each α, there is an n, $1\le n\le N$, such that $\bar{y}_n\in B_\alpha^j$. Consider an arbitrary α and notice that for some n there is a subnet $\{x_p:p\in D'\}$ such that $y_n^p\in B_\alpha^j$ for all $p\in D'$. By the projection theorem for product spaces (see [2], page 91), $\{y_n^p:p\in D'\}$ is a net in B_α^j converging to $\bar{y}_n$. Since B_α^j is closed in the $M^*(N)$ topology, $\bar{y}_n\in B_\alpha^j$ and $\bar{x}\in G_j$. Thus G_j is closed. If $m\le n$ then $G_m\supset G_n$, so the G_j's have the finite intersection property. E^N is compact by Tychonoff's theorem, so $\bigcap_{j=1}^\infty G_j\neq\varnothing$. Every element in this intersection is a best possible N-net, so existence is established.

THEOREM 2. *If E is a conjugate space, then for each natural number N, E is an $M(N)$ space. Hence every bounded set in a conjugate space has a best N-net.*

Proof. Any $M(N)$ topology is weaker than the weak* topology for E. Let $\{\mathcal{O}_\beta:\beta\in B\}$ be a covering of E by elements of the subbase for the $M(N)$ topology. By Alexander's theorem, E is compact if such a covering always has a finite subcovering. Let $\mathcal{O}^*$ be any element of the covering. Then $\{\mathcal{O}_\beta:\beta\in B\}-\mathcal{O}^*$ covers $C(\mathcal{O}^*)$, a norm closed ball. Norm closed balls are compact in the weak* topology for E, so there is a finite subcovering of $C(\mathcal{O}^*)$. This finite sub-collection plus $\mathcal{O}^*$ then covers E, and E is compact.

There are non-conjugate spaces that are $M(N)$ spaces. c_0, the space of all sequences of complex numbers converging to zero, is such a space. To establish this, we need two geometric lemmas.

LEMMA 1. *Let $\{D_\alpha:\alpha\in A\}$ be a collection of discs in the complex plane, with radii $\{r_\alpha:\alpha\in A\}$ such that for some $R>0$, $r_\alpha<R$ for all $\alpha\in A$. If $I=\bigcap_{\alpha\in A}D_\alpha\neq\varnothing$ and $\inf_{z\in I}|z|\ge\varepsilon>0$, then there is a disc D^* in the collection such that $\inf_{z\in D^*}|z|\ge\delta(R,\varepsilon)>0$ where δ depends only on R and ε and $\delta(R,\varepsilon)<1$.*

Proof. Fix $R>0$ and $\varepsilon>0$, and suppose for a contradiction that for each $\delta>0$ there is a collection of discs with $\inf_{z\in I}|z|\ge\varepsilon$ and $\inf_{z\in D}|z|<\delta$ for each D in the

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collection. Consider a disc with center the origin and radius some $\delta>0$ and construct two more discs of radius R, so that the three discs are mutually tangent and no disc is contained by another. Let the distance from the tangent point of the last two discs to the origin be p. Then, using the supposition, for a collection of discs corresponding to δ, $\inf_{z\in I}|z|\leq p$, since if any point z_1 outside a distance p is in all the discs, so also is the point z_2 located a distance of p units from the origin and on the ray from the origin through z_1. But $p\to 0$ as $\delta\to 0$, a contradiction.

LEMMA 2. *Let $\{D_\alpha:\alpha\in A\}$ be a collection of discs in the complex plane with radii $\{r_\alpha:\alpha\in A\}$ such that for some $R>0$, $r_\alpha\leq R$ for all $\alpha\in A$. If $\bigcap_{\alpha\in A} D\neq\varnothing$ and $\inf_{z\in I}|z|\geq\varepsilon>0$, then there is a $\delta(R,\varepsilon)$ with $0<\delta(R,\varepsilon)<1$ such that $I'=\bigcap_{\alpha\in A} D'_\alpha\neq\varnothing$ and $\inf_{z\in I'}|z|\geq\varepsilon/2$ where the D'_α's are discs, concentric with the D_α's and with radii $\{r'_\alpha:\alpha\in A\}$ such that $r_\alpha\leq r'_\alpha\leq r_\alpha+\delta(R,\varepsilon)$.*

Proof. This is a consequence of the fact that the length of the longest line segment within an annulus with outer radius bounded above by R, approaches zero as the thickness of the annulus approaches zero.

THEOREM 3. *c_0 is an $M(N)$ space, where N is any natural number.*

Proof. We show that c_0 is compact in every $M(N)$ topology by showing that for a fixed M, every covering of the space by elements of the corresponding subbase has a finite subcover. Let $\{\mathcal{O}_\gamma:\gamma\in\Gamma\}$ be such a covering where $\mathcal{O}_\gamma=C(B(x_\gamma, r_\gamma))$ and for each γ, $x_\gamma=\langle x^1_\gamma, x^2_\gamma,\ldots, x^k_\gamma,\ldots\rangle$ is a sequence in M. There are two possibilities:

(1) For some natural number ℓ, $\{\mathcal{O}^l_\gamma:\gamma\in\Gamma\}$ is a cover of the complex plane, where $\mathcal{O}^l_\gamma=C(B(x^l_\gamma, r_\gamma))$.

(2) $\bigcap_{\gamma\in\Gamma} B(x^l_\gamma, r_\gamma)\neq\varnothing$ for each natural number ℓ but every sequence $\langle y_1, y_2,\ldots, y_l,\ldots\rangle$ such that $y_l\in\bigcap_{\gamma\in\Gamma} B(x^l_\gamma, r_\gamma)$ for each ℓ, fails to converge to zero.

The compactness in the norm topology of closed balls in the complex plane yields a finite subcover for c_0 in case 1. We show that case 2 is impossible. We therefore suppose that case 2 holds. Then given any $\varepsilon>0$, there are an infinite number of natural numbers m such that $\inf_{z\in I(m)}|z|\geq\varepsilon$, where $I(m)=\bigcap_{\gamma\in\Gamma} B(x^m_\gamma, r_\gamma)$. Set $r'_\gamma=\max\{r_\gamma, R_N(M)+\delta(R_N(M)+1,\varepsilon)\}$ where $\delta(R_N(M)+1,\varepsilon)$ is given by lemma 2. Then the lemma implies that $\{C(B(x_\gamma, r'_\gamma)):\gamma\in\Gamma\}$ covers c_0. Note that $\bigcap_{\gamma\in\Gamma} B(x^l_\gamma, r'_\gamma)\neq\varnothing$ for each ℓ. Lemma 1 implies the existence of a $\delta^*>0$ such that for each m with $\inf_{z\in I(m)}|z|\geq\varepsilon$, a ball $B(m)$ can be chosen with the property that $\inf_{z\in B(m)}|z|\geq\delta^*$. Here $B(m)=B(x^m_{f(m)}, r'_{f(m)})$ where f is an appropriate choice function. Now the collection of balls $\{B(x_{f(m)}, r'_{f(m)})\}$ is infinite. By the definition of the $M(N)$ topology, an infinite subset must have non-empty intersection. This is impossible, since no sequence converging to zero can be in an infinite number of the above balls. This concludes the proof.

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We note that any space that is compact in the topology generated by the complements of all the norm closed balls will be an $M(N)$ space for every N. It is apparently unknown if there are non-conjugate spaces with this property. It is easily verified that c_0 is not compact in such a strong topology.

4. When $N=1$, theorem 1 becomes both necessary and sufficient.

THEOREM 4. *Every bounded set M in the space E has a Čebyčev center if and only if E is an $M(1)$ space.*

Proof. It remains to show that if bounded sets always have Čebyčev centers, then the space is an $M(1)$ space. But this is clear since for each M, the closed balls defining the topology all contain a point in common (any Čebyčev center), so compactness follows immediately.

5. In [1] it is stated that the bounded set of functions

$$f_n(t) = \begin{cases} (a+t)\sin\dfrac{1}{t} & \text{for } \dfrac{1}{2\pi n} \le t \le 1 \\ 0 & \text{for } 0 \le t \le \dfrac{1}{2\pi n} \end{cases}$$

where $n=1, 2, \ldots$, has no Čebyčev center in $C[0, 1]$. This is fallacious since for $a>0$, choose $c<a$ and of the form $1/2\pi k$ where k is a positive integer greater than or equal to two, and define

$$g(t) = \begin{cases} (a+t)\sin\dfrac{1}{t} & \text{for } \dfrac{1}{2\pi} \le t \le 1 \\ \dfrac{a+t}{2}\sin\dfrac{1}{t} & \text{for } c \le t \le \dfrac{1}{2\pi} \\ t\sin\dfrac{1}{t} & \text{for } 0 < t \le c \\ 0 & \text{for } t = 0. \end{cases}$$

Then $g(t)$ is a Čebyčev center for the collection of functions. If $a=0$, take $c=0$ and define $g(t)$ similarly.

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