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ON DETERMINING THE GROWTH OF MEROMORPHIC SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS HAVING ARBITRARY ENTIRE COEFFICIENTS(1)

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1. Introduction: In this paper, we treat the problem of determining the rate of growth of meromorphic functions on the plane, which are solutions of n^{th} order algebraic differential equations whose coefficients are arbitrary entire functions (i.e., equations of the form, $\Omega(z, y, dy/dz, \cdots, d^n y/dz^n) = 0$, where Ω is a polynomial in $y, dy/dz, \cdots, d^n y/dz^n$ whose coefficients are arbitrary entire functions of z.)

One attack on this problem has been to restrict the class of equations considered. For example, in [8; pp. 221–223], Valiron (and also Wittich [9; pp. 70–71]) considered a very special class of n^{th} order algebraic differential equations having polynomial coefficients, and it was shown that all entire solutions of equations in this special class were of finite order of growth. Of course, for $n > 1$, arbitrary n^{th} order equations with polynomial coefficients may possess entire solutions of infinite order. (See also Nikolaus [7; p. 625].) More recently, Yang [10; p. 6] treated a special class of n^{th} order equations (with restrictions similar to those imposed by Valiron and Wittich), having arbitrary coefficients, and he obtained results on the growth of the logarithmic derivative of certain solutions.

In our investigation here, no restrictions are imposed on the form of the equations we treat, and we seek to determine what factors affect the growth of a solution. The second fundamental theorem of Nevanlinna [5; p. 69] (or [6; p. 261, Formula (1.1)] shows that the growth of an arbitrary meromorphic function $f(z)$ in the plane (regard-

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less of whether it solves an algebraic differential equation) can be estimated if for three distinct values of λ (finite or infinity), one knows the growth of the counting functions $N(r, \lambda)$ for the λ-points of f. However, when $f(z)$ is a solution of a first-order algebraic differential equation with entire coefficients, it was shown in [3] that the growth of f can be estimated in terms of the counting function $N(r, \lambda)$ for just the *one* value $\lambda = \infty$, and the growth of the coefficients in the equation. (For the reader's convenience, this result from [3] is stated in §3 below. For the special case of entire solutions (i.e., $N(r, \infty) = 0$), this result can be found in [1; p. 109], in a slightly different formulation). The fact that $N(r, \infty)$ must be involved in the estimate for the growth of an arbitrary meromorphic solution is indicated by the following phenomenon: In the special case of first-order equations whose coefficients are entire functions of finite order, it was shown in [1; p. 116] that for any entire solution, or more generally, for any meromorphic solution f whose sequence of poles has a finite exponent of convergence, the estimate, $T(r, f) = O(\exp r^A)$ holds for some constant A as $r \to +\infty$. However, it was shown in [4], that no such uniform growth estimate exists for *arbitrary* meromorphic solutions of such equations, since for any preassigned function $\Phi(r)$ on $(0, +\infty)$, one can construct a meromorphic solution of such an equation, whose Nevanlinna characteristic dominates $\Phi(r)$ at a sequence of r tending to $+\infty$.

In the case of equations $\Omega = 0$ of order higher than 1, having arbitrary entire coefficients, it was shown in [2; §3] (see §5 below), that regardless of the order of the equation, the growth of certain solutions can be estimated in terms of the counting functions $N(r, \lambda)$ for the *two* values $\lambda = 0, \infty$, and the growth of the coefficients in the equation. This result holds for those solutions of an n^{th} order equation $\Omega = 0$, which fail to be solutions of some equation $\Omega_q = 0$, where Ω_q is the homogeneous part of Ω of total degree q in the indeterminates $y, dy/dz, \cdots, d^n y/dz^n$. One of the main results of the present paper (§6 below) shows that the above result can be extended to *any* meromorphic solution of any second-order algebraic differential equation with entire coefficients (i.e., the growth of any solution can be estimated in terms of the two counting functions $N(r, 0)$ and $N(r, \infty)$ and the growth of the coefficients in the equation. For the precise statement of how the growth can be so estimated, see §6 below.)

In §9, we treat equations $\Omega = 0$ of order higher than two. As mentioned above, the growth of those solutions which fail to be solutions of some equation $\Omega_q = 0$ can be estimated in terms of $N(r, 0)$, $N(r, \infty)$ and the growth of the coefficients. It is easy to see that this result cannot be extended to all solutions of all algebraic differential equations. For example, if we set $g_1(z) = e^z$, and $g_{n+1}(z) = \exp\left(\int_0^z g_n(\zeta)d\zeta\right)$ for $n \geq 1$, then by induction it is easy to verify that each g_n satisfies an n^{th} order equation with constant coefficients and that g_n has no zeros and no poles (i.e., $N(r, 0) = 0$, $N(r, \infty) = 0$). Hence since the growth of g_{n+1} is roughly like the exponential of the growth of g_n, it is clear that just knowing $N(r, 0)$, $N(r, \infty)$ and the growth of the coefficients in the equation, cannot lead to an estimate on the growth of a solution. In §9, we investigate the other quantities which are required in estimating the growth of a meromorphic solution f of an n^{th} order equation where $n > 2$. As the above example indicates, these other quantities involve the counting functions for the zeros of certain successive logarithmic derivatives, $f_1 = f'/f$, $f_2 = f_1'/f_1, \cdots, f_k = f_{k-1}'/f_{k-1}$ for some $k \leq n-2$. In §9, we discuss the precise determination of k, that is, the number of these counting functions that actually can play a role in determining the growth of the solution f.

2. Notation: For a meromorphic function $f(z)$ on the plane, we will use the standard notation for the Nevanlinna functions $m(r, f)$, $N(r, f)$ and $T(r, f)$ introduced in [5; pp. 6, 12]. We will also use the notation $n(r, f)$ to denote the number of poles (counting multiplicity) of f in $|z| \leq r$. As in [2], we shall say that a certain property $P(r)$ holds "nearly everywhere" (briefly, n.e.) if $P(r)$ holds for all $r \geq 0$ with the possible exception of a set of finite measure. We will make use of the following fact: If $g(r)$ and $h(r)$ are monotone nondecreasing functions on $(0, +\infty)$ such that $g(r) \leq h(r)$ n.e., then for any $a > 1$ there exists $r_0 > 0$ such that $g(r) \leq h(ar)$ for all $r > r_0$. (This follows very easily, for if σ is the measure of the exceptional set E, then for any $r > \sigma/(a-1)$, the interval $[r, ar]$ cannot be contained in E.)

3. The following result was proved in [3]:

THEOREM: *Let* $\Lambda(z, y, y') = \sum f_{kj}(z) y^k (y')^j$ *be a polynomial in* y *and* y' *whose coefficients* $f_{kj}(z)$ *are entire functions. Let* $p = \max\{k + j:$

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$f_{kj} \not\equiv 0\}$. *Let* $M_1(r)$ *and* $M_2(r)$ *be monotone nondecreasing functions on* $[0, +\infty)$ *such that the following conditions hold n.e.:*

(A) $$M_1(r) \geq M_2(r) \geq 1 .$$

(B) $$|f_{kj}(z)| \leq M_1(r) \quad on \quad |z| = r \quad if \quad k + j < p .$$

(C) $$|f_{kj}(z)| \leq M_2(r) \quad on \quad |z| = r \quad if \quad k + j = p .$$

Let $m = \max\{j: f_{p-j,j} \not\equiv 0\}$ *and let* $A(r)$ *be a monotone nonincreasing function on* $[0, +\infty)$ *which satisfies* $A(r) > 0$ *on* $[0, +\infty)$ *and for which the following condition holds n.e.:*

(D) $$|f_{p-m,m}(z)| \geq A(r) \quad on \quad |z| = r .$$

Let $v(z)$ *be a meromorphic function on the plane which satisfies* $\Lambda(z, v(z), v'(z)) \equiv 0$. *Then for any real number* $a > 1$, *there exist positive constants* K *and* r_0 *such that for all* $r > r_0$, *we have*

(1) $$T(r, v) \leq K(J(ar)) ,$$

where

$$J(r) = \log^+ M_1(r) + r^2 M_2(r)/A(r) + rN(r, v) .$$

4. Definition and Notation:

(a) Under the hypothesis and notation of § 3, we will say that the triple of functions, (M_1, M_2, A) is a *bounding triple* for the first order differential polynomial Λ.

(b) If $\Omega(z, y, y', \cdots, y^{(n)})$ is a differential polynomial of order $\leq n$ (i.e., a polynomial in $y, y', \cdots, y^{(n)}$ whose coefficients are entire functions of z), then for each nonnegative integer q, we denote by Ω_q the homogeneous part of Ω of total degree q in the indeterminates $y, y', \cdots, y^{(n)}$. We say Ω is *non-trivial* if at least one coefficient of Ω is not identically zero. Now by induction, it is easy to see that if we set $w = y'/y$, then for each $j \geq 1, y^{(j)}/y$ can be written as a polynomial in $w, w', \cdots, w^{(j-1)}$ with nonnegative integer coefficients. Hence if $q \geq 0$, and we divide the homogeneous polynomial Ω_q by y^q, and set $w = y'/y$, it easily follows that we obtain a differential polynomial in $w, w', \cdots, w^{(n-1)}$, whose coefficients belong to the additive group generated by the coefficients of Ω_q. We denote this differential polynomial of order $\leq n - 1$ by $[\Omega_q]$. We require the following fact: If Ω_q is non-trivial, then $[\Omega_q]$

is non-trivial. This can be seen as follows: Since Ω_q is non-trivial, we can assume that some coefficient of Ω_q is non-vanishing at the origin (by dividing Ω_q, if necessary, by some positive power of z). If $[\Omega_q]$ were trivial, then clearly every meromorphic w solves $[\Omega_q]=0$. But then if y is any meromorphic function (not identically zero), $w=y'/y$ would solve $[\Omega_q]=0$ and hence clearly y would solve $\Omega_q=0$. (Since $q>0$, $y\equiv 0$ also solves $\Omega_q=0$.) Thus every polynomial $Q(z)$ would solve $\Omega_q=0$. Hence if $x_0,\cdots,x_n$ are any complex numbers, then $\sum_{j=0}^{n} x_j z^j$ would solve $\Omega_q=0$. Substituting into Ω_q and evaluating at $z=0$, we would obtain a polynomial in $x_0, x_1,\cdots,x_n$, where the coefficient of any term $x_0^{j_0}x_1^{j_1}\cdots x_n^{j_n}$ is $(2!)^{j_2}\cdots(n!)^{j_n}$ times the value at $z=0$ of the coefficient in Ω_q of the term $y^{j_0}(y')^{j_1}\cdots(y^{(n)})^{j_n}$. Since the polynomial in x_0, $x_1,\cdots,x_n$ is identically zero, it follows that each coefficient of Ω_q would vanish at the origin which contradicts our initial assumption. Hence $[\Omega_q]$ must be non-trivial.

5. The result proved in [2; §3], when combined with the fact stated in §2 above, can be stated as follows:

THEOREM: *Let $\Omega(z,y,y',\cdots,y^{(n)})$ be a non-trivial differential polynomial whose coefficients are any entire functions of z. For each $r>0$, let $\Phi(r)$ be the maximum of the Nevanlinna characteristics of the coefficients of Ω. Let $u(z)$ be a meromorphic function on the plane which is not identically zero and which satisfies the equation $\Omega=0$, but which for some nonnegative integer q does not satisfy the equation $\Omega_q=0$. Then for any real number $a>1$, there exist positive constants K_1 and r_1 such that for all $r>r_1$, we have*

$$T(r,u)\leq K_1[N(ar,u)+N(ar,1/u)+\Phi(ar)+\log r]\,. \tag{2}$$

6. We now state our main result for second-order algebraic differential equations:

THEOREM: *Let $\Omega(z,y,y',y'')$ be a non-trivial differential polynomial whose coefficients are entire functions of z. For each $r>0$, let $\Phi(r)$ be the maximum of the Nevanlinna characteristics of the coefficients of Ω. Let $y_0(z)$ be a meromorphic function on the plane which is not identically zero and which satisfies the equation $\Omega=0$, Then:*

(a) *If for some nonnegative integer q, $y_0(z)$ does not satisfy the*

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equation $\Omega_q = 0$, *then for any* $a > 1$, *there exist constants* K_1 *and* r_1 *such that for all* $r > r_1$,

$$(3) \qquad T(r, y_0) \leq K_1[N(ar, y_0) + N(ar, 1/y_0) + \Phi(ar) + \log r] .$$

(b) *If for all nonnegative integers* q, $y_0(z)$ *is a solution of* $\Omega_q = 0$, *then let* (M_1, M_2, A) *be any bounding triple for any non-trivial* $[\Omega_q]$. *(At least one such* q *exists by* § 4(b) *since some* Ω_q *is non-trivial.) Then for any* $a > 1$, *there exist positive constants* K, K_1 *and* r_0 *such that for all* $r > r_0$,

$$(4) \qquad T(r, y_0) \leq K(\exp(K_1 E(ar))) ,$$

where

$$E(r) = \log^+ M_1(r) + r^2 M_2(r)/A(r) + (N(r, y_0) + N(r, 1/y_0)) (r + \log^+ N(r, y_0) + \log^+ N(r, 1/y_0)) .$$

Hence, in any case, the growth of a solution y_0 *can be estimated in terms of the counting functions* $N(r, y_0)$ *and* $N(r, 1/y_0)$ *for the poles and zeros respectively, and the growth of the coefficients in the equation.*

Before proving the above result, we first prove a lemma which estimates the growth of an arbitrary meromorphic function in terms of the growth of its logarithmic derivative.

7. LEMMA: *Let* $y(z)$ *be any meromorphic function in the plane which is not identically zero, and let* $w = y'/y$. *Then for any* $a > 1$, *there exist positive constants* c, c_1 *and* r_0 *such that for all* $r > r_0$,

$$(5) \qquad T(r, y) \leq c(rN(ar, y) + r^2 \exp(c_1 \Psi(ar))) ,$$

where

$$\Psi(r) = T(r, w) + N(r, w) \log r + N(r, w) \log^+ N(r, w) .$$

Proof. Clearly we can assume $w \not\equiv 0$.

Given $a > 1$, let $\sigma > 1$ be such that $\sigma^3 = a$. Let $\{a_n\}$ and $\{b_m\}$ be the sequences of zeros and poles respectively of w in the plane (each arranged in order of increasing moduli). Let $r > 0$, and let $z = re^{i\theta}$ be any point on $|z| = r$ which is not a zero or pole of w. Then if $R = \sigma r$, we have by the Poisson-Jensen formula [5; p. 3] that,

$$\begin{aligned}\log|w(z)| &= (1/2\pi)\int_0^{2\pi} \log|w(Re^{i\varphi})|\, G(r,R,\theta,\varphi)d\varphi \\ &\quad - \sum_{|a_n|<R} \log\left|\frac{R^2-\bar{a}_n z}{R(z-a_n)}\right| + \sum_{|b_m|<R} \log\left|\frac{R^2-\bar{b}_m z}{R(z-b_m)}\right| ,\end{aligned} \tag{6}$$

where $G = (R^2 - r^2)/(R^2 + r^2 - 2Rr\cos(\theta - \varphi))$. Clearly $G \leq (R^2 - r^2)/(R - r)^2$. Also, since $|z| = r < R$, it follows that if $|a_n| < R$, then the term, $\log|R^2 - \bar{a}_n z)/R(z - a_n)|$ is positive. Hence from (6), we obtain,

$$\log|w(z)| \leq \frac{R+r}{R-r}\, m(R, w) + \sum_{|b_m|<R} \log\left|\frac{R^2-\bar{b}_m z}{R(z-b_m)}\right| . \tag{7}$$

For $|b_m| < R$, we have $|R^2 - \bar{b}_m z| \leq 2R^2$. Furthermore, if $r \neq |b_m|$, then $|z - b_m| \geq |r - |b_m||$. Since $R = \sigma r$, we have thus shown that if r is not equal to any $|a_n|$ or $|b_m|$, then on $|z| = r$, we have

$$\log|w(z)| \leq \frac{\sigma+1}{\sigma-1}\, m(\sigma r, w) + \sum_{|b_m|<\sigma r} \log 2\sigma r + \sum_{|b_m|<\sigma r} \log \frac{1}{|r - |b_m||} . \tag{8}$$

Now from the definition of $N(s, w)$, it follows easily that if $s \geq e/\sigma$, we have,

$$n(s, w) \leq ((2\sigma - 1)/(\sigma - 1))N(\sigma s, w) . \tag{9}$$

For the moment, let us assume that the sequence of poles $\{b_m\}$ is non-empty. If this sequence is infinite, let $m_0 \geq 1$ be an index such that $|b_m| > e/\sigma$ for $m \geq m_0$. If the sequence is finite, say $\{b_1, \cdots, b_t\}$, set $m_0 = t$. Now for any $m, n(|b_m|, w) \geq m$. Hence if we set, $\alpha_m = (N(\sigma|b_m|, w))^{-\sigma}$ for $m > m_0$, then since $\sigma > 1$, it follows from (9) that $\sum_{m>m_0} \alpha_m$ converges. Let E_1 be the union of all intervals $[|b_m| - \alpha_m, |b_m| + \alpha_m]$ for $m > m_0$, together with the set $\{|a_n|: n \geq 0\}$. Hence E_1 is of finite measure. We refer now to the last term in (8). If $r \geq |b_{m_0}| + 1$, and $r \notin E_1$, then clearly $|r - |b_m|| > \alpha_m$ for $m > m_0$ and $|r - |b_m|| \geq 1$ for $m \leq m_0$. Hence the last term in (8) is $\leq \sum \sigma \log N(\sigma|b_m|, w)$, the sum being over all $m > m_0$ for which $|b_m| < \sigma r$. Since $N(-, w)$ is increasing, and since there are $\leq n(\sigma r, w)$ terms in this sum, we see that the last term in (8) is at most $\sigma n(\sigma r, w)(\log^+ N(\sigma^2 r, w))$. Since the second term on the right of (8) has at most $n(\sigma r, w)$ terms, we see that this term is $\leq n(\sigma r, w)\log(2\sigma r)$. Thus, if we let E be the union of E_1 and $[0, |b_{m_0}| + 1]$, then E is of finite measure, and we have shown that if

$r \notin E$, then on $|z| = r$, we have

$$\log|w(z)| \leq \frac{\sigma+1}{\sigma-1} m(\sigma r, w) + n(\sigma r, w)\log(2\sigma r) + \sigma n(\sigma r, w)(\log^+ N(\sigma^2 r, w)) \tag{10}$$

This was derived under the assumption that the sequence $\{b_m\}$ was non-empty. However, if this sequence is empty, then the last two terms on the right of (8) are zero, and so clearly (10) holds in this case too, if we take $E = \{|a_n|: n \geq 0\}$.

Now let $V(r)$ denote the right side of the inequality (10). Let ε be a positive number such that y has no zeros or poles on $0 < |z| \leq \varepsilon$. By Jensen's formula [6; p. 166], there is constant $\lambda > 0$ such that for all $r > 0$,

$$T(r, 1/y) = T(r, y) + h(r), \tag{11}$$

where

$$|h(r)| \leq \lambda .$$

Set $b = n(0, y) + n(0, 1/y)$.

We now assert that if $r \geq 1$ and $r \notin E$, then on $|z| = r$ we have,

$$\log^+ |y(z)| \leq B(r), \tag{12}$$

where

$$B(r) = (r/\varepsilon)(2n(r, y) + r(\exp V(r))) + \lambda + b(\log r) + 2\pi r(\exp V(r)) .$$

To prove (12), we assume the contrary. Hence there exists $r \geq 1$ with $r \notin E$, and a point $z_0 = re^{i\theta_0}$ on $|z| = r$ such that $\log^+ |y(z_0)| > B(r)$. Since $B(r) > 0$,

$$\log|y(z_0)| > B(r) . \tag{13}$$

Now let $z_1 = re^{i\theta_1}$ (where $\theta_0 < \theta_1 < \theta_0 + 2\pi$) be any point on $|z| = r$ distinct from z_0, and let Γ be the arc $\zeta = re^{-i\varphi}$, $-\theta_1 \leq \varphi \leq -\theta_0$. Now by construction of the set E and the fact that $r \notin E$, w has no poles on $|z| = r$. Thus clearly y is analytic and nowhere zero on some simply-connected neighborhood of the arc Γ. Hence there exists an analytic branch g of $\log y$ on this neighborhood. Since $g' = y'/y = w$, we have,

(14) $$g(z_0) - g(z_1) = \int_{\Gamma} w(\zeta)d\zeta\ .$$

Taking the exponential of (14), we see that,

(15) $$|y(z_0)| \leq |y(z_1)| \exp\left|\int_{\Gamma} w(\zeta)d\zeta\right|\ .$$

Hence in view of (10) and (13), we obtain,

(16) $$\log|y(z_1)| > B(r) - 2\pi r(\exp V(r))\ .$$

Of course by (13), (16) also holds for $z_1 = z_0$ so that (16) is valid for all points z_1 on $|z| = r$. Hence,

(17) $$m(r, y) > B(r) - 2\pi r(\exp V(r))\ .$$

However, from the definition of $B(r)$ it follows that the right side of (16) is positive so that $|y(z_1)| \geq 1$ for all points z_1 on $|z| = r$. Thus,

(18) $$m(r, 1/y) = 0\ .$$

Now from the definitions of $N(r, y)$ and ε, we have,

(19) $$N(r, y) \leq (r/\varepsilon)n(r, y) + n(0, y)\log r\ .$$

Similarly, we have,

(20) $$N(r, 1/y) \leq (r/\varepsilon)n(r, 1/y) + n(0, 1/y)\log r\ .$$

But since y has no zeros or poles on $|z| = r$, we have by the argument principle,

(21) $$n(r, 1/y) - n(r, y) = (1/2\pi i)\int_{|z|=r} w(\zeta)d\zeta\ ,$$

and hence in view of (10),

(22) $$n(r, 1/y) \leq n(r, y) + r(\exp V(r))\ .$$

However, by (11) and (18), we clearly have,

(23) $$m(r, y) \leq N(r, y) + N(r, 1/y) + \lambda\ .$$

Using the estimates (19), (20) and (22), and the definitions of b and $B(r)$, we easily obtain from (23) an inequality which is in direct contradiction to (17). This contradiction proves the assertion (12).

In view of (12), we have,

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$$m(r, y) \leq B(r) \quad \text{if} \quad r \geq 1 \quad \text{and} \quad r \notin E\ . \tag{24}$$

Since $V(r) \geq 0$, it follows easily from the definition of $B(r)$, that there exists $r_1 > 0$ such that for $r \geq r_1$,

$$B(r) \leq (2r/\varepsilon)n(r, y) + (4r^2/\varepsilon) \exp V(r)\ . \tag{25}$$

Now examining $V(r)$, it follows easily from (9) that there are positive constants c_1 and r_2 such that,

$$V(r) \leq c_1 \Psi(\sigma^2 r) \qquad \text{for} \qquad r \geq r_2\ , \tag{26}$$

where $\Psi(r)$ is as defined in the statement of the lemma (see (5)). Adding $N(r, y)$ to both sides of (24), and using (9) for y instead of w, in the estimate for $B(r)$ in (25), it follows in view of (26), that there exist positive constants c and r_3 such that the conclusion (5) holds with σ^2 in place of a for all $r \geq r_3$ for which $r \notin E$. Since both sides of (5) are monotone nondecreasing and since $\sigma > 1$ and E is of finite measure, it follows from the fact stated in § 2, that there exists r_0 such that the conclusion (5) holds for all $r \geq r_0$ without exception, with σ^3 in place of a and $\sigma^2 c$ in place of c. Since $\sigma^3 = a$, the proof is complete.

8. Proof of the Theorem of § 6: We need only prove Part (b), since Part (a) follows from the result in [2] stated in § 5. Hence we suppose that the solution y_0 of $\Omega = 0$ is also a solution of each equation $\Omega_q = 0$. Let (M_1, M_2, A) be any bounding triple for any non-trivial $[\Omega_q]$, and given $a > 1$, let $\sigma > 1$ be such that $\sigma^2 = a$. By construction of $[\Omega_q]$, the function $w_0 = y_0'/y_0$ is a solution of the first-order equation $[\Omega_q] = 0$. Since $\sigma > 1$, it follows from the result proved in [3] which is stated in § 3 above, that there exist positive constants K and r_0 such that for all $r > r_0$, the inequality (1) holds with w_0 replacing v and σ replacing a. Using this estimate for $T(r, w_0)$, together with the fact that $N(r, w_0) \leq N(r, y_0) + N(r, 1/y_0)$, it easily follows that for the quantity $\Psi(r)$ defined in the statement of the previous lemma, the following is true: There exist constants K_1 and $r_1 > r_0$ such that for $r > r_1$,

$$\Psi(r) \leq K_1 E(\sigma r)\ , \tag{27}$$

where $E(r)$ is as defined in (4). Hence by (5) of the previous lemma, (using σ for a), there are positive constants c, c_1 and r_2 such that for $r > r_2$,

$$T(r, y_0) \leq c(rN(\sigma r, y_0) + r^2 \exp(c_1 E(\sigma^2 r))) . \tag{28}$$

Since (M_1, M_2, A) is a bounding triple, it follows (see (C) and (D) in § 3) that $M_2(r)/A(r) \geq 1$ n.e. Hence for all sufficiently large $r, (M_2(\sigma^2 r)/A(\sigma^2 r) \geq 1$ by § 2. Thus $E(\sigma^2 r) \geq r^2$ for all sufficiently large r, and hence clearly, $r^2 \leq \exp(c_1 E(\sigma^2 r))$ for all r greater than some r_3. Furthermore, for $r \geq 1$, clearly $E(r) \geq rN(r, y_0)$. Since $E(r)$ is increasing and tends to $+\infty$ as $r \to +\infty$, it easily follows that $rN(\sigma r, y_0) \leq \exp(2c_1 E(\sigma^2 r))$ for all r greater than some r_4. Hence from (28), for all r greater than some r_5, we have

$$T(r, y_0) \leq (2c) \exp(2c_1 E(\sigma^2 r)) . \tag{29}$$

Since $a = \sigma^2$, this proves Part (b) of the theorem.

9. Higher Order Equations: In this section, we investigate the growth of meromorphic solutions of algebraic differential equations of order $n > 2$. The actual estimates on the growth of solutions that one obtains in these cases (i.e., the analogues of (3) and (4)), can be quite complicated, and hence we will content ourselves with determining those quantities which enter into the growth estimates. However, we emphasize that the actual growth estimates themselves can be derived by following the method outlined below.

Let $y_0(z)$ be a meromorphic solution of an n^{th} order algebraic differential equation $\Omega = 0$, with $n > 2$, and let $y_0 \not\equiv 0$. We set $y_1 = y_0'/y_0$, and by induction, we set $y_{k+1} = y_k'/y_k$ if $y_k \not\equiv 0$. (We can exclude from consideration here, any solution y_0 for which $y_k \equiv 0$ for some k with $1 \leq k \leq n-2$. Such functions y_0 can be treated by solving successively the first-order equations $y_m' = y_{m+1} y_m$ for $m = k-1, k-2, \cdots, 0$, and it is easy to see that any such y_0 is entire, and its growth satisfies the condition that for some constant K, the maximum modulus $M(r, y_0)$ is $\leq \exp_{k-1}(Kr)$ for all sufficiently large r, where $\exp_{k-1}$ is the $(k-1)^{st}$ iterate of the exponential function. Hence we can assume that $y_k \not\equiv 0$ for $1 \leq k \leq n-2$.) From the fact that $N(r, y_{k+1}) \leq N(r, y_k) + N(r, 1/y_k)$, it follows easily by induction that for each $k \geq 1$,

$$N(r, y_k) \leq N(r, y_0) + \sum_{j=0}^{k-1} N(r, 1/y_j) . \tag{30}$$

Let $\Phi(r)$ be an unbounded monotone nondecreasing function on $(0, +\infty)$

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with the property that the maximum of the Nevanlinna characteristics of the coefficients of Ω is $O(\Phi(r))$ as $r \to +\infty$. Let $\sigma > 1$. If y_0 fails to solve some equation $\Omega_q = 0$, then by § 5, $T(r, y_0)$ can be estimated in terms of $N(\sigma r, y_0)$, $N(\sigma r, 1/y_0)$ and $\Phi(\sigma r)$. If y_0 is a solution of each equation $\Omega_q = 0$, then clearly y_1 solves each equation $[\Omega_q] = 0$. There are again two possibilities. If for some q and q_1, y_1 fails to solve the equation $[\Omega_q]_{q_1} = 0$, then in view of § 5 and (30), $T(r, y_1)$ can be estimated in terms of $N(\sigma r, y_0)$, $N(\sigma r, 1/y_0)$, $N(\sigma r, 1/y_1)$ and $\Phi(\sigma r)$. Since $y_1 = y_0'/y_0$, it follows from § 7, that $T(r, y_0)$ can be estimated in terms of $N(\sigma^2 r, y_0)$, $N(\sigma^2 r, 1/y_0)$, $N(\sigma^2 r, 1/y_1)$ and $\Phi(\sigma^2 r)$. The second possibility is that for all choices of q and q_1, y_1 solves each $[\Omega_q]_{q_1} = 0$. Then, of course, y_2 solves each equation $[[\Omega_q]_{q_1}] = 0$. These equations are of order $\leq n - 2$. If $n - 2 = 1$, then by § 3, $T(r, y_2)$ can be estimated in terms of $N(\sigma r, y_2)$, $\log^+ M_1(\sigma r)$, and $M_2(\sigma r)/A(\sigma r)$, where (M_1, M_2, A) is any bounding triple for any nontrivial equation $[[\Omega_q]_{q_1}] = 0$. Using § 7 to estimate $T(r, y_1)$ in terms of $T(\sigma r, y_2)$, and using it again to estimate $T(r, y_0)$ in terms of $T(r, y_1)$ we obtain (in view of (30)) that $T(r, y_0)$ can be estimated in terms of $N(\sigma^3 r, y_0)$, $N(\sigma^3 r, 1/y_0)$, $N(\sigma^3 r, 1/y_1)$, $\log^+ M_1(\sigma^3 r)$, and $M_2(\sigma^3 r)/A(\sigma^3 r)$ if $n - 2 = 1$. If $n - 2 > 1$, then the case where y_2 solves each equation $[[\Omega_q]_{q_1}] = 0$, leads again to two possibilities. If for some q, q_1, q_2, the function y_2 fails to solve the equation $[[\Omega_q]_{q_1}]_{q_2} = 0$, then by § 5, $T(r, y_2)$ can be estimated in terms of $N(\sigma r, y_2)$, $N(\sigma r, 1/y_2)$ and $\Phi(\sigma r)$. Using § 7 twice as above (and (30)), we see that $T(r, y_0)$ can be estimated in terms of $N(\sigma^3 r, y_0)$, $N(\sigma^3 r, 1/y_0)$, $N(\sigma^3 r, 1/y_1)$, $N(\sigma^3 r, 1/y_2)$ and $\Phi(\sigma^3 r)$. The other possibility is that for all choices of q, q_1, q_2, the function y_2 solves each equation $[[\Omega_q]_{q_1}]_{q_2} = 0$. Then of course y_3 solves each equation $[[[\Omega_q]_{q_1}]_{q_2}] = 0$ and we are faced with either $n - 3 = 1$ or the usual two possibilities. (Of course, if the second of these two possibilities continually holds, it is clear that we are eventually led to a first-order situation where § 3 is applicable.) Continuing in this manner (and for a given $a > 1$, taking $\sigma > 1$ such that $\sigma^n = a$), we clearly obtain the following result:

THEOREM: *Let $\Omega(z, y, y', \cdots, y^{(n)})$ be a non-trivial differential polynomial with $n > 2$, whose coefficients are entire functions of z. Let $\Phi(r)$ be an unbounded monotone nondecreasing function on* $(0, +\infty)$ *with the property that the maximum of the characteristics of the coeffi-*

cients of Ω *is* $O(\Phi(r))$ *as* $r \to +\infty$. *Let* $y_0(z)$ *be a meromorphic solution of the equation* $\Omega = 0$, *and inductively let* $y_{j+1} = y'_j/y_j$ *assuming that* $y_k \not\equiv 0$ *for* $0 \leq k \leq n-2$. *Let* $a > 1$. *Then*:

(a) *If for some* k, $0 \leq k \leq n-2$, *and some choice of nonnegative integers* $q_0, q_1, \cdots, q_k$, *the function* y_k *is not a solution of the equation*,

$$[\cdots[[\Omega_{q_0}]_{q_1}]_{q_2}\cdots]_{q_k} = 0\ ,$$

then $T(r, y_0)$ *can be estimated in terms of* $N(ar, y_0)$, $N(ar, 1/y_0)$, $N(ar, 1/y_1), \cdots, N(ar, 1/y_k)$ *and* $\Phi(ar)$.

(b) *If for all choices of nonnegative integers*, $q_0, q_1, \cdots, q_{n-2}$, *the function* y_{n-2} *is a solution of the equation*,

$$[\cdots[[\Omega_{q_0}]_{q_1}]_{q_2}\cdots]_{q_{n-2}} = 0\ ,$$

and if (M_1, M_2, A) *is any bounding triple for any non-trivial equation*,

$$[[\cdots[[\Omega_{q_0}]_{q_1}]_{q_2}\cdots]_{q_{n-2}}] = 0\ ,$$

then $T(r, y_0)$ *can be estimated in terms of* $N(ar, y_0)$, $N(ar, 1/y_0)$, $N(ar, 1/y_1)$, $\cdots, N(ar, 1/y_{n-2})$, $\log^+ M_1(ar)$ *and* $M_2(ar)/A(ar)$.

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