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PARTIAL REGULARITY OF STABLE p-HARMONIC MAPS INTO SPHERES

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In this paper we prove partial regularity for a weakly stable p-harmonic map from Ω into S^k when $k > 2p - 1$.

1. INTRODUCTION

Let n and k be positive integers with $n \geqslant 3$. Let Ω be a bounded smooth domain in the n-dimensional Euclidean space $\mathbb{R}^n$ and let $N \subset \mathbb{R}^l$ be a compact k-dimensional Riemannian manifold without boundary for some integer l.

For a map $u \in W^{1,p}(\Omega, N) := \{v \in W^{1,p}(\Omega, \mathbb{R}^l) \mid v \in N \text{ for almost everywhere } x \in \Omega\}$, its p-energy is given by

$$E_p(u, \Omega) = \int_\Omega |\nabla u|^p \, dx, \tag{1.1}$$

where ∇u is the gradient of u.

A map $u \in W^{1,p}(\Omega, N)$ is said to be a p-harmonic map if u satisfies

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + |\nabla u|^{p-2}A(u)(\nabla u, \nabla u) = 0, \tag{1.2}$$

in the distribution sense, where $A(\cdot)(\cdot,\cdot)$ is the second fundamental form of N in $\mathbb{R}^l$.

A p-harmonic map $u \in W^{1,p}(\Omega, N)$ is called stable if the 2nd variation of the p-energy functional $E_p(u) = \int_\Omega |\nabla u|^p \, dx$ is nonnegative (see [9]).

The study of partial regularity of various classes of weakly harmonic maps has been of great interest for a number of years. Schoen–Uhlenbeck in [17] and Giaquinta-Giusti in [5] established that an energy minimising map $u : M \to N$ between Riemannian manifolds is smooth in M away from a singular set Σ that has Hausdorff dimension $\leqslant n - 3$, where n is the dimension of M. Evans [3] and Bethuel [1] proved that a weak stationary harmonic map $u : M \to N$ is smooth away from a singular set of vanishing $(n-2)$-dimensional Hausdorff measure. Lin [10] proved an important result that if there is no non-constant harmonic map from S^2 to N, then the singular set of

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any stationary harmonic map into N has to be $(n-4)$-rectifiable. Lin's paper led to a series of interesting results on harmonic maps by Lin–Rivière [11] and on the heat flow of harmonic maps by Lin–Wang [12, 13, 14].

Without any assumption on weak harmonic maps, Rivière in [15] gave an example to show that weakly harmonic maps may have singularities everywhere. Motivated by the result of stable minimal hypersurfaces in [16], some optimal results about the estimate of the set of singularities of stationary-stable harmonic maps were obtained by the author in [8] and with Wang in [9].

In this paper, we shall prove partial regularity for a new class of weakly harmonic maps, which are stable, but not necessarily stationary. We restrict ourselves to the case that $N = S^k$, where S^k is the unit sphere in $\mathbb{R}^{k+1}$.

The main result of this paper is the following.

Theorem A. *Let $u \in W^{1,p}(\Omega; S^k)$ be a weakly stable p-harmonic map from Ω into S^k. Then, for $k > 2p-1$, u is belong to $C^{1,\alpha}(\Omega\backslash\Sigma)$, where Σ is the singular set of u. Moreover, we have $\mathcal{H}^{n-p-\delta}(\Sigma) = 0$ for some $\delta > 0$, where $\mathcal{H}^{n-p-\delta}$ denotes the Hausdorff measure of dimension $n-p-\delta$.*

For $p = 2$, Theorem A yields that when $k > 3$, a stable harmonic map $u \in W^{1,2}(\Omega; S^k)$ is smooth in an open subset Ω_0 of Ω and $\mathcal{H}^{n-2-\delta}(\Omega\backslash\Omega_0) = 0$ for some $\delta > 0$. When $k = 2$, the weakly harmonic map in [15] having singularities in Ω everywhere is also stable, so we can not expect to have the partial regularity of a stable harmonic map from B^3 into S^2.

In Section 2, we present a proof of Theorem A. The key to the proof of Theorem A is to prove that a stable harmonic map is a quasi-minimiser in $W^{1,p}(\Omega, S^k)$. Combining this with Hardt-Lin's extension Lemma, we obtain a Caccioppoli's inequality for such maps. Then it follows from a well-known result that weakly p-harmonic maps satisfying a Caccioppoli inequality have partial regularity.

2. Proof of Theorem A

We recall that a function $u = (u^1, \dots, u^{k+1})$ belongs to $W^{1,p}(\Omega, S^k)$ for $p \geqslant 2$ if u belongs to the standard Sobolev space $W^{1,p}(\Omega, \mathbb{R}^{k+1})$ and $|u| = 1$ almost everywhere in Ω.

A map $u : \Omega \to S^k$ is called weakly p-harmonic if $u \in W^{1,p}(\Omega, S^k)$ satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi\, dx = \int_\Omega |\nabla u|^p u \cdot \phi\, dx \tag{2.1}$$

for all functions $\phi \in W_0^{1,p} \cap L^\infty(\Omega, \mathbb{R}^{k+1})$.

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We say that a p-harmonic map u is stable if the second variation of E_p of u is non-negative. Then the stability of u implies

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0}\int_\Omega |\nabla u_t|^p\,dx &= p\int_\Omega |\nabla u|^{p-2}\Big[|\nabla\phi|^2 - |\nabla u|^2\phi^2 - \big|\nabla(u\cdot\phi)\big|^2\Big]\,dx \\ &+ p\int_\Omega \big[(p+2)|\nabla u|^p(u\cdot\phi)^2 - 2p|\nabla u|^{p-2}(\nabla u\cdot\nabla\phi)(u\cdot\phi)\big]\,dx \\ &+ p(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u\cdot\nabla\phi)^2\,dx \geqslant 0\end{aligned} \tag{2.2}$$

for all $\phi\in C_0^1(\Omega,\mathbb{R}^{k+1})$, where $u_t=(u+t\phi)/|u+t\phi|$.

Using (2.2), we have (see [**8**]):

Lemma 1. *Assume that $k>p$. For a stable p-harmonic map u into S^k, we have*

$$\int_\Omega |\nabla u|^p\phi^2\,dx \leqslant \frac{k+p-2}{k-p}\int_\Omega |\nabla\phi|^2|\nabla u|^{p-2}\,dx \tag{2.3}$$

for all smooth ϕ with support in Ω.

Next, we prove that a stable p-harmonic map is a quasi-minima of the energy functional E_p in $W^{1,p}(\Omega;\mathbb{R}^{k+1})$ for a sufficiently large k.

Definition: A function $u\in W^{1,p}(\Omega;S^k)$ is a quasi-minimiser of E_p in $W^{1,p}(\Omega,S^k)$ if there exists a constant Q such that

$$E_p(u;\widetilde\Omega)\leqslant QE_p(w;\widetilde\Omega)$$

for all sub-domains $\widetilde\Omega\subset\Omega$ and for all functions $w\in W^{1,p}(\Omega,S^k)$ with

$$u-w\in W_0^{1,p}(\widetilde\Omega;\mathbb{R}^{k+1}).$$

Applying Lemma 1, we have the following.

Proposition 2. *When $k>2p-1$, a stable p-harmonic map is a quasi-minimiser of the energy functional E_p in $W^{1,p}(\Omega;S^k)$.*

Proof: Let w be a map in $H^{1,p}(\Omega;S^k)$ with $u-w\in W_0^{1,p}(\Omega;\mathbb{R}^{k+1})$.

Setting $\phi=[u\cdot(u-w)]w=(1-u\cdot w)w$ on Ω, one notes

$$\nabla\phi=(1-u\cdot w)\nabla w-\nabla(u\cdot w)w.$$

Taking the above ϕ as a test function in (2.1), we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla w(1-u\cdot w)\,dx - \int_\Omega |\nabla u|^{p-2}(\nabla u\cdot w)\cdot\nabla(u\cdot w)\,dx = \int_\Omega |\nabla u|^p u\cdot w(1-u\cdot w)\,dx.$$

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This implies

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}|\nabla u\cdot w|^2\,dx &= \int_\Omega |\nabla u|^p(u\cdot w)^2\,dx - \int_\Omega |\nabla u|^p(u\cdot w)\,dx\\ &\quad + \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla w(1-u\cdot w)\,dx\\ &\quad - \int_\Omega |\nabla u|^{p-2}(\nabla u\cdot w)(u\cdot\nabla w)\,dx.\end{aligned}\tag{2.4}$$

Since u is a stable p-harmonic map into S^4, it follows from Lemma 1 that

$$\int_\Omega |\nabla u|^p\eta^2\,dx \leqslant \frac{k+p-2}{k-p}\int_\Omega |\nabla u|^{p-2}|\nabla\eta|^2\,dx$$

for all smooth function η with support in Ω. Taking $\eta = u\cdot(u-w) = 1-u\cdot w$ in the above inequality, we obtain

$$\begin{aligned}&\int_\Omega |\nabla u|^p\left[1+(u\cdot w)^2-2u\cdot w\right]dx\\ &\quad\leqslant \frac{k+p-2}{k-p}\int_\Omega \left|\nabla(u\cdot w)\right|^2|\nabla u|^{p-2}\,dx\\ &\quad= \frac{k+p-2}{k-p}\int_\Omega |\nabla u\cdot w|^2|\nabla u|^{p-2}\,dx + \frac{k+p-2}{k-p}\int_\Omega |u\cdot\nabla w|^2|\nabla u|^{p-2}\,dx\\ &\qquad + \frac{k+p-2}{k-p}\int_\Omega (\nabla u\cdot w)(u\cdot\nabla w)|\nabla u|^{p-2}\,dx.\end{aligned}\tag{2.5}$$

It follows from (2.4) with (2.5) that

$$\begin{aligned}&\int_\Omega \left[|\nabla u|^2 - \frac12(u\cdot w)^2\left|\nabla u\right|^2 - \frac12\left|\nabla u\cdot w|^2\right]|\nabla u|^{p-2}\,dx\\ &\qquad + \left(2-\frac{k+p-2}{k-p}\right)\int_\Omega |\nabla u|^2(u\cdot w)^2|\nabla u|^{p-2}\,dx\\ &\quad\leqslant \left(\frac{k+p-2}{k-p}-\frac12\right)\int_\Omega \nabla u\cdot\nabla w\,(1-u\cdot w)|\nabla u|^{p-2}\,dx\\ &\qquad - \left(\frac{k+p-2}{k-p}-\frac52\right)\int_\Omega |\nabla u|^2u\cdot w|\nabla u|^{p-2}\,dx\\ &\qquad + \left(\frac{k+p-2}{k-p}-\frac12\right)\int_\Omega |u\cdot\nabla w|^2|\nabla u|^{p-2}\,dx\\ &\qquad + \left(\frac{k+p-2}{k-p}-\frac12\right)\int_\Omega (\nabla u\cdot w)(u\cdot\nabla w)|\nabla u|^{p-2}\,dx.\end{aligned}\tag{2.6}$$

Letting $\phi = u-w$ in equation (2.1), we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u-w)\,dx = \int_\Omega |\nabla u|^p u\cdot(u-w)\,dx$$

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implying

$$\int_\Omega \nabla u\cdot\nabla w|\nabla u|^{p-2}\,dx=\int_\Omega |\nabla u|^2(u\cdot w)|\nabla u|^{p-2}\,dx.$$

Since $(k+p-2)/(k-p)<3$ for $k>2p-1$, we have

$$\begin{aligned}(2.7)\qquad &\int_\Omega\Big[|\nabla u|^2-\frac{1}{2}(u\cdot w)^2|\nabla u|^2-\frac{1}{2}|\nabla u\cdot w|^2\Big]|\nabla u|^{p-2}\,dx\\ &+\left(2-\frac{k+p-2}{k-p}\right)\int_\Omega|\nabla u|^2(u\cdot w)^2|\nabla u|^{p-2}\,dx\\ &\leqslant C_2\int_\Omega|\nabla w|^p+\varepsilon\int_\Omega|\nabla u|^p\,dx,\end{aligned}$$

where C_2 is a constant depending only on k and ε is a sufficiently small constant which will be determined later.

For a fixed point x_0, let $\lambda=w(x_0)$. Then we claim

$$(2.8)\qquad |\nabla u\cdot\lambda|^2+|u\cdot\lambda|^2|\nabla u|^2\leqslant|\nabla u|^2.$$

In fact, since $\lambda\in S^k$, there exists a k-dimensional tangent plane to S^k at λ. Assume that $\widetilde{e}_i$, $i=1,\dots,k$, is an orthonormal basis of the tangent plane. Then λ and $\widetilde{e}_i$, $i=1,\dots,k$, form a new basis of $\mathbf{R}^{k+1}$. We write $u=(u\cdot\lambda)\lambda+\sum_{i=1}^k(u\cdot\widetilde{e}_i)\widetilde{e}_i$, then

$$|u|^2=|u\cdot\lambda|^2+\sum_{i=1}^k|u\cdot\widetilde{e}_i|^2=1$$

and

$$|\nabla u|^2=|\nabla u\cdot\lambda|^2+\sum_{i=1}^k|\nabla u\cdot\widetilde{e}_i|^2.$$

Using the fact that $|u|=1$, we obtain

$$(u\cdot\lambda)\,(\nabla u\cdot\lambda)=-\sum_{i=1}^k u\cdot\widetilde{e}_i\,(\nabla u\cdot\widetilde{e}_i).$$

By the Cauchy inequality, we have

$$\begin{aligned}(u\cdot\lambda)^2|\nabla u\cdot\lambda|^2&=\left(\sum_i^k u\cdot\widetilde{e}_i\,(\nabla u\cdot\widetilde{e}_i)\right)^2\\ &\leqslant\sum_{i=1}^k(u\cdot\widetilde{e}_i)^2\sum_{i=1}^k(\nabla u\cdot\widetilde{e}_i)^2=(1-|u\cdot\lambda|^2)(|\nabla u|^2-|\nabla u\cdot\lambda|^2).\end{aligned}$$

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This proves our claim (2.8).

It follows from (2.4), (2.6) and (2.8) that

$$\left(3-\frac{k+p-2}{k-p}\right)\int_\Omega |\nabla u|^p (u\cdot w)^2\,dx \leqslant \varepsilon\int_\Omega |\nabla u|^p\,dx + C\int_\Omega |\nabla w|^p\,dx. \tag{2.9}$$

Choosing a sufficiently small ε in (2.7) and (2.9), Proposition 2 is proved. ∎

We modify a lemma in [**7**, Appendix] to obtain:

Lemma 3. (Hardt–Lin's Extension Lemma) *Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^n$ and assume $2 \leqslant p \leqslant k$. For any $v \in W^{1,p}(\widetilde{\Omega}, \mathbb{R}^{k+1})$ with $|v| = 1$ on $\partial\widetilde{\Omega}$, there exists a function $w \in W^{1,p}(\widetilde{\Omega}; S^k)$ such that*

$$w - v \in W_0^{1,p}(\widetilde{\Omega}; \mathbb{R}^{k+1}),$$
$$\int_{\widetilde{\Omega}} |\nabla w|^p\,dx \leqslant C\int_{\widetilde{\Omega}} |\nabla v|^p\,dx, \tag{2.7}$$

for a constant C independent of u and $\widetilde{\Omega}$.

Proof: The proof is essentially due to one in [**7**]. Without loss of generality, we assume that $\widetilde{\Omega}$ is Ω. For any $a \in \mathbb{R}^{k+1}$ with $|a| \leqslant 1/2$, consider the function

$$w_a(x) = \frac{v(x)-a}{|v(x)-a|}, \quad x \in \Omega.$$

Then

$$\nabla w_a = |v-a|^{-1}\nabla v - |v-a|^{-3}(v-a)\otimes(v-a)\nabla v.$$

Integrating over Ω with respect to x and over $B_{1/2}$ with respect to a, we obtain

$$\int_{B_{1/2}}\int_\Omega |\nabla w_a|^p\,dx\,da = \int_\Omega\int_{B_{1/2}} |\nabla w_a|^p\,dx\,da \leqslant C\int_\Omega |\nabla u|^p\,dx.$$

due to the fact that

$$\int_{B_{1/2}} |v-a|^{-p}\,da \leqslant K,$$

where K is a positive constant depending on k. Hence there exists a point a_0 with $|a_0| \leqslant 1/2$, such that

$$\int_\Omega |\nabla w_{a_0}|^p\,dx \leqslant C\int_\Omega |\nabla v|^p\,dx. \tag{2.8}$$

Let

$$\Pi_a(\xi) = \frac{\xi - a}{|\xi - a|}.$$

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Π_a is a C^1-bilipshitz diffeomorphism of S^k onto itself. Indeed,

$$\Pi_a^{-1}(\eta) = a + \left[(a\cdot\eta)^2 + \left(1-|a|^2\right)\right]^{1/2}\eta.$$

with

$$\left|\nabla\Pi_a^{-1}(\eta)\right| \leqslant \Lambda,$$

for a constant Λ uniformly independent of a with $|a| \leqslant 1/2$. Thus taking

$$w = \Pi_{a_0}^{-1}\circ w_{a_0},$$

we have

$$|\nabla w| \leqslant C(\Lambda)|\nabla w_{a_0}|. \tag{2.9}$$

Our claim (2.7) follows from (2.8) and (2.9). □

PROPOSITION 4. (Caccioppoli's inequality)

Let u be a quasi-minimiser of E_p in $W^{1,p}_\gamma(\Omega, S^2)$. Then for all $x_0\in\Omega$ and R with $0 < R < \operatorname{dist}(x_0,\partial\Omega)$, we have

$$\int_{B_{R/2}(x_0)} |\nabla u|^p \leqslant CR^{-p}\int_{B_R(x_0)} |u-u_{x_0,R}|^p\,dx. \tag{2.10}$$

PROOF: Note that u is a quasi-minimiser of E_p in $W^{1,p}_\gamma(\Omega,\mathbb{R}^3)$, that is,

$$E_p(u;\widetilde{\Omega}) \leqslant QE_p(v;\widetilde{\Omega}),$$

for any $v\in W^{1,p}_\gamma(\widetilde{\Omega},\mathbb{R}^3)$, where $\widetilde{\Omega}$ is a sub-domain of Ω. Taking $\widetilde{\Omega} = B_s$ and using Lemma 1, we have

$$\int_{B_s} |\nabla u|^p\,dx \leqslant QC(\Lambda)\int_{B_s}|\nabla v|^p\,dx \tag{2.11}$$

for any $v\in W^{1,p}_u(B_s)$.

Let $x_0\in\Omega$ and $R>0$ such that $B_R(x_0)\subset\Omega$. For any two positive numbers t, s with $R/2\leqslant t < s\leqslant R$, we choose a cut-off function $\eta\in C_0^\infty(B_s)$ such that $0\leqslant\eta\leqslant 1$ with $\eta\equiv 1$ in B_t and $|\nabla\eta|\leqslant C/(s-t)$. Taking $v = u-\eta(u-u_{x_0,R})$ in (2.11), we see

$$\nabla v = (1-\eta)\nabla u - \nabla\eta(u-u_{x_0,R}).$$

By the standard filling hole trick, there exists a positive $\theta < 1$ such that

$$\int_{B_t}|\nabla u|^p \leqslant \theta\int_{B_s}|\nabla u|^p + C(s-t)^{-p}\int_{B_R}|u-u_{x_0,R}|^p\,dx$$

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for all t, s with $R/2 \leqslant t < s \leqslant R$. It implies from a lemma in [**4**, Lemma 3.1 in Chapter V] that for all $x_0 \in \Omega$ and R with $0 < R < \operatorname{dist}(x_0, \partial\Omega)$, we have

$$\int_{B_{R/2}} |\nabla u|^p \leqslant CR^{-p} \int_{B_R} |u - u_{x_0,R}|^p \, dx.$$

This proves our claim. ∎

For any function f on $B_R(x_0)$, we write

$$\fint_{B_R(x_0)} f \, dx := \left|B_R(x_0)\right|^{-1} \int_{B_R(x_0)} f \, dx.$$

By Proposition 3, it easily follows from the standard reverse Hölder inequality that there exists an exponent $q > p$ such that $u \in W^{1,q}_{loc}(\Omega, \mathbb{R}^{k+1})$; that is, for all $x_0 \in \Omega$ and $R < \operatorname{dist}(x_0, \partial\Omega)$, we have

$$\left(\fint_{B_{R/2}(x_0)} |\nabla u|^q \, dx\right)^{1/q} \leqslant C\left(\fint_{B_R(x_0)} |\nabla u|^p \, dx\right)^{1/p}, \tag{2.12}$$

where C is a constant independent of u.

Theorem 5. *Let u be a weakly p-harmonic maps from Ω into S^k, satisfying a Caccioppoli's inequality, that is, inequality (2.4) holds for any positive R with $R \leqslant \operatorname{dist}(x_0, \Omega)$. Then there exists a subset Ω_0 of Ω such that u is $C^{1,\alpha}(\Omega_0; S^k)$. Moreover,*

$$\mathcal{H}^{n-p-\delta}(\Omega \backslash \Omega_0) = 0,$$

for some $\delta > 0$.

Proof: The proof is standard by using the reverse Hölder inequality (2.12) (see [**5**]). See a different proof in [**2**]. ∎

Theorem A follows from Theorem 2, Proposition 4 and Theorem 5.

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