Structuralism and Conceptual Change in Mathematics

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Professor Grosholz packs a lot into her interesting and suggestive paper "Formal Unities and Real Individuals" (Grosholz 1990b). In the limited space available I can comment briefly on its several parts, or direct more substantive comments at a single issue. I will opt for the latter; specifically, I want to address her critique of mathematical structuralism, as found especially in the writings of Michael Resnik.

I begin with a brief, hence necessarily caricatured, summary of Resnik's influential view. According to structuralism, the subject matter of a mathematical theory is a given *pattern*, or *structure*, and the objects of the theory are intrinsically unstructured points, or *positions*, within that pattern. Mathematical objects thus have no identity, and no intrinsic features, outside of the patterns in which they occur. Hence, they cannot be given in isolation but only in their role within an antecedently given pattern, and are distinguishable from one another only in virtue of the relations they bear to one another in the pattern (see, e.g., Resnik (1981)).

Grosholz's judgment on Resnik's structuralism is rather severe: "Neither points or positions, nor structures, seem to be the kind of thing about which one could pose an interesting mathematical problem." The reason for this, in the case of points, is simply that they have no intrinsic features; there simply is nothing to say about them *per se*. But since there is nothing to say about points, she continues, then because there is nothing more to structures than relations between points, there won't be anything interesting to say about structures, since these are relations among entities about which there is nothing to say.

The problem Resnik has run into, then, is that he has failed to locate a "middle ground" of genuine mathematical objects that can be given independently, and which are internally structured, internally complex. For, as any mathematician knows, "the true objects of a mathematical domain are those entities about which problems arise." And in contrast to Resnik's bland, unstructured points, these objects are "profoundly interesting," at once recalcitrant, mysterious, and revelatory; they are complex unities that are greater than the sums of their parts. A consequence of this, Grosholz notes, is that structuralism is woefully inadequate vis-a-vis the history of mathematics. For "the founding of research programs in mathematics, indeed the very possibility of

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mathematical knowledge, requires the existence of objects that are yet isolable, internally articulated unities."

Though I share some of Grosholz's concerns, I think her critique seriously misrepresents both the character and richness of Resnik's view. First of all, Grosholz reads too much into the centrality of points in Resnik's account. For I think the featurelessness of points leads her to conclude that, despite the importance of internal complexity exhibited by a structure in his theory, insofar as we try to objectify that structure, we must identify it with a point, and hence whatever structure we had was lost. Thus, she infers, Resnik finds himself hoist on his own petard: despite the obvious conceptual centrality of internal structure in mathematical research, it must be relegated to a secondary role in mathematical ontology, just out of Resnik's theoretical reach. Quite to the contrary, however, Resnik doesn't lose internal structure, at least nothing that is important in the notion; he just places it elsewhere.

Grosholz is correct to a certain extent: structures are in a precise sense secondary with respect to mathematical theories, if we say that the primary objects of a theory are those things it quantifies over, those objects in its intended domain. A structure is thus, so to say, what a theory is *about*, its subject matter, but not an object of the theory, as Peano Arithmetic is about, but does not quantify over, the natural number structure, and ZF is about, but does not quantify over, the iterative hierarchy of sets. In Wittgenstein-ese, a theory's subject matter is *shown*, but not spoken of. Grosholz's objection seems to be that structures thus remain forever shown, never spoken of, and hence, on Resnik's account, cannot play an appropriately primary role in mathematical research. However, this would be so only if the secondary character of structures were absolute. But in Resnik's account there is also a precise sense in which structures are nonetheless *primary* mathematical objects, hence structureless points, and yet with all the information Grosholz packs into the notion of internal complexity intact.

How so? It is a well known fact of model theory that what is shown at one level can be spoken of at another. That is, a structure S that constitutes the subject matter of one theory, and hence is not an object of that theory, can in fact become an object of a further theory capable of overtly expressing the relations between S and its points. But then what is the subject matter of *this* theory? It is a *new* structure that contains all the points of S, and a further point as well that plays the role of S itself qua mathematical object, in addition to new relations that depict the relations between S and its points. The internal complexity of mathematical objects does not disappear, but simply manifests itself as external complexity between further structureless points. The original structure itself becomes "positionalized," as Resnik has put it in a recent paper (Resnik 1988, p. 410).

Let's fix ideas with some examples. Consider the natural number structure (N,<), which we might represent pictorially as follows (taking the arrow to be transitive):

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \ldots$$

Now, Grosholz wants the structure itself to be an internally articulated mathematical object, which we might represent so:

$$[\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \ldots]$$

Since there are no such internally structured objects for Resnik, this kind of representation is unavailable. But internal structure is just more relations. Thus, when we

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want explicitly to objectify the natural number structure, we consider it not in itself as an *isolated* point, but in relation to other points in a new pattern:

Intrinsically, then, qua mathematical object, (N,<) is indeed a structureless point; but the information that Grosholz packs into the internal structure of her mathematical objects is nonetheless perfectly transparent within the larger pattern.

A second example: the right triangle. For Grosholz, a triangle has an

> endlessly interesting internal composition,...the points...that bound its sides, the lines that join its vertices, the angles that exist between those lines. That whole has the unity of shape that allows it to be given in a diagram, in isolation from all the other possible objects of geometrical study.

And again comes the charge that, because the objects of structuralism are featureless points, the internal composition of the right triangle that is exhibited so clearly in concrete diagrams is lost. But this attack against the structuralist is similarly misguided. *Qua* structure, the right triangle pattern has a rich and complex internal composition with vertex positions, line positions, angle positions, and their associated relations. Considered as part of a further pattern, *qua* object, the right triangle is indeed just a featureless point within the larger structure, but one which nonetheless bears all the appropriate structural relations to the vertex, line, and angle positions of the original structure—if the larger structure really does incorporate the right triangle, then it must exhibit all of the complexity of the triangle's structure as well.

From this perspective, there seems little that distinguishes Grosholz's picture from Resnik's in regard to the matter of internal complexity. The general point is that the complexity we associate with a mathematical object lies not in the object considered in isolation, but in *relations* that hold between several (perhaps many) associated objects. Given that we can objectify patterns, whether we choose to build this complexity directly into the object itself, and call this its internal structure, or reserve complexity for patterns alone, seems more a matter of taste than substance.

This failure to appreciate the richness and flexibility of Resnik's structuralism lies behind Grosholz's charges of explanatory inadequacy vis-a-vis the history of mathematics. If all mathematical objects are mere homogenous points, the argument seems to be, then we lose what is unique to each mathematical domain, and hence we cannot explain how distinct domains differ in a way that allows for mathematical progress and change. Her arguments here tie in with her own intriguing notion of a hybrid (Grosholz 1985, 1990a). According to Grosholz, a *hybrid* is a mathematical object that, as it were, straddles two or more distinct mathematical domains. Such objects typically arise when problems insoluble in one domain can be approached within another, overlapping domain. Thus, the tools of analytic geometry can solve problems relating to curves, e.g., parabolas, that are insoluble in classical Euclidean geometry. But the curves of analytic geometry are hybrids, neither wholly geometrical, nor wholly algebraic. The crucial point in this context is that such curves must be considered *autonomous and irreducible*, related to, but distinct from, Euclidean curves. To reduce curves generally to structures or positions in structures is to strip them of this ambiguous character that explains mathematical progress. Thus, Grosholz charges, Resnik

> cannot explain the synthesis of geometry and algebra in the work of Descartes,...or the synthesis of geometry, algebra, number theory and mechanics in the late seventeenth century writings of Leibniz, for those partial unifications were posed and elaborated in terms of isolable, internally complex mathematical objects.

But in light of the discussion above, the charge here will not stick; to the contrary, structuralism can provide an appealing account of mathematical change, in particular the specific sorts of change that lead Grosholz to her hybrids. The overlap of distinct domains, for the structuralist, is just co-occurrence of patterns.[1] The distinctive character of a co-occurring pattern in different contexts is explained by differences in the ways the pattern occurs (e.g., as a spatially located diagram or a point set), the tools that are used to study it in these different settings (ruler and compass vs. algebra or differential calculus), and the problems broached. Advances in tools especially can open up hitherto inexpressible avenues of research in the study of a given pattern or cluster of patterns. Mathematical progress generally thus consists in the acquisition of deeper insight into the nature of a certain pattern, or cluster of patterns; progress and change of the sort Grosholz seizes upon in particular occurs when such new insight is acquired by recognition of pattern co-occurrence across distinct domains, and the corresponding development of new and more powerful tools and systems of representation. For the structuralist there is therefore no need to postulate a special hybrid object "hovering ambiguously" between distinct domains that share similar structure. One need not go beyond the structure *simpliciter*. This is not to say that the notion of a hybrid has no significant philosophical role to play. It is especially noteworthy as a device for characterizing the above sort of conceptual advance in mathematics. Grosholz errs, I think, in investing it with ontological, rather than simply conceptual, significance.

Now, of course, to analyze the advances of past mathematical pioneers in structuralist terms is not to say that they understood what they were doing along overtly structuralist lines. However, Grosholz appears to saddle the structuralist with such a thesis when she points out that

> Descartes' Cartesian parabola [and] Leibniz' tractrix...were not treated as congruences in which the positions or points of one pattern are mapped onto or occur within the positions or points of another pattern.

Structuralism is a *metaphysical* thesis, a thesis about what mathematical objects *are*, not a historical thesis. Though we should certainly be able to understand the historical development of mathematics in structuralist terms—e.g., along the lines sketched in the previous paragraph—nothing at all follows about the views of Descartes, Leibniz, and their fellows in regard to the nature of mathematical objects. How Descartes and the other pioneers did in fact conceive the denizens of the mathematical universe is no doubt crucial for understanding the actual historical development of mathematics.[2] All the structuralist needs to argue is that these conceptions, however prominent in the minds of the pioneers, might diverge from the metaphysical facts of the matter.

Finally, it should be mentioned that the structuralist—Resnik in particular—can do more than simply defend against Grosholz's charges. For in addition, structuralism promises—if still imperfectly—some significant advantages over Grosholz's position. First, it brings an appealing unity to mathematics: all mathematics is ultimately grounded in patterns. Its diversity, as argued, springs from the great multiplicity of patterns there are, the way they cluster into separate domains, the great variety of ways in which they can be represented, and the great variety of tools by means of which they can be studied. Second, perhaps more important, structuralism potentially yields considerable epistemological dividends, since knowledge of patterns generally can be grounded in simple perceptual situations. Given the ontological unity of mathematics according to structuralism, this in turn promises an epistemological ground for all of mathematics. Resnik thus purports to tell us what mathematical objects are, and to do so in a way that does justice to the richness of mathematics while yet making them epistemologically tractable. Grosholz, by contrast, seems to be subject to the usual problems that plague her—to all appearances—rather traditional variety of platonism.

Resnik's structuralism is not without its difficulties; but it appears to be undiminished by Grosholz's attacks.

Notes

[1]Grosholz (forthcoming) seems to admit as much herself: of Leibniz' application of the calculus to the sphere of mechanics she writes, "[t]hus another domain is brought into alignment with number theory, algebra and geometry, *sharing structure with them*...but maintaining a partial independence as well" (my emphasis).

[2]Indeed, the notion of a hybrid might be useful here as well. In treating a curve as an infinite-sided polygon, Leibniz arguably wasn't conceiving of it *simply* as a Euclidean plane figure, or an object of analytic geometry, since those domains do not permit such a representation. By the same token, it would be anachronistic to characterize him as conceiving of his discovery as consisting in deeper knowledge of a single pattern. It might thus be appropriate to characterize his own conception as directed toward a new object—a hybrid—closely related to its fellows, but nonetheless distinct, with a similar but distinct internal articulation.

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