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A NOTE ON ADDITIVE MAPPINGS IN NONCOMMUTATIVE FIELDS

J. VUKMAN

In this paper we prove a result concerning the Cauchy functional equation, that is the functional equation $f(x + y) = f(x) + f(y)$, in skew fields with characteristic not two.

This research has been inspired by the work of S. Kurepa [2].

THEOREM. *Let* F *be a skew field of characteristic not two and let* $f: F \to F$ *be an additive mapping such that the relation*

$$f(a) = -a^2 f(a^{-1}) \tag{1}$$

holds for all nonzero $a \in F$. *Then we have* $f(a) = 0$ *for all* $a \in F$.

Proof. We intend to prove that

$$(ab - ba)af(a) = a(ab - ba)f(a) \tag{2}$$

holds for all pairs $a,b \in F$. For the proof of (2) we need several steps. The first step is to prove that

$$f(a^2) = 2af(a) \tag{3}$$

holds for all $a \in F$. Since the characteristic of the field is not two it follows immediately that

$$f(1) = 0 . \tag{4}$$

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We have also $f(0) = 0$. Hence we may assume that $a \neq 0$ and $a \neq 1$. In this case we have

$$a^2 = a - (a^{-1} + (1 - a)^{-1})^{-1} .$$

Then from the additivity of f and from (1) it follows that

$$f(a^2) = f(a) - f((a^{-1} + (1 - a)^{-1})^{-1}) = f(a)$$
$$+ (a^{-1} + (1 - a)^{-1})^{-2} f(a^{-1} + (1 - a)^{-1})$$
$$= f(a) - (1 - a)^2 a^2 a^{-2} f(a) - a^2 (1 - a)^2 (1 - a)^{-2} f(1 - a)$$
$$= f(a) - (1 - a)^2 f(a) + a^2 f(a) = 2af(a) .$$

Thus the relation (3) is proved. Replacing a by $a + b$ in (3) one obtains easily that

(5) $$f(ab + ba) = 2af(b) + 2bf(a), \quad a,b \in F .$$

Let us prove that

(6) $$f(aba) = a^2 f(b) + 3abf(a) - baf(a)$$

holds for all pairs $a,b \in F$. From (5) it follows that

$$f(a(ab + ba) + (ab + ba)a) = 2af(ab + ba) + 2(ab + ba)f(a)$$
$$= 4a^2 f(b) + 6abf(a) + 2baf(a) .$$

On the other hand we obtain, using (2) and (5), that

$$f(a(ab + ba) + (ab + ba)a) = f(a^2 b + ba^2 + 2aba)$$
$$= 2a^2 f(b) + 4baf(a) + 2f(aba).$$

By comparing these equations, we obtain relation (6). Let us write $a + c$ instead of a in (6). Then we have

$$f((a + c)b(a + c)) = (a + c)^2 f(b) + 3(a + c)bf(a + c)$$
$$- b(a + c)f(a + c)$$

and

$$f(aba) + f(cbc) + f(abc + cba) = a^2 f(b) + 3abf(a)$$
$$- baf(a) + c^2 f(b) + 3cbf(c) - bcf(c) + (ac + ca)f(b)$$
$$+ 3abf(c) + 3cbf(a) - baf(c) - bcf(a) .$$

Using (6) we obtain

(7) $$f(abc + cba) = (ac + ca)f(b) + 3abf(c) + 3cbf(a) - baf(c) - bcf(a)$$

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where a,b,c are arbitrary elements from F. All is prepared to prove that the relation

$$(8) \qquad (ab-ba)f(ab) = a(ab-ba)f(b) + b(ab-ba)f(a)$$

holds for all pairs $a,b \in F$. Let us write A for $f(ab(ab) + (ab)ba)$. Then from (7) we obtain

$$A = (a(ab) + (ab)a)f(b) + 3abf(ab) - baf(ab) + 3ab^2f(a) - babf(a) .$$

On the other hand since $A = f((ab)^2 + ab^2a)$ we obtain, using (3) and (6), that

$$A = f((ab)^2) + f(ab^2a) = 2abf(ab) + a^2f(b^2) + 3ab^2f(a)$$
$$- b^2af(a) = 2abf(ab) + 2a^2bf(b) + 3ab^2f(a) - b^2af(a) .$$

By comparing these equations, we obtain (8). Let us write $a + c$ instead of a in (8). We have

$$((a + c)b - b(a + c))f((a + c)b)$$
$$= (a + c)((a + c)b - b(a + c))f(b)$$
$$+ b((a + c)b - b(a + c))f(a + c)$$

which implies

$$((ab - ba) + (cb - bc))(f(ab) + f(cb))$$
$$= (a(ab - ba) + c(ab - ba) + a(cb - bc) + c(cb - bc))f(b)$$
$$+ (b(ab - ba) + b(cb - bc))(f(a) + f(c)) .$$

Now it is obvious that from (8) we obtain

$$(cb - bc)f(ab) + (ab - ba)f(cb) = c(ab - ba)f(b)$$
$$+ a(cb - bc)f(b) + b(cb - bc)f(a) + b(ab - ba)f(c) .$$

If we put $b = a$ in the relation above, we obtain

$$(ca - ac)f(a^2) = 2a(ca - ac)f(a)$$

which proves (2) since (3) holds and since the characteristic of the field is not two.

Relation (2) makes it possible to use Lemma 1.1.9 of [1]. Let us assume that $f(a) \neq 0$ for some $a \in F$. Then from (2) it follows that $(ab - ba)a = a(ab - ba)$ holds for all $b \in F$. Hence since a commutes with all its own commutators $ab - ba$ we can conclude that a is in the centre of F by Lemma 1.1.9 in [1]. Let us take $b \in F$ which is not in the centre of F. Then $a + b$ is not in the centre. Now we have $f(b) = 0$ and $f(a + b) = 0$, otherwise b and $a + b$ would be in the centre of F. Since f is additive we have finally

$0 = f(a + b) = f(a) + f(b) = f(a)$ which contradicts $f(a) \neq 0$. The proof of the theorem is complete.

As we have mentioned, this work has been inspired by the work of S. Kurepa [2] where additive mappings with the additional requirement (1) on the real field are considered. One can prove that in the case where f is an additive mapping with the addition requirement (1) on a commutative field with characteristic not two it follows that f is a derivation (that is, an additive mapping such that $f(ab) = f(a)b + af(b)$ for all a and b). This is obvious from the beginning of the proof of the Theorem (see also [2]). Therefore, since it is well-known that there exist commutative fields with nonzero derivations, the assumption that the field is noncommutative is essential in the Theorem.

References

[1] I.N. Herstein, *Rings with involution*, (University of Chicago Press, 1976).

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University of Maribor
VEKŠ
Razlagova 14
62000 MARIBOR
YUGOSLAVIA

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