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SK_2 AND K_3 OF DIHEDRAL GROUPS

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ABSTRACT. New computations of birelative K_2 groups and recent results on K_3 of rings of algebraic integers are combined in generalized Mayer-Vietoris sequences for algebraic K-theory. Upper and lower bounds for $SK_2(\mathbb{Z}G)$ and lower bounds for $K_3(\mathbb{Z}G)$ are deduced for G a dihedral group of square-free order, and for some other closely related groups G.

0. **Introduction.** Shortly after J. Milnor introduced the definition of K_2 of a ring in 1968–69, J. B. Wagoner discovered that pseudo-isotopy obstructions can be found in $K_2(\mathbb{Z}G)$, where G is the fundamental group of a smooth manifold (see [28]). Since then there have been several attempts to compute $K_2(\mathbb{Z}G)$, but this has proven difficult, even for the smallest groups G.

To date, the only complete calculations of $K_2(\mathbb{Z}G)$ have been those of M. Dunwoody [5], when G has order 2 or 3, and M. Stein [26], when G is cyclic of order 4 or 5, or dihedral of order 6 or 10. Beyond these computations, only lower bounds for the order of $K_2(\mathbb{Z}G)$ have been found. In 1976, K. Dennis, M Keating and M. Stein published lower bounds for the order of $K_2(\mathbb{Z}G)$ when G is an elementary abelian p-group, p a prime (see [4]). These were based on surjectivity of K_2 of the map reducing $\mathbb{Z}G$ modulo p, and grow exponentially with the rank of G. These results were complemented by S. Chaladus in 1979 (see [2], [3]) who produced lower bounds when G is a cyclic p-group by using iterated Mayer-Vietoris sequences to establish surjectivity of K_2 of the inclusion of $\mathbb{Z}G$ into its integral closure in $\mathbb{Q}G$.

In the mid-1980's (see [20], [21]), as part of his conquest of $SK_1(\mathbb{Z}G)$, R. Oliver generalized the bounds of Dennis-Keating-Stein to arbitrary finite p-groups, replacing reduction mod p by completion at p, and conjectured lower bounds for these same p-groups involving the cyclic homology of $\mathbb{Z}G$. In addition, Oliver generalized and improved the bounds obtained by Chaladus, obtaining lower bounds for the case of finite cyclic groups of arbitrary order.

In this paper we focus on the kernel, $SK_2(\mathbb{Z}G)$, of K_2 of the inclusion of $\mathbb{Z}G$ into $\mathbb{Q}G$. This kernel has trivial intersection with the part of $K_2(\mathbb{Z}G)$ detected by Chaladus. We obtain lower and upper bounds for $SK_2(\mathbb{Z}G)$ when G is cyclic or dihedral of square-free even order. Our upper bounds show that Oliver's bounds in the cyclic case detect more than just $SK_2(\mathbb{Z}G)$; the precise relationship of SK_2 to those bounds is unclear. Our lower

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bounds for both classes of groups include bounds on the minimum number of generators of $SK_2(\mathbb{Z}G)$, not merely on its order. Our bounds for square-free order dihedral groups, and for certain of their extensions including dicyclic groups (see Theorem 9.10), are the first ever obtained for $K_2(\mathbb{Z}[-])$ of these groups.

Our approach continues a line of work begun by M. Stein in [26], using Mayer-Vietoris sequences which incorporate a measure of the failure of excision for relative K_2. That measure is the birelative K_2 introduced by D. Guin-Walery and J.-L. Loday in [7] and by F. Keune in [10]. In [13] and [14], the first author of our paper constructed long exact generalized Mayer-Vietoris sequences for the K-theory $\mathbb{Z}G$, which exist for a large class of finite groups G, including those of square-free order. The underlying idea is that, if G has square-free order, then $\mathbb{Z}G$ is a subdirect product of a hereditary order in $\mathbb{Q}G$. The inclusion map from $\mathbb{Z}G$ into the hereditary order induces the long exact Mayer-Vietoris sequence. The third term in the sequence involves K_3-groups of various semisimple quotients of the hereditary order, as well as birelative K_2-groups associated to all the fiber squares occurring in the description of $\mathbb{Z}G$ as a subdirect product.

To use this sequence to get a complete computation of $K_2(\mathbb{Z}G)$, one needs to know the K_2 of rings of integers in certain number fields. This is a very hard problem, but there has been substantial progress toward its solution (e.g., see [11]). However, to compute $SK_2(\mathbb{Z}G)$ from the sequence requires only the determination of birelative K_2-groups, the K_3-groups of the hereditary order and semisimple quotients, and the boundary map in the sequence from dimension 3 to dimension 2.

In Sections 1–3 we construct a filtration of $SK_2(\mathbb{Z}G)$ with one filtration quotient associated to each prime factor of the order of G. Sections 4–6 contain the birelative K_2 computations. Section 7 is devoted to the determination of K_3 of the hereditary order, using M. Keating's work on tiled orders to reduce to the determination of K_3 of rings of integers, the latter having been independently completed by M. Levine in [15], and A. Merkurjev with A. Suslin in [18]. Finally, Section 9 contains the upper and lower bounds for $SK_2(\mathbb{Z}G)$, and lower bounds for $K_3(\mathbb{Z}G)$. We anticipate that a better understanding of the K_3-level maps in the sequence will eventually close the gap between our upper and lower bounds.

Before we begin, a few remarks about notation are in order. If R is the ring of integers in a number field F, and Λ is an R-order in a finite dimensional semisimple F-algebra Σ, then for $n \geq 0$, $SK_n(\Lambda)$ denotes the kernel of K_n of the inclusion map from Λ into Σ. Second, we use $\oplus$ to denote the direct (cartesian) product of rings, as well as the direct sum of abelian groups. Third, the term "fiber square" refers to any commutative square of ring homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & U \end{array}$$

for which the induced map

$$R \longrightarrow S \oplus T$$

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is injective, and for which any pair of elements in S and T with the same image in U have a common pre-image in R. This is also called a "cartesian square" or a "pullback" in the category of rings.

1. **Filtrations of $SK_n(\mathbb{Z}G)$.** If $A_1, \ldots, A_m$ are rings, it is a property of algebraic K-functors $K_n (n \geq 0)$ that projections to each coordinate induce a natural isomorphism:

$$K_n(\bigoplus_{i=1}^{m} A_i) \cong \bigoplus_{i=1}^{m} K_n(A_i).$$

This property is useful in calculating $K_n(\mathbb{Q}G)$ for a finite group G, since $\mathbb{Q}G$ is a direct product of its simple components. But the topological invariants which provide major applications of K-groups take their values in $K_n(\mathbb{Z}G)$ or in closely related groups, and $\mathbb{Z}G$ does not decompose, having no central idempotents aside from 0 and 1 (see [27], Corollary 8.1).

To take advantage of the decomposition:

$$\mathbb{Q}G = \bigoplus_{i=1}^{m} \Sigma_i$$

of $\mathbb{Q}G$ into its simple components Σ_i, consider $\mathbb{Z}$-orders defined as follows: If $\tau \subseteq \{1, \ldots, m\}$ let $\Lambda(\tau)$ denote the image of the projection:

$$\mathbb{Z}G \longrightarrow \bigoplus_{i \in \tau} \Sigma_i.$$

If $p = \{\tau_1, \ldots, \tau_r\}$ is a partition of $\{1, \ldots, m\}$:

$$\tau_1 \cup \cdots \cup \tau_r = \{1, \ldots, m\}, \quad \tau_i \cap \tau_j = \emptyset \text{ if } i \neq j.$$

then

$$R(p) = \bigoplus_{j=1}^{r} \Lambda(\tau_j)$$

is a $\mathbb{Z}$-order in $\mathbb{Q}G$ containing $\mathbb{Z}G$. If p' is a refinement of p, then $R(p) \subseteq R(p')$.

If, in passing from a partition p to a refined partition p', a part $\tau_j = \tau$ is divided into two parts $\tau' \cup \tau''$, there is a fiber square of ring homomorphisms:

$$\begin{array}{ccc} \Lambda(\tau) & \longrightarrow & \Lambda(\tau') \\ \downarrow & & \downarrow \\ \Lambda(\tau'') & \longrightarrow & S \end{array}$$

where the top and left sides are projections. (This is described more fully below.) Under some conditions there are Mayer-Vietoris sequences in K-theory which can be used to study K_n of the inclusion:

$$\Lambda(\tau) \longrightarrow \Lambda(\tau') \oplus \Lambda(\tau'').$$

Beginning with $\{1,\ldots,m\}$, define a sequence of partitions, each refining its predecessor by dividing each part into at most two parts. Denote the associated orders by:

$$\mathbb{Z}G = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_k = \bigoplus_{i=1}^{m} \Lambda(i).$$

Then K_n of each inclusion $R_{i-1} \longrightarrow R_i$ is the direct product of maps:

$$K_n\big(\Lambda(\tau)\big) \longrightarrow K_n\big(\Lambda(\tau')\big) \oplus K_n\big(\Lambda(\tau'')\big)$$

for each τ that is divided, and identity maps on $K_n\big(\Lambda(\tau)\big)$ for each τ that is left alone.

For $0 \le i \le k$, let F^i denote the kernel of $K_n(\mathbb{Z}G \longrightarrow R_i)$. The inclusion $\mathbb{Z}G \longrightarrow \mathbb{Q}G$ factors through the inclusion $\mathbb{Z}G \longrightarrow R_i$; so each F^i is a subgroup of $SK_n(\mathbb{Z}G)$. Thus there is a filtration:

$$0 = F^0 \subseteq F^1 \subseteq \cdots \subseteq F^k \subseteq SK_n(\mathbb{Z}G).$$

The last layer $SK_n(\mathbb{Z}G)/F^k$ is isomorphic to the image of $SK_n(\mathbb{Z}G)$ under $K_n(\mathbb{Z}G \longrightarrow R_k)$, which is contained in $SK_n(R_k)$. When each $\Lambda(i)$, and hence also R_k, is hereditary, and n is even,

$$SK_n(R_k) = SG_n(R_k) = 0$$

by [12], Theorem 1.1. This happens, in particular, when all Sylow subgroups of G are cyclic.

To compute the other layers, we use the following rephrased version of [14], Proposition 2.3, presented here with a corrected proof:

PROPOSITION 1.1. *Suppose $R_0 \subseteq \cdots \subseteq R_k$ are $\mathbb{Z}$-orders in a $\mathbb{Q}$-algebra and $q_1,\ldots,q_k$ are pairwise relatively prime positive integers with $q_iR_i \subseteq R_{i-1}$ for $1 \le i \le k$. Then for $0 \le i < j \le k$ and $n > 0$,*

$$\frac{\ker K_n(R_0 \longrightarrow R_j)}{\ker K_n(R_0 \longrightarrow R_i)} \cong \ker K_n(R_i \longrightarrow R_j)$$

where the arrows are inclusions.

PROOF. Taking $A = R_0$, $B = R_i$ and $C = R_j$, it is sufficient to prove that if

$$A \xrightarrow{f} B \xrightarrow{g} C$$

are inclusions of $\mathbb{Z}$-orders in a $\mathbb{Q}$-algebra, and $pC \subseteq B$, $qB \subseteq A$ for relatively prime positive integers p and q, then the image of $K_n(f)$ contains the kernel of $K_n(g)$ for all $n > 0$. For then the map $K_n(f)$ induces an isomorphism:

$$\frac{\ker K_n(gf)}{\ker K_n(f)} \cong \ker K_n(g),$$

as required.

Let $K(-) = BQP(-)$ be the K-theory space functor, so that

$$K_n(-) = \pi_{n+1}\big(K(-), 0\big).$$

Consider the commutative square of homotopy fiber sequences:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd, "\gamma"'] & & K(A) \arrow[rr] \arrow[dr] \arrow[dd] & & K(B) \arrow[dr] \arrow[dd] & \\
& X' \arrow[rr] \arrow[dd] & & K(A[1/p]) \arrow[rr] \arrow[dd] & & K(B[1/p]) \arrow[dd] \\
Y \arrow[rr] \arrow[dr] & & K(A) \arrow[rr] \arrow[dr] & & K(C) \arrow[dr] & \\
& Y' \arrow[rr] & & K(A[1/p]) \arrow[rr] & & K(C[1/p])
\end{tikzcd} \tag{1.2}$$

where the maps between K-theory spaces are induced by inclusions, and the spaces X, X', Y, Y' are appropriate homotopy fibers.

Let S denote $\{p^n : n \geq 0\}$. Since $qB \subseteq A$ and $p\mathbb{Z} + q\mathbb{Z} = \mathbb{Z}$, the inclusion $A \subseteq B$ induces an isomorphism: $A/sA \cong B/sB$ for each $s \in S$. In other words, the inclusion of A into B is an analytic isomorphism along S. By a theorem of M. Karoubi (see [8], Appendix 5, and [30], Theorem 1.1), $B \otimes_A (-)$ is a natural equivalence of categories $H^1_S(A) \longrightarrow H^1_S(B)$, where, for any ring R, $H^1_S(R)$ is the category of finitely generated S-torsion R-modules of projective dimension ≤ 1. Let $\alpha : Z \longrightarrow Z'$ denote the induced map between homotopy fibers of the vertical maps in the square:

$$\begin{tikzcd}
K(A) \arrow[r] \arrow[d] & K(B) \arrow[d] \\
K(A[1/p]) \arrow[r] & K(B[1/p])
\end{tikzcd}$$

which is part of diagram (1.2). Combining the above category equivalence with the localization theorem for projective modules (see [6], Example 1), one obtains α as a composite of weak homotopy equivalences:

$$Z \simeq BQH^1_S(A) \simeq BQH^1_S(B) \simeq Z'.$$

Further, putting in all homotopy fibers, one obtains a commutative diagram:

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & Z \arrow[r, "\alpha"] \arrow[d] & Z' \arrow[d] \\
X \arrow[r] \arrow[d, "\beta"'] & K(A) \arrow[r] \arrow[d] & K(B) \arrow[d] \\
X' \arrow[r] & K(A[1/p]) \arrow[r] & K(B[1/p])
\end{tikzcd}$$

with every row and column a homotopy fiber sequence. Since α is a weak homotopy equivalence, X'' is weakly contractible: so $\pi_n(\beta)$ is an isomorphism for all $n > 0$.

Since $pC \subseteq B$, the front vertical maps in (1.2) are identity maps. So for every $n > 0$, the map $\pi_n(\gamma)$ is split injective (follow the left side of (1.2)). From a chase through the commutative ladder of homotopy exact sequences obtained from the back of (1.2), one sees that $\ker\big(K_n(B) \longrightarrow K_n(C)\big)$ is contained in $\mathrm{image}\big(K_n(A) \longrightarrow K_n(B)\big)$, for all $n \geq 0$. ∎

Returning to the discussion prior to Proposition 1.1, each inclusion $R_{i-1} \longrightarrow R_i$ in the filtration described there is the direct product of identity maps and inclusions:

$$\Lambda(\tau) \longrightarrow \Lambda(\tau') \oplus \Lambda(\tau'')$$

in which $\Lambda(\tau)$ projects onto each direct factor, $\Lambda(\tau')$ and $\Lambda(\tau'')$. For notational convenience we write Λ, Λ', Λ'' for $\Lambda(\tau)$, $\Lambda(\tau')$, $\Lambda(\tau'')$ respectively. If I is the kernel of $\Lambda \longrightarrow \Lambda'$ and J is the kernel of $\Lambda \longrightarrow \Lambda''$, then $I \cap J = \{0\}$; so there is a fiber square of canonical ring homomorphisms:

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda/I \cong \Lambda' \\ \downarrow & & \downarrow \\ \Lambda'' \cong \Lambda/J & \longrightarrow & \Lambda/(I+J), \end{array}$$

in which $\Lambda/(I+J)$ is a finite ring (being a quotient of $\mathbb{Z}G$ whose tensor with $\mathbb{Q}$ must vanish because $\mathbb{Q} \otimes (-)$ of the above square is still a surjective pullback). In fact $I+J$ is the conductor from $\Lambda' \oplus \Lambda''$ into Λ, and the characteristic of $\Lambda/(I+J)$ generates the ideal $(I+J) \cap \mathbb{Z}$ of $\mathbb{Z}$.

By [13], Theorem 2.1, if that characteristic is a prime p and, for all $n > 0$, $K_n\big(\Lambda/(I+J)\big)$ is a torsion group with no p-torsion, then there is a long exact Mayer-Vietoris sequence:

$$\begin{aligned} \cdots \longrightarrow K_{n+1}(\Lambda') \oplus K_{n+1}(\Lambda'') \longrightarrow K_{n+1}\big(\Lambda/(I+J)\big) \oplus K_n(\Lambda; I, J) \\ \longrightarrow K_n(\Lambda) \longrightarrow K_n(\Lambda') \oplus K_n(\Lambda'') \longrightarrow \cdots \end{aligned}$$

where $K_n(\Lambda; I, J)$ are the birelative K-groups (see [7]) associated to the fiber square. The kernel of $K_n(R_{i-1} \longrightarrow R_i)$ is the direct product of the kernels of the maps:

$$K_n(\Lambda) \longrightarrow K_n(\Lambda' \oplus \Lambda'') \simeq K_n(\Lambda') \oplus K_n(\Lambda'')$$

in these Mayer-Vietoris sequences, as Λ ranges over the direct factors which are split up as we pass from R_{i-1} to R_i.

To provide such sequences for each $\Lambda \longrightarrow \Lambda' \oplus \Lambda''$ from R_{i-1} to R_i, we shall require that each q_i is a prime p_i. To avoid p_i-torsion in $K_n\big(\Lambda/(I+J)\big)$ it is sufficient that $\Lambda/(I+J)$ be semisimple, hence a product of matrix rings over finite fields of characteristic p_i; for D. Quillen's formula (see [22]) for K_n of such a field yields a cyclic group of order relatively prime to p_i. These constraints lead us to consider decompositions of $\mathbb{Q}G$ derived from square-free order cyclic normal subgroups of G, with index relatively prime to their order.

2. **Filtrations based on cyclic normal subgroups.** For the rest of this paper, assume G is a finite group with a cyclic normal subgroup H generated by an element a of order m, and $b_1, \dots, b_s$ is a full list of coset representatives for G/H, with $b_1 = 1$. The group algebra $\mathbb{Q}G$ is a (left and right) free $\mathbb{Q}[a]$ $(= \mathbb{Q}H)$-module with basis $b_1, \dots, b_s$. Its multiplication is determined by relations:

$$b_i a = a^{n(i)} b_i, \quad b_i b_j = a^{\ell(i,j)} b_k$$

where $n(i)$, $\ell(i,j) \in \mathbb{Z}$ and $n(i)$ is relatively prime to m.

If d is a positive divisor of m, let ζ_d denote the primitive d-th root of unity $e^{2\pi/d}$. Replacing a by ζ_d defines a surjective ring homomorphism,

$$\psi_d \colon \mathbb{Q}G \longrightarrow \Sigma(d),$$

where $\Sigma(d)$ is a $\mathbb{Q}$-algebra with the above description, but with ζ_d substituted for a. As in [16], Section 7, there is a $\mathbb{Q}$-algebra isomorphism:

$$\mathbb{Q}G \cong \bigoplus_{d|m} \Sigma(d)$$

which is ψ_d in each d-component.

If $\mathcal{D}$ is a set of positive divisors of m, let $O(\mathcal{D})$ denote the image of the projection:

$$\mathbb{Z}G \longrightarrow \bigoplus_{d \in \mathcal{D}} \Sigma(d).$$

Let $\alpha = \alpha_{\mathcal{D}}$ denote the image $(\zeta_d)_{d \in \mathcal{D}}$ of a in $O(\mathcal{D})$. Then $O(\mathcal{D})$ has the same description as $\mathbb{Q}G$, but with $\mathbb{Q}$ replaced by $\mathbb{Z}$ and a replaced by α: $O(\mathcal{D})$ is a (left and right) free $\mathbb{Z}[\alpha]$-module with basis $b_1, \dots, b_s$; its multiplication is determined by the relations:

$$b_i \alpha = \alpha^{n(i)} b_i, \quad b_i b_j = \alpha^{\ell(i,j)} b_k.$$

Note that the minimal polynomial of $\alpha_{\mathcal{D}}$ over $\mathbb{Q}$ is:

$$\gamma_{\mathcal{D}}(x) = \prod_{d \in \mathcal{D}} \Phi_d(x)$$

where $\Phi_d(x)$ is the minimal polynomial of ζ_d over $\mathbb{Q}$. Since each $\Phi_d(x)$ is monic with integer coefficients, so is $\gamma_{\mathcal{D}}(x)$. So $\mathbb{Z}[\alpha]$ has $\mathbb{Z}$-basis $1, \alpha, \alpha^2, \dots, \alpha^{\delta-1}$, where

$$\delta = \delta_{\mathcal{D}} = \sum_{d \in \mathcal{D}} \varphi(d)$$

is the degree of $\gamma_{\mathcal{D}}(x)$.

Now suppose p is a prime factor of m and p does not divide any element of $\mathcal{D}$. Then there is a fiber square of surjective ring homomorphisms:

$$\text{(2.1)} \qquad \begin{array}{ccc} O(\mathcal{D} \cup p\mathcal{D}) & \xrightarrow{\pi_{p\mathcal{D}}} & O(p\mathcal{D}) \\ {\scriptstyle \pi_{\mathcal{D}}}\downarrow & & \downarrow \\ O(\mathcal{D}) & \xrightarrow[\bmod p]{} & O(\mathcal{D})/pO(\mathcal{D}) \end{array}$$

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where $\pi_{\mathcal{D}}$, $\pi_{p\mathcal{D}}$ are projections and the right vertical map may be defined by commutativity of the square. To justify this, note that there is a surjective fiber square beginning with these projections, and by [16], sections 8–9, the bottom map in such a square has kernel $pO(\mathcal{D})$.

Note that the left vertical map $\pi_{\mathcal{D}}$ in the square (2.1) is a *split* surjective ring homomorphism. To see this, note that for each $d \in \mathcal{D}$, $\mathbb{Z}[\zeta_d] \subseteq \mathbb{Z}[\zeta_{pd}]$; and by compatibility of the multiplicative relations, $\Sigma(d)$ is a subring of $\Sigma(pd)$, and then $O(\mathcal{D})$ is a subring of $O(p\mathcal{D})$. The splitting of $\pi_{\mathcal{D}}$ is the ring homomorphism taking x to $\big(x, \theta(x)\big)$, where

$$\theta : O(\mathcal{D}) \longrightarrow O(p\mathcal{D})$$

is the q-power map on $\alpha_{\mathcal{D}}$ followed by inclusion, where q is the inverse of p under multiplication modulo $r = \operatorname{lcm}(\mathcal{D})$.

To produce Mayer-Vietoris sequences for all such squares, we need to assume m (= order of a) and s (= index of $\langle a \rangle$) are relatively prime, so that p never divides s:

LEMMA 2.2. *If $\mathcal{D}$ is a set of positive divisors of m with least common multiple r, and p is a prime not dividing rs, then the ring $O(\mathcal{D})/pO(\mathcal{D})$ is semisimple.*

PROOF. The map $\mathbb{Z}G \longrightarrow O(\mathcal{D})$ $(a \mapsto \alpha_{\mathcal{D}})$ factors through $\mathbb{Z}G_r$, where $G_r = G/\langle a^r \rangle$. So there is a surjective homomorphism:

$$\mathbb{F}_p G_r \longrightarrow O(\mathcal{D})/pO(\mathcal{D}).$$

Since G_r has order rs, $\mathbb{F}_p G_r$ is semisimple artinian by Maschke's Theorem. ∎

If m is *square-free*, we obtain a filtration:

$$\mathbb{Z}G = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_t = \bigoplus_{d|m} O(d)$$

as follows. Say $m = p_1 p_2 \cdots p_t$ for distinct primes p_i. Let $\mathcal{D}_i$ denote the set of all positive divisors of $m/(p_1 \cdots p_i)$. Define

$$R_i = \bigoplus_{d|p_1\cdots p_i} O(d\mathcal{D}_i).$$

Note that if $d | p_1 \cdots p_{i-1}$, then $d\mathcal{D}_{i-1} = d\mathcal{D}_i \cup dp_i\mathcal{D}_i$, and the sets $d\mathcal{D}_i$ and $dp_i\mathcal{D}_i$ do not overlap, and are in bijective correspondence via multiplication by p_i.

The inclusion $R_{i-1} \longrightarrow R_i$ is just the direct product of inclusions:

$$O(d\mathcal{D}_{i-1}) \longrightarrow O(d\mathcal{D}_i) \oplus O(dp_i\mathcal{D}_i).$$

Considering the squares (2.1) above, we see that $p_i R_i \subseteq R_{i-1}$ for each i.

Let $B_n(\mathcal{D}, p\mathcal{D})$ denote the birelative K_n-group associated to the square (2.1). With the R_i just defined and with

$$F^i = \ker K_n(\mathbb{Z}G \longrightarrow R_i)$$

as in Section 1, we have machinery in place to analyze $SK_n(\mathbb{Z}G)$:

COROLLARY 2.3. *Suppose m is the product of distinct primes $p_1 \cdots p_t$ and is relatively prime to s. There is a filtration:*

$$0 = F^0 \subseteq F^1 \subseteq \cdots \subseteq F^t \subseteq SK_n(\mathbb{Z}G)$$

in which $SK_n(\mathbb{Z}G)/F^t$ is isomorphic to a subgroup of $\oplus_{d|m} SK_n\big(O(d)\big)$, and F^i/F^{i-1} is the direct product over all divisors d of $p_1 \cdots p_{i-1}$ of the cokernels of the maps:

$$K_{n+1}\big(O(p_i d\mathcal{D}_i)\big) \longrightarrow K_{n+1}\big(O(d\mathcal{D}_i)/p_i O(d\mathcal{D}_i)\big) \oplus B_n(d\mathcal{D}_i, p_i d\mathcal{D}_i)$$

in the Mayer-Vietoris sequences of squares (2.1) with $p = p_i$ and $\mathcal{D} = d\mathcal{D}_i$.

PROOF. From Proposition 1.1 we know that F^i/F^{i-1} is the direct product of the kernels of the separation maps:

$$K_n\big(O(d\mathcal{D}_{i-1})\big) \longrightarrow K_n\big(O(d\mathcal{D}_i)\big) \oplus K_n\big(O(p_i d\mathcal{D}_i)\big)$$

in the specified Mayer-Vietoris sequences. Now use the exactness of those sequences and the split surjectivity of $O(d\mathcal{D}_{i-1}) \longrightarrow O(d\mathcal{D}_i)$. ∎

Note that for each ordering of the prime factors of m, one obtains a different filtration of $SK_n(\mathbb{Z}G)$. The easiest layer to compute is F^t/F^{t-1}, since in this case $d\mathcal{D}_t = d$, and the $\Sigma(d\mathcal{D}_t) = \Sigma(d)$ are closest to simple components of $\mathbb{Q}G$. However, the $\Sigma(d)$ need not be simple, and this impedes the computations.

3. **The filtration for square-free G.** The filtration of $SK_n(\mathbb{Z}G)$ described in Section 2 does not reach all the way to the direct product of images of $\mathbb{Z}G$ in the simple components of $\mathbb{Q}G$, unless $G = H$ is cyclic. But in the special case where $|G| = ms$ is square-free and G/H is abelian, we can extend this filtration to have a step for each prime factor of ms, and thereby reach the simple components.

Since m and s are relatively prime, we can choose $b_1, \ldots b_s$ to be an abelian subgroup B of G. If B acts faithfully on $\mathbb{Z}[\zeta_d]$, then $\Sigma(d)$ is a crossed product $\mathbb{Q}(\zeta_d) \circ B$, which is simple.

On the other hand, if the kernel of $B \longrightarrow \mathrm{Aut}(\mathbb{Z}[\zeta_d])$ has an element b of prime order p, then in $G_d = G/\langle a^d \rangle$, the element ab generates a cyclic normal subgroup of order dp. Thus, replacing G by G_d and a by ab, the previous $\Sigma(d)$ and $O(d)$ become $\Sigma(d, pd)$ ($= \Sigma(\{d, pd\})$) and $O(d, pd)$ ($= O(\{d, pd\})$). Now we can form R_{t+1} by replacing $O(d, pd)$ by $O(d) \oplus O(pd)$ for each d for which p divides the order of the kernel of $B \longrightarrow \mathrm{Aut}(\mathbb{Z}[\zeta_d])$. Then $pR_{t+1} \subseteq R_t$, and we obtain a new subquotient F^{t+1}/F^t of $SK_n(\mathbb{Z}G)$, isomorphic to the direct product of the kernels of separation maps in the mayer-Vietoris sequences of squares:

$$\begin{array}{ccc} O(d,pd) & \longrightarrow & O(pd) \\ \downarrow & & \downarrow \\ O(d) & \longrightarrow & O(d)/pO(d) \end{array}$$

Iterating this process for each prime factor of s, we eventually reach the decomposition of $\mathbb{Q}G$ into simple components.

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EXAMPLE. Suppose G is the dihedral group of order 70, with generating rotation a of order 35, and with $b_1 = 1$, $b_2 = b$, where $b^2 = 1$ and $ba = a^{-1}b$. Take $p_1 = 5$ and $p_2 = 7$. Then

$$R_0 = \mathbb{Z}G = \mathcal{O}(1,5,7,35),$$
$$R_1 = \mathcal{O}(1,7) \oplus \mathcal{O}(5,35),$$
$$R_2 = \mathcal{O}(1) \oplus \mathcal{O}(7) \oplus \mathcal{O}(5) \oplus \mathcal{O}(35).$$

Under conjugation, $\{1,b\}$ acts faithfully on $\mathbb{Z}[\zeta_d]$ for $d = 5,7$, and 35. But the kernel of $\{1,b\} \longrightarrow \mathrm{Aut}(\mathbb{Z})$ is of order $p = 2$. Replacing G by $G_1 = \{1,b\}$ and $a \in G$ by $ab = b \in G_1$, the previous $\mathcal{O}(1) = \mathbb{Z}[\{1,b\}]$ becomes $\mathcal{O}(1,2)$, where $\mathcal{O}(1) = \mathbb{Z}$ and $\mathcal{O}(2) = \mathbb{Z}$. And then

$$R_3 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{O}(7) \oplus \mathcal{O}(5) \oplus \mathcal{O}(35),$$

corresponding to the decomposition:

$$\mathbb{Q}G = \mathbb{Q} \oplus \mathbb{Q} \oplus \Sigma(7) \oplus \Sigma(5) \oplus \Sigma(35)$$

of $\mathbb{Q}G$ into its simple components.

The squares produced in this process all have the same form as (2.1). In the next section we compute the birelative K_2-groups of the squares of type (2.1).

4. **Birelative K_2 computation.** In this section we put no restrictions on the positive integers m and s in the description of the group G in Section 2. If R is a ring with ideals I and J, where $I \cap J = \{0\}$, the birelative K_2-group $K_2(R; I, J)$ has been determined (in [7] and [10]) to be

$$I/I^2 \otimes_{R^e} J/J^2,$$

where R^e is additively the same as $R \otimes_{\mathbb{Z}} R$, and its multiplication is extended $\mathbb{Z}$-bilinearly from

$$(r_1 \otimes s_1)(r_2 \otimes s_2) = r_1 r_2 \otimes s_2 s_1$$

for all $r_i, s_i \in R$. Here I is a right R^e-module with:

$$m \cdot (r \otimes s) = smr$$

for $m \in I$, $r, s \in R$; and J is a left R^e-module with

$$(r \otimes s) \cdot m = rms$$

for $m \in J$, $r, s \in R$. So $I \otimes_{R^e} J$ is an abelian group. Since $IJ \subseteq I \cap J = 0$, this group is equal to $I/I^2 \otimes_{R^e} J/J^2$.

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Putting in the kernels, the square (2.1) of Section 2 above is part of the commutative diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & I \arrow[r, "\cong"] \arrow[d] & I' \arrow[d] & \\
0 \arrow[r] & J \arrow[r] \arrow[d, "\cong"] & \mathcal{O}(\mathcal{D} \cup p\mathcal{D}) \arrow[r] \arrow[d] & \mathcal{O}(p\mathcal{D}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J' \arrow[r] & \mathcal{O}(\mathcal{D}) \arrow[r] \arrow[d] & \mathcal{O}(\mathcal{D})/p\mathcal{O}(\mathcal{D}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
\tag{4.1}
$$

with exact rows and columns. Every object in this diagram is a $\mathbb{Z}G$-bimodule, where $\mathbb{Z}G$ acts through the maps from $\mathbb{Z}G$ to $\mathcal{O}(\mathcal{D} \cup p\mathcal{D})$, $\mathcal{O}(\mathcal{D})$ and $\mathcal{O}(p\mathcal{D})$; and every map in the diagram is $\mathbb{Z}G$-linear on each side, and multiplicative. So the projections $\pi_{p\mathcal{D}}$, $\pi_{\mathcal{D}}$ induce $\mathbb{Z}G$-linear multiplicative isomorphisms:

$$I/I^2 \cong I'/I'^2, \quad J/J^2 \cong J'/J'^2,$$

respectively.

As shown in [16], Section 7, if $\mathcal{D} \subseteq \mathcal{E}$ are sets of positive divisors of m, the kernel of the projection $\pi_{\mathcal{D}}\colon \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{D})$ is:

$$\gamma_{\mathcal{D}}(\alpha_{\mathcal{E}})\mathcal{O}(\mathcal{E}) = \mathcal{O}(\mathcal{E})\gamma_{\mathcal{D}}(\alpha_{\mathcal{E}}),$$

where

$$\gamma_{\mathcal{D}}(x) = \prod_{d \in \mathcal{D}} \Phi_d(x)$$

as in Section 2, and $\alpha_{\mathcal{E}}$ is the image of a.

To see concretely why $\gamma_{\mathcal{D}}(\alpha_{\mathcal{E}})$ generates the same ideal on the left or right, note that since every $d \in \mathcal{D}$ is relatively prime to each $n(i)$ (from $b_i a = a^{n(i)} b$),

$$\Phi_d(x^{n(i)}) = \prod_{e \mid n(i)} \Phi_{de}(x)$$

because these have the same roots and degree, and the left side is separable. So

$$b_i \gamma_{\mathcal{D}}(\alpha_{\mathcal{E}}) = \gamma_{\mathcal{D}}(\alpha_{\mathcal{E}}) \prod_{d \in \mathcal{D}} \prod_{\substack{e \mid n(i) \\ e > 1}} \Phi_{de}(\alpha_{\mathcal{E}}) b_i. \tag{4.2}$$

A similar expression for sliding b_i to the left is obtained by replacing $n(i)$ with its multiplicative inverse mod m.

LEMMA 4.3. *Right multiplication by $\gamma_{\mathcal{D}}(\alpha_{p\mathcal{D}})$ induces a left $\mathbb{Z}G$-linear isomorphism:*

$$O(p\mathcal{D})/I' \cong I'/I'^2.$$

Right multiplication by $\gamma_{p\mathcal{D}}(\alpha_{\mathcal{D}})$ induces a left $\mathbb{Z}G$-linear isomorphism:

$$O(\mathcal{D})/J' \cong J'/J'^2.$$

PROOF. Let O denote $O(p\mathcal{D})$ (resp. $O(\mathcal{D})$) and γ denote $\gamma_{\mathcal{D}}(\alpha_{p\mathcal{D}})$ (resp. $\gamma_{p\mathcal{D}}(\alpha_{\mathcal{D}})$). Since each projection is an isomorphism on kernels, I' (resp. J') $= O\gamma = \gamma O$. So I'^2 (resp. J'^2) $= O\gamma^2$. Right multiplication by γ induces a left $\mathbb{Z}G$-linear surjection:

$$O/O\gamma \to O\gamma/O\gamma^2.$$

Since p does not divide any $d \in \mathcal{D}$, γ is nonzero in every $\mathbb{Z}[\zeta]$-coordinate, hence is not a zero-divisor in O. Therefore the above surjection is also injective. ∎

Since $O(\mathcal{D})$ has the $\mathbb{Z}$-basis

$$\{\alpha^i b_j : 0 \le i < \delta, 1 \le j \le s\},$$

where

$$\delta = \sum_{d\in\mathcal{D}} \varphi(d),$$

cosets of these elements form an $\mathbb{F}_p$-basis of $O(\mathcal{D})/pO(\mathcal{D})$. We have established left $\mathbb{Z}G$-linear isomorphisms:

$$\begin{array}{ccc} & & O(p\mathcal{D})/I' \cong I'/I'^2 \cong I/I^2 \\ & \cong & \\ O(\mathcal{D})/pO(\mathcal{D}) & & \\ & \cong & \\ & & O(\mathcal{D})/J' \cong J'/J'^2 \cong J/J^2 \end{array}$$

which are additive, hence $\mathbb{F}_p$-linear. So I/I^2 has $\mathbb{F}_p$-basis:

$$\{\overline{\alpha^i b_j \gamma_{\mathcal{D}}(\alpha)} : 0 \le i < \delta, 1 \le j \le s\},$$

and J/J^2 has $\mathbb{F}_p$-basis:

$$\{\overline{\alpha^i b_j \gamma_{p\mathcal{D}}(\alpha)} : 0 \le i < \delta, 1 \le j \le s\},$$

where $\alpha = \alpha_{\mathcal{D}\cup p\mathcal{D}}$. Therefore we have proved:

PROPOSITION 4.4. *If I and J are the kernels of the projections in the square (2.1), then $I/I^2 \otimes_{\mathbb{Z}} J/J^2$ is an $\mathbb{F}_p$-vector space with basis:*

$$\{\overline{a^i b_k \cdot \gamma_{\mathcal{D}}(\alpha)} \otimes \overline{a^j b_\ell \cdot \gamma_{p\mathcal{D}}(\alpha)} : 0 \le i,j < \delta, 1 \le k, \ell \le s\}.$$ ∎

NOTE. Since $\gamma_{\mathcal{D}}(a)$ annihilates both I/I^2 and J/J^2, if r is the least common multiple $\operatorname{lcm}(\mathcal{D})$, the actions of $\mathbb{Z}G$ on I/I^2 and J/J^2 factor through $\mathbb{Z}G_r$, where $G_r = G/\langle a^r\rangle$ has the same descriptions as G, except with m replaced by r. So in the tensors which form the basis of $I/I^2 \otimes_{\mathbb{Z}} J/J^2$, we may assume that a has order r. The range $0 \le i,j < \delta$ is unaffected, since

$$\delta = \sum_{d\in\mathcal{D}} \varphi(d) \le \sum_{d|r} \varphi(d) = r.$$

The birelative K_2 of the fiber squares we are considering is

$$I/I^2 \otimes_{R^e} J/J^2,$$

where we may take R to be either $O(\mathcal{D} \cup p\mathcal{D})$ or $\mathbb{Z}G$ (which acts through its projection to $O(\mathcal{D} \cup p\mathcal{D})$), or, in view of the preceding note, $\mathbb{Z}G_r$. To compute this birelative group, which we denote $B_2(\mathcal{D}, p\mathcal{D})$, we need only reduce the $\mathbb{F}_p$-vector space $I/I^2 \otimes_{\mathbb{Z}} J/J^2$ modulo the subspace generated by the additional relators:

$$\big((x\cdot z)\otimes y\big) - \big(x\otimes(z\cdot y)\big)$$

where $z \in \mathbb{Z}G_r^e$. This expression is additive in z, so we can generate all additional relators by using those with $z = g\otimes h$ (for $g,h\in G_r$). If z is a product, such a relator is a sum of relators in which one factor at a time is moved across $\otimes$; so we only need those relators with $z = a\otimes 1$, $b_i\otimes 1$, $1\otimes a$ or $1\otimes b_i (1\le i\le s)$. The relators are also additive in x and y, so we only need those relators where x and y come from the $\mathbb{F}_p$-bases of I/I^2 and J/J^2, respectively. This reduces us to a finite list of relators:

$$(4.5)\qquad \begin{aligned}
&\big(a^ib_k\cdot\overline{\gamma_{\mathcal{D}}(\alpha)}\cdot a\otimes a^jb_\ell\cdot\overline{\gamma_{p\mathcal{D}}(\alpha)}\big) - \big(a^ib_k\cdot\overline{\gamma_{\mathcal{D}}(\alpha)}\otimes a^{j+1}b_\ell\cdot\overline{\gamma_{p\mathcal{D}}(\alpha)}\big)\\
&\big(a^{i+1}b_k\cdot\overline{\gamma_{\mathcal{D}}(\alpha)}\otimes a^jb_\ell\cdot\overline{\gamma_{p\mathcal{D}}(\alpha)}\big) - \big(a^ib_k\cdot\overline{\gamma_{\mathcal{D}}(\alpha)}\otimes a^jb_\ell\cdot\overline{\gamma_{p\mathcal{D}}(\alpha)}\cdot a\big)\\
&\big(a^ib_k\cdot\overline{\gamma_{\mathcal{D}}(\alpha)}\cdot b_u\otimes a^jb_\ell\cdot\overline{\gamma_{p\mathcal{D}}(\alpha)}\big) - \big(a^ib_k\cdot\overline{\gamma_{\mathcal{D}}(\alpha)}\otimes b_ua^jb_\ell\cdot\overline{\gamma_{p\mathcal{D}}(\alpha)}\big)\\
&\big(b_ua^ib_k\cdot\overline{\gamma_{\mathcal{D}}(\alpha)}\otimes a^jb_\ell\cdot\overline{\gamma_{p\mathcal{D}}(\alpha)}\big) - \big(a^ib_k\cdot\overline{\gamma_{\mathcal{D}}(\alpha)}\otimes a^jb_\ell\cdot\overline{\gamma_{p\mathcal{D}}(\alpha)}\cdot b_u\big)
\end{aligned}$$

where $0\le i,j<\delta$ and $1\le k,\ell,u\le s$. One need only express these relations as $\mathbb{F}_p$-linear combinations of the basis of $I/I^2\otimes_{\mathbb{Z}} J/J^2$, using relations such as (4.2), and then mechanically determine the quotient $B_2(\mathcal{D},p\mathcal{D})$.

Notice that the relators (4.5) generate all relators of the same form, but with i and j unrestricted integers; for this larger set of relators is zero in $B_2(\mathcal{D},p\mathcal{D})$. Since we may take $a\in G_r$, the exponents i and j can be understood as elements of $\mathbb{Z}/r\mathbb{Z}$, where $r=\operatorname{lcm}(\mathcal{D})$.

When $\mathcal{D}$ consists of a single divisor d of m, the situation simplifies somewhat:

$$\gamma_{\mathcal{D}}(x) = \Phi_d(x),\quad \gamma_{p\mathcal{D}}(x) = \Phi_{pd}(x),$$
$$\alpha_{\mathcal{D}} = \zeta_d,\quad \alpha_{p\mathcal{D}} = \zeta_{pd}.$$

The computation of $B_2(\mathcal{D},p\mathcal{D})$ can always be reduced to this case:

PROPOSITION 4.6. *If the prime p does not divide any member of $\mathcal{D}$, the birelative K_2-group $B_2(\mathcal{D}, p\mathcal{D})$ of the square (2.1) satisfies:*

$$B_2(\mathcal{D}, p\mathcal{D}) \cong \bigoplus_{d \in \mathcal{D}} B_2(d, pd).$$

PROOF. Let S denote $\mathbb{Z} - p\mathbb{Z}$. By localizing the diagram (4.1) at p, we obtain the surjective fiber square:

$$\begin{array}{lccc} (4.7) & S^{-1}O(\mathcal{D} \cup p\mathcal{D}) & \longrightarrow & S^{-1}O(p\mathcal{D}) \\ & \downarrow & & \downarrow \\ & S^{-1}O(\mathcal{D}) & \longrightarrow & S^{-1}\big(O(\mathcal{D})/pO(\mathcal{D})\big) \end{array}$$

with birelative K_2-group

$$\begin{aligned} B_2^S(\mathcal{D}, p\mathcal{D}) &= S^{-1}(I/I^2) \otimes_{S^{-1}O(\mathcal{D} \cup p\mathcal{D})^e} S^{-1}(J/J^2) \\ &= \big(S^{-1}(I/I^2) \otimes_{\mathbb{Z}} S^{-1}(J/J^2)\big)/R, \end{aligned}$$

where R is the subgroup generated by the elements:

$$(xc \otimes y) - (x \otimes cy)$$

for $c \in S^{-1}O(\mathcal{D} \cup p\mathcal{D})^e$. Since multiplication by an element of S is bijective on I/I^2 and J/J^2, localizations of these at p are $S^{-1}\mathbb{Z}$-linear isomorphisms, inducing a group isomorphism:

$$I/I^2 \otimes_{\mathbb{Z}} J/J^2 \cong S^{-1}(I/I^2) \otimes_{\mathbb{Z}} S^{-1}(J/J^2).$$

Under this map, the extra relations for $B_2(\mathcal{D}, p\mathcal{D})$ are mapped onto R, so

$$B_2(\mathcal{D}, p\mathcal{D}) \cong B_2^S(\mathcal{D}, p\mathcal{D}).$$

Since p does not divide any $d \in \mathcal{D}$, the square (4.7) is isomorphic to the square:

$$\begin{array}{ccc} S^{-1}\mathcal{H}(\mathcal{D} \cup p\mathcal{D}) & \longrightarrow & S^{-1}\mathcal{H}(p\mathcal{D}) \\ \downarrow & & \downarrow \\ S^{-1}\mathcal{H}(\mathcal{D}) & \longrightarrow & S^{-1}\big(\mathcal{H}(\mathcal{D})/p\mathcal{H}(\mathcal{D})\big) \end{array}$$

where

$$\mathcal{H}(\mathcal{E}) = \bigoplus_{d \in \mathcal{E}} O(d);$$

this follows from the fact that $r\mathcal{H}(\mathcal{E}) \subseteq O(\mathcal{E})$ if $r = \mathrm{lcm}(\mathcal{E})$, and [16], Section 9. Since all maps in the latter square operate coordinatewise, its birelative K_2-group is

$$\bigoplus_{d \in \mathcal{D}} B_2(d, pd). \qquad \blacksquare$$

If Proposition 4.6 is used to determine $B_2(\mathcal{D}, p\mathcal{D})$, one may obtain a complicated description of its generators. In the computations to follow, where G is dihedral or cyclic a more direct determination of $B_2(\mathcal{D}, p\mathcal{D})$ is expedient.

5. **Birelative K_2 for dihedral groups of order $2m, m$ odd.** To express the relators (4.5) in terms of the $\mathbb{F}_p$-basis given in Proposition 4.4, one must move each b_i past $\gamma_{\mathcal{D}}(\alpha)$ and $\gamma_{p\mathcal{D}}(\alpha)$. This can be done with the aid of (4.2), but the extra factor from $\mathbb{Z}[\alpha]$ that is produced as a by-product is rather complicated. A simpler formula is available when G is dihedral of order $2m$. Of course if G is cyclic of order m, matters are even easier, since the only b_i is $b_1 = 1$. In this section we study the dihedral case. Having done that, we describe in the next section the parallel but simplified arguments which derive the birelative K_2 for cyclic groups.

For the rest of this section, assume G is dihedral, with presentation:

$$(a, b : a^m = 1, b^2 = 1, ba = a^{-1}b),$$

and take $b_1 = 1$ and $b_2 = b$. Then for any set $\mathcal{E}$ of positive divisors of m,

$$b\gamma_{\mathcal{E}}(a) = \gamma_{\mathcal{E}}(a^{-1})b, \quad \gamma_{\mathcal{E}}(a)b = b\gamma_{\mathcal{E}}(a^{-1}).$$

To rewrite $\gamma_{\mathcal{E}}(a^{-1})$, we use the symmetry of cyclotomic polynomials: In $\mathbb{Z}[x, x^{-1}]$,

$$\Phi_d(x^{-1}) = \begin{cases} x^{-\varphi(d)}\Phi_d(x), & \text{if } d > 1, \\ -x^{-\varphi(d)}\Phi_d(x), & \text{if } d = 1. \end{cases}$$

This is easily verified by induction on the number of prime factors of d, using the standard cyclotomic identities:

$$\Phi_{pd}(x) = \begin{cases} \Phi_d(x^p), & \text{if } p \mid d, \\ \Phi_d(x^p) / \Phi_d(x), & \text{if } p \nmid d, \end{cases}$$

for every prime p.

So if $\mathcal{D}$ is a nonempty set of positive divisors of m, p is a prime not dividing $r = \operatorname{lcm}(\mathcal{D})$, and $\delta = \sum_{d \in \mathcal{D}} \varphi(d)$, then

$$\overline{\gamma_{\mathcal{D}}(\alpha)} \cdot b = \begin{cases} ba^{-\delta} \cdot \overline{\gamma_{\mathcal{D}}(\alpha)}, & \text{if } 1 \notin \mathcal{D}, \\ -ba^{-\delta} \cdot \overline{\gamma_{\mathcal{D}}(\alpha)}, & \text{if } 1 \in \mathcal{D}, \end{cases}$$

and

$$\overline{\gamma_{p\mathcal{D}}(\alpha)} \cdot b = ba^{-\delta(p-1)} \cdot \overline{\gamma_{p\mathcal{D}}(\alpha)},$$

sicne $1 \notin p\mathcal{D}$.

Now we simplify notation by writing (g, h) for

$$g \cdot \overline{\gamma_{\mathcal{D}}(\alpha)} \otimes h \cdot \overline{\gamma_{p\mathcal{D}}(\alpha)}$$

in $I/I^2 \otimes_{\mathbb{Z}} J/J^2$, where g and h come from $\mathbb{Z}G_r$. The birelative K_2-group $B_2(\mathcal{D}, p\mathcal{D})$ of the square (2.1) in the dihedral case is generated by all $(a^i b^k, a^j b^\ell)$ with $0 \le i, j < r$ and $0 \le k, \ell \le 1$. (Replacing r by δ, we get an $\mathbb{F}_p$-basis of $I/I^2 \otimes_{\mathbb{Z}} J/J^2$.)

The relations (4.5) become:

$$(5.1)\quad \begin{array}{rl}
\text{(i)} & (a^{i+1}, a^j) = (a^i, a^{j+1}) \\
\text{(ii)} & (a^{i-1}b, a^j) = (a^ib, a^{j+1}) \\
\text{(iii)} & (a^{i+1}, a^jb) = (a^i, a^{j+1}b) \\
\text{(iv)} & (a^{i-1}b, a^jb) = (a^ib, a^{j+1}b) \\
\\
\text{(v)} & (a^{i+1}, a^j) = (a^i, a^{j+1}) \\
\text{(vi)} & (a^{i+1}b, a^j) = (a^ib, a^{j+1}) \\
\text{(vii)} & (a^{i+1}, a^jb) = (a^i, a^{j-1}b) \\
\text{(viii)} & (a^{i+1}b, a^jb) = (a^ib, a^{j-1}b) \\
\\
\text{(ix)} & \pm(a^{i+\delta}b, a^j) = (a^i, a^{-j}b) \\
\text{(x)} & \pm(a^{i-\delta}, a^j) = (a^ib, a^{-j}b) \\
\text{(xi)} & \pm(a^{i+\delta}b, a^jb) = (a^i, a^{-j}) \\
\text{(xii)} & \pm(a^{i-\delta}, a^jb) = (a^ib, a^{-j}) \\
\\
\text{(xiii)} & (a^{-i}b, a^j) = (a^i, a^{j+\delta(p-1)}b) \\
\text{(xiv)} & (a^{-i}, a^j) = (a^ib, a^{j+\delta(p-1)}b) \\
\text{(xv)} & (a^{-i}b, a^jb) = (a^i, a^{j-\delta(p-1)}) \\
\text{(xvi)} & (a^{-i}, a^jb) = (a^ib, a^{j-\delta(p-1)})
\end{array}$$

where the $\pm$ sign means + if $1 \notin \mathcal{D}$, and $-$ if $1 \in \mathcal{D}$. Among those generators with $0 \leq i,j < \delta$, these are a full set of defining relations for $B_2(\mathcal{D}, p\mathcal{D})$. And among the generators with $0 \leq i,j < r$ (which means for any $i,j \in \mathbb{Z}$ since $a^r = 1$ in G_r), the equations (5.1) are true for all integers i and j.

From the relations (5.1), we deduce:

$$\begin{aligned}
(a^i, a^j) &= (1, a^{i+j}) && \text{(by (i))} \\
(a^i, a^jb) &= (1, a^{i+j}b) && \text{(by (iii))} \\
&= (1, a^{j-i}b) && \text{(by (vii))} \\
(a^ib, a^j) &= (1, a^{1+j+\delta(p-1)}b) && \text{(by (vi), (xiii))} \\
(a^ib, a^jb) &= (1, a^{j-i-\delta(p-1)}) && \text{(by (viii), (xv)).}
\end{aligned}$$

By proper choices of i and j, the integers $i+j$ and $i-j$ can be made into any two integers congruent modulo 2. So $B_2(\mathcal{D}, p\mathcal{D})$ is generated by the elements $(1, b)$, $(1, ab)$ and $(1, a^i)$, for $0 \leq i < r$.

For the rest of this section, *assume the order m of the rotation subgroup is odd*, so that $r = \operatorname{lcm}(\mathcal{D})$ is odd, and the prime p is also odd. Then

$$(1, ab) = (1, a^{1+r}b) = (1, b).$$

So for all integers i,

$$(a^i, b) = (1, b).$$

By linearity of $\otimes$ in the first variable,

$$0 = \big(\gamma_{\mathcal{D}}(a), b\big) = \gamma_{\mathcal{D}}(1)(1, b).$$

If $1 \notin \mathcal{D}$, then by Diederichsen's formula (see [16], Lemma 9.3), $\gamma_{\mathcal{D}}(1)$ is relatively prime to p. Since $p(1, b) = 0$, it follows that $(1, b) = 0$. On the other hand, if $1 \in \mathcal{D}$, then by (5.1) (ix),

$$\begin{aligned} 0 &= (a^{i+\delta} b, a^j) + (a^i, a^{-j} b) \\ &= 2(1, b), \end{aligned}$$

since

$$(a^t b, a^u) = (a^t, a^u b) = (1, b)$$

for all $t, u \in \mathbb{Z}$. Since p is odd and $p(1, b) = 0$, it follows that $(1, b) = 0$ in this case too.

THEOREM 5.2. *Suppose G is dihedral of order $2m$, m is odd, $B_2(\mathcal{D}, p\mathcal{D})$ is the birelative K_2 of the square (2.1), $\alpha = \alpha_{\mathcal{D}}$, and $\delta = \sum_{d \in \mathcal{D}} \varphi(d)$. There is a surjective additive homomorphism*

$$f \colon \mathbb{F}_p[\alpha] \longrightarrow B_2(\mathcal{D}, p\mathcal{D})$$

taking α^i to $(1, a^i)$ for all $i \in \mathbb{Z}$, and the kernel of f is the subgroup $\mathcal{R}$ generated by the elements:

$$\begin{cases} \alpha^i - \alpha^{-\delta p - i}, & \text{if } 1 \notin \mathcal{D}, \\ \alpha^i + \alpha^{-\delta p - i}, & \text{if } 1 \in \mathcal{D}. \end{cases}$$

So $B_2(\mathcal{D}, p\mathcal{D}) \cong \mathbb{F}_p[\alpha] / \mathcal{R}$ is an elementary abelian p-group of rank*:*

$$\begin{cases} \delta / 2, & \text{if } 1 \notin \mathcal{D}, \\ (\delta - 1)/2, & \text{if } 1 \in \mathcal{D}. \end{cases}$$

PROOF. Among the generators $(1, a^i)$, $0 \leq i < r$, of $B_2(\mathcal{D}, p\mathcal{D})$, relations (5.1) (x), (xv) imply

$$\begin{aligned} (1, a^i) &= (bb, a^i) \\ &= \pm(1, a^{-i-\delta-\delta(p-1)}) \\ &= \pm(1, a^{-\delta p - i}) \end{aligned} \tag{5.3}$$

where + applies if $1 \notin \mathcal{D}$ and $-$ applies if $1 \in \mathcal{D}$. Since $\gamma_{\mathcal{D}}(a)$ annihilates J / J^2, we are led to another set of relations: If, for $t \in \mathbb{Z}$,

$$x^t \gamma_{\mathcal{D}}(x) \sum_u c_{tu} x^u \quad (c_{tu} \in \mathbb{Z}),$$

then

$$\sum_u c_{tu}(1, a^u) = \big(1, a^t \gamma_{\mathcal{D}}(a)\big) = (1, 0) = 0.$$

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So the homomorphism of additive groups:

$$\mathbb{Z}[x] \longrightarrow B_2(\mathcal{D}, p\mathcal{D}),$$

taking x^i to $(1, a^i)$ for each i, has $\gamma_{\mathcal{D}}(x)\mathbb{Z}[x] + p\mathbb{Z}[x]$ in its kernel, so induces a (surjective) homomorphism

$$f: \mathbb{F}_p[\alpha] \longrightarrow B_2(\mathcal{D}, p\mathcal{D})$$

taking α^i to $(1, a^i)$ for all integers i (recall $\alpha^r = 1$ where $r = \operatorname{lcm}(\mathcal{D})$). And by the relations (5.3), $f(\mathcal{R}) = 0$; so we have an induced homomorphism,

$$\bar{f}: \mathbb{F}_p[\alpha]/\mathcal{R} \longrightarrow B_2(\mathcal{D}, p\mathcal{D}).$$

Next we construct an inverse to $\bar{f}$. Define V to be the $\mathbb{F}_p$-linear span of the elements $(1, a^i)$ for $0 \leq i < \delta$ in $I/I^2 \otimes_{\mathbb{Z}} J/J^2$. Define

$$F_1: I/I^2 \otimes_{\mathbb{Z}} J/J^2 \longrightarrow V$$

to be the $\mathbb{F}_p$-linear map taking:

$$\begin{aligned} (a^i, a^j) &\longrightarrow (1, a^{i+j}) \\ (a^i b, a^j) &\longrightarrow 0 \\ (a^i, a^j b) &\longrightarrow 0 \\ (a^i b, a^j b) &\longrightarrow (1, a^{j-i-\delta(p-1)}) \end{aligned} \tag{5.4}$$

for those i and j with $0 \leq i, j < \delta$. The images lie in V because each power of a is a $\mathbb{Z}$-linear combination of $1, a, \ldots, a^{\delta-1}$ modulo $\gamma_{\mathcal{D}}(a)$.

Since $\mathbb{Z}[\alpha]$ ($\subseteq O(\mathcal{D})$) has $\mathbb{Z}$-basis $1, \alpha, \ldots, \alpha^{\delta-1}$, the quotient

$$\mathbb{F}_p[\alpha] = \mathbb{Z}[\alpha]/p\mathbb{Z}[\alpha]$$

has an $\mathbb{F}_p$-basis consisting of the cosets of these elements. So there is an $\mathbb{F}_p$-linear isomorphism,

$$F_2: V \longrightarrow \mathbb{F}_p[\alpha]$$

taking $(1, a^i)$ to α^i for $0 \leq i < \delta$.

Let

$$F_3: \mathbb{F}_p[\alpha] \longrightarrow \mathbb{F}_p[\alpha]/\mathcal{R}$$

be the canonical map. Then define

$$F: I/I^2 \otimes_{\mathbb{Z}} J/J^2 \longrightarrow \mathbb{F}_p[\alpha]/\mathcal{R}$$

to be the composite $F_3 F_2 F_1$.

Since $\gamma_{\mathcal{D}}(a)$ annihilates both I/I^2 and J/J^2, the effects (5.4) of F_1 are true for *all* integers i and j. (The fourth effect in the list is verified by using the involution $\alpha \longrightarrow \alpha^{-1}$ on $\mathbb{Z}[\alpha]$.) Thus F_1 has the same image when applied to both sides of each of the relations

(5.1) except for the relations (x) and (xi); and $F_3F_2F_1$ has the same image when applied to both sides of (x) and (xi). Therefore F induces an $\mathbb{F}_p$-linear map

$$\bar{F}: B_2(\mathcal{D}, p\mathcal{D}) \longrightarrow \mathbb{F}_p[\alpha]/\mathcal{R}$$

inverse to $\bar{f}$.

For the assertion about rank, note that complex conjugation in each d-coordinate defines the ring automorphism of $\mathbb{Z}[\alpha]$ taking α to α^{-1}, and induces an automorphism θ of the ring $\mathbb{F}_p[\alpha]$. The maps

$$T^+: \mathbb{F}_p[\alpha] \longrightarrow \mathbb{F}_p[\alpha], \quad x \longrightarrow x + \theta(x),$$
$$T^-: \mathbb{F}_p[\alpha] \longrightarrow \mathbb{F}_p[\alpha], \quad x \longrightarrow x - \theta(x),$$

are $\mathbb{F}_p$-linear, and using the fact that p is odd, so 2 is a unit in $\mathbb{F}_p$, it is straight-forward to show that:

$$\text{kernel } (T^+) = \text{ image } (T^-), \text{ and}$$
$$\text{kernel } (T^-) = \text{ image } (T^+).$$

By the linearity of these maps, image(T^-) is spanned by the elements $\alpha^i - \alpha^{-i}$ for $i \geq 1$, and image(T^+) is spanned by 2 and the $\alpha^i + \alpha^{-i}$ for $i \geq 1$. Since $\alpha^r + \alpha^{-r} = 2$, we can drop the initial 2 from the spanning set of image(T^+).

By [16], Proposition 9.1, since $p \nmid r = \operatorname{lcm}(\mathcal{D})$, the inclusion:

$$\mathbb{Z}[\alpha] \longrightarrow \bigoplus_{d \in \mathcal{D}} \mathbb{Z}[\zeta_d]$$

induces an isomorphism of rings:

$$\mathbb{F}_p[\alpha] \cong \bigoplus_{d \in \mathcal{D}} \mathbb{Z}[\zeta_d]/p\mathbb{Z}[\zeta_d].$$

And T^+ respects this decomposition, so has image:

$$\bigoplus_{d \in \mathcal{D}} \mathbb{Z}[\zeta_d + \zeta_d^{-1}]/p\mathbb{Z}[\zeta_d + \zeta_d^{-1}].$$

thus the $\mathbb{F}_p$-rank of image(T^+)

$$= \begin{cases} \sum_{d \in \mathcal{D}} \varphi(d)/2 = \delta/2, & \text{if } 1 \notin \mathcal{D}, \\ 1 + \sum_{1 < d \in \mathcal{D}} \varphi(d)/2 = (\delta + 1)/2, & \text{if } 1 \in \mathcal{D}, \end{cases}$$

and the $\mathbb{F}_p$-rank of $\mathbb{F}_p[\alpha]$ is δ. So if $1 \notin \mathcal{D}$,

$$\mathbb{F}_p[\alpha]/\langle \alpha^i - \alpha^{-i} : i \geq 1 \rangle \cong \mathbb{F}_p^{\delta/2};$$

and if $1 \in \mathcal{D}$,

$$\mathbb{F}_p[\alpha]/\langle \alpha^i + \alpha^{-i} : i \geq 1 \rangle \cong \mathbb{F}_p^{(\delta-1)/2}.$$

Recall that $\alpha_{\mathcal{D}}^r = 1$, so the exponents of α can be taken from $\mathbb{Z}/r\mathbb{Z}$. Let t denote the multiplicative inverse of 2 in $\mathbb{Z}/r\mathbb{Z}$. Multiplication of the ring $\mathbb{F}_p[\alpha]$ by the unit $\alpha^{-\delta pt}$ is an invertible $\mathbb{F}_p$-linear map taking the above denominators to $\mathcal{R}$; so

$$\mathbb{F}_p[\alpha]/\mathcal{R} \cong \begin{cases} \mathbb{F}_p^{\delta/2}, & \text{if } 1 \notin \mathcal{D}, \\ \mathbb{F}_p^{(\delta-1)/2}, & \text{if } 1 \in \mathcal{D}. \end{cases} \qquad \blacksquare$$

NOTE. In the filtration of Section 2, derived when m is square-free, each $\mathcal{D}$ has the form $q_1 \cdots q_i \mathcal{D}_i$, where $q_1, \ldots, q_i, q_{i+1}, \ldots, q_n$ are distinct primes and $\mathcal{D}_i$ is the set of all positive divisors of $q_{i+1} \cdots q_n$. In this case,

$$\delta = \sum_{d \in \mathcal{D}} \varphi(d) = (q_1 - 1) \cdots (q_i - 1) q_{i+1} \cdots q_n.$$

Also, multiplication by $\alpha^{-\delta pt}$ in the above argument is unnecessary exactly when $\alpha^{-\delta pt} = 1$, that is, when $r | \delta$. Since

$$\delta = \sum_{d \in \mathcal{D}} \varphi(d) \leq \sum_{d|r} \varphi(d) = r,$$

this adjustment is unnecessary if and only if $\mathcal{D}$ consists of *all* positive divisors of $r = \operatorname{lcm}(\mathcal{D})$; and this is equivalent to $O(\mathcal{D}) = \mathbb{Z} G_r$, where $G_r = G/\langle a^r \rangle$. For such squares:

$$\begin{array}{ccc} \mathbb{Z} G_{rp} & \longrightarrow & \mathbb{Z} G_{rp}/\langle \Phi_p(a^r) \rangle \\ \downarrow & & \downarrow \\ \mathbb{Z} G_r & \longrightarrow & \mathbb{F}_p G_r, \end{array}$$

we obtain $B_2(\mathcal{D}, p\mathcal{D}) \cong \mathbb{F}_p^{(r-1)/2}$.

6. Birelative K_2 for finite cyclic groups.

Suppose G is cyclic, generated by an element a of order m. Then the birelative K_2-group $B_2(\mathcal{D}, p\mathcal{D})$ of square (2.1) can be computed just as in Section 5; but the details are simpler.

THEOREM 6.1. *If G is cyclic of finite order m generated by a, then:*

$$B_2(\mathcal{D}, p\mathcal{D}) \cong \mathbb{F}_p^{\delta}, \quad \delta = \sum_{d \in \mathcal{D}} \varphi(d),$$

with $\mathbb{F}_p$-basis:

$$\{ \overline{\gamma_{\mathcal{D}}(\alpha)} \otimes a^i \cdot \overline{\gamma_{p\mathcal{D}}(\alpha)} : 0 \leq i < \delta \},$$

where $\alpha = \alpha_{\mathcal{D} \cup p\mathcal{D}}$.

PROOF. As in the proof of Theorem 5.2, $B_2(\mathcal{D}, p\mathcal{D})$ is generated by all elements (a^i, a^j) for $1 \leq i, j \leq r$, and these satisfy:

$$(a^i, a^j) = (1, a^{i+j}), \text{ and}$$
$$\sum_u c_{tu}(1, a^u) = 0, \text{ whenever } x^t \gamma_{\mathcal{D}}(x) = \sum_u c_{tu} x^u \text{ with } c_{tu} \in \mathbb{Z}.$$

So $\alpha^i_{\mathcal{D}} \longrightarrow (1, a^i)$ induces an $\mathbb{F}_p$-linear map

$$\bar{f}\colon \mathbb{F}_p[\alpha_{\mathcal{D}}] \longrightarrow B_2(\mathcal{D}, p\mathcal{D}).$$

To define an inverse to $\bar{f}$, we proceed as in the proof of Theorem 5.2 to define:

$$\begin{aligned} B_2(\mathcal{D}, p\mathcal{D}) &\longrightarrow V \cong \mathbb{F}_p[\alpha_{\mathcal{D}}] \\ (a^i, a^j) &\longrightarrow (1, a^{i+j}) \longrightarrow \alpha^{i+j}_{\mathcal{D}}. \end{aligned}$$

■

7. K_3 **of some orders.** To see how much of each birelative K_2 survives in $SK_2(\mathbb{Z}G)$, one must examine the preceding groups $K_3\big(O(\mathcal{D})\big)$ and $K_3\big(O(p\mathcal{D})\big)$ in the Mayer-Vietoris sequences for squares such as (2.1). Direct computations of these are not yet accessible unless $\mathcal{D}$ contains only one element d.

We begin with no assumptions except the following: Suppose $d \geq 1$ and ζ_d is a primitive d-th root of unity. Suppose B is a subgroup of $\mathrm{Aut}\big(\mathbb{Q}(\zeta_d)\big)$ of order s, relatively prime to d. Define Λ_d to be the twisted group ring $\mathbb{Z}[\zeta_d] \circ B$ with trivial factor set, F_d to be the fixed field $\mathbb{Q}(\zeta_d)^B$, and R_d to be alg. int. (F_d). Where convenient, we drop the subscript d.

Any basis of both $\mathbb{Q}(\zeta)$ over F and $\mathbb{Z}[\zeta]$ over R yields a matrix representation:

$$\mathbb{Q}(\zeta) \circ B \cong M_s(F)$$

taking Λ into $M_s(R)$. (The $\mathbb{Q}(\zeta) \circ B$-module defining this representation is $\mathbb{Q}(\zeta)$, on which B acts by evaluation.) Since d and s are relatively prime, and the only primes ramified in $\mathbb{Z}[\zeta]$ are those dividing d, while their ramification index from R to $\mathbb{Z}[\zeta]$ divides s, the $\mathbb{Z}$-order Λ is a tamely ramified twisted group ring. By a theorem of M. Rosen (see [23], Theorem 40.13), Λ is therefore a hereditary order.

Thus Λ falls within the class of "tiled orders" for which M. K. Keating has computed the K-theory in terms of the K-groups of their center and of residue rings of their completions at primes of the center. The center of Λ is R.

Using the matrix description of hereditary orders over a complete discrete valuation ring (see [23], Theorem 39.14), we find that, for each maximal ideal μ of R, if $J = \mathrm{rad}\,\hat{\Lambda}_\mu$, then $\hat{\Lambda}_\mu / J$ is a direct product of r matrix rings over R/μ, where

$$J^r \mathbb{Z}[\zeta] = \mu \mathbb{Z}[\zeta]$$

(see [23], Corollary 39.18). If $\mu \nmid d$, then $r = 1$, for as in [16] (proof of Proposition 10.2),

$$dM_s(\hat{R}_\mu) \subseteq \hat{\Lambda}_\mu,$$

and by the matrix description referred to above,

$$\mu M_s(\hat{R}_\mu) \subseteq \hat{\Lambda}_\mu;$$

so, since $1 \in dR + \mu$, $M_s(\hat{R}_\mu) = \hat{\Lambda}_\mu$. On the other hand, if $\mu | d$, so that $\mu \nmid s$, then, as in [17] (p. 182),

$$J = P_1 \cdots P_g \hat{\Lambda}_\mu,$$

where $P_1, \dots, P_g$ are the primes of $\mathbb{Z}[\zeta]$ over μ. Then

$$J^r\mathbb{Z}[\zeta] = (P_1 \cdots P_g)^r\mathbb{Z}[\zeta] = \mu\mathbb{Z}[\zeta]$$

if and only if $r = e =$ the ramification index of μ in $\mathbb{Z}[\zeta]$.

Using this determination of r in Keating's theorem ([9], Theorem 2), we get:

$$K_n(\Lambda) \cong K_n(M) \oplus \bigoplus_{\mu \in \max R} \big(e(\mu) - 1\big) K_n(R/\mu) \tag{7.1}$$

for all $n \geq 0$, where M is a maximal R-order containing Λ, and $e(\mu)$ is the ramification index of μ in $\mathbb{Z}[\zeta]$. By [23], Corollary 21.7, M is Morita equivalent to R; so $K_n(M) \cong K_n(R)$.

Now consider $K_3(\Lambda)$. In independent work, M. Levine [15], and A. S. Merkurjev with A. A. Suslin [18], have established formula

$$K_3(R) \cong \mathbb{Z}^{r_2} \otimes \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r_1-1} \otimes \mathbb{Z}/2w_2(F)\mathbb{Z}, & r_1 > 0 \\ \mathbb{Z}/w_2(F)\mathbb{Z}, & r_1 = 0, \end{cases}$$

where F is a number field with r_1 real embeddings and $2r_2$ imaginary embeddings, and $R =$ alg. int. (F). The number $w_2(F)$, which figures in the Birch-Tate conjecture, is the order of the etale cohomology group

$$H^0_{et}\big(F, \mathbb{Q}/\mathbb{Z}(2)\big).$$

For each prime ℓ, the ℓ-primary part of this group is

$$\varinjlim_n H^0(F, \mu_{\ell^n} \otimes \mu_{\ell^n}) = \varinjlim_n (\mu_{\ell^n} \otimes \mu_{\ell^n})^{\mathrm{Aut}(F^s/F)},$$

where F^s is the separable closure of F and μ_{ℓ^n} is the group of ℓ^n-roots of unity in F^s (see [25], Section III). Here the action of $\mathcal{G} = \mathrm{Aut}(F^s/F)$ is diagonal, so there is a $\mathcal{G}$-isomorphism

$$\mu_{\ell^n} \otimes \mu_{\ell^n} \cong \mu_{\ell^n},$$

where $\mathcal{G}$ acts on the latter by $g \cdot \zeta = g^2(\zeta)$. An element ζ of μ_{ℓ^n} is fixed by that action if and only if $\mathrm{Aut}\big(F(\zeta)/F\big)$ has exponent at most 2 (see [18], Section 4.19.1). So the ℓ-primary factor of $w_2(F)$ is ℓ^n, where n is the largest integer for which $\mathrm{Aut}\big(F(\zeta)/F\big)$ has exponent at most 2.

In the two examples of direct relevance to the present analysis, a straight-forward computation shows:

$$\begin{aligned} w_2\big(\mathbb{Q}(\zeta_d)\big) &= w_2\big(\mathbb{Q}(\zeta_d + \zeta_d^{-1})\big) \\ &= \mathrm{lcm}(24, 2d). \end{aligned}$$

Assembling these facts, we have:

$$K_3(\mathbb{Z}[\zeta_d]) \cong \begin{cases} \mathbb{Z}/48\mathbb{Z}, & \text{if } d = 1 \text{ or } 2, \\ \mathbb{Z}^{\varphi(d)/2} \oplus \mathbb{Z}/\,\mathrm{lcm}(24, 2d)\mathbb{Z}, & \text{if } d > 2. \end{cases} \tag{7.2}$$

Further, if G is dihedral with rotation subgroup $H = \langle a \rangle$, then for odd d greater than 1, $O(d)$ is the tamely ramified twisted group ring $\Lambda_d = \mathbb{Z}[\zeta_d] \circ B$ described above, with $F_d = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$, $R_d = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$ and $s = |B| = |\{1, b\}| = 2$. By [16], pp. 408–409, if this odd d is divisible by two distinct odd primes, then all the ramification indices $e(\mu)$ are 1; so

$$K_n\big(O(d)\big) \cong K_n(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]).$$

But if, instead, $d = p^r$ for an odd prime p and $r \geq 1$, then p is the only prime ramified in $\mathbb{Z}[\zeta_d]$, and there, it ramifies totally. So there is only one prime μ of $\mathbb{Z}[\zeta_d + \zeta_d^{-1}]$ ramified in $\mathbb{Z}[\zeta_d]$, and its ramification index is the degree $s = 2$ of $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}(\zeta_d + \zeta_d^{-1})$. Also

$$\mathbb{Z}[\zeta_d + \zeta_d^{-1}]/\mu \cong \mathbb{F}_p.$$

So

$$K_n\big(O(d)\big) \cong K_n(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]) \oplus K_n(\mathbb{F}_p).$$

This proves:

PROPOSITION 7.3. *For odd $d > 1$ and G dihedral with rotation subgroup $H = \langle a \rangle$,*

$$K_3\big(O(d)\big) \cong$$
$$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^{(\varphi(d)/2)-1} \oplus \mathbb{Z}/2\,\mathrm{lcm}(24, 2d)\mathbb{Z} \oplus \mathbb{Z}/(p^2-1)\mathbb{Z}, & \text{if } d = p^t \text{ for } p \text{ an odd prime, } t > 0. \\ (\mathbb{Z}/2\mathbb{Z})^{(\varphi(d)/2)-1} \oplus \mathbb{Z}/2\,\mathrm{lcm}(24, 2d)\mathbb{Z}, & \text{otherwise.} \end{cases}$$

∎

8. **K_3 of the semisimple corner.** To complete our inventory of the pieces of the Mayer-Vietoris sequence of a square of the type (2.1), we now consider $K_3\big(O(\mathcal{D})/pO(\mathcal{D})\big)$. Since p does not divide any member of $\mathcal{D}$,

$$O(\mathcal{D})/pO(\mathcal{D}) \cong \bigoplus_{d \in \mathcal{D}} O(d)/pO(d), \tag{8.1}$$

by [16], Proposition 10.1.

For now, fix a choice of $d > 2$, and take G to dihedral, with rotation subgroup $H = \langle a \rangle$. By [16], Proposition 10.2,

$$O(d)/pO(d) \cong M_2(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]/p\mathbb{Z}[\zeta_d + \zeta_d^{-1}]).$$

Since $p \nmid d$, p is unramified in the Galois extension $\mathbb{Q}(\zeta_d + \zeta_d^{-1})$ over $\mathbb{Q}$. So there are positive integers f and g for which $fg = \varphi(d)/2$, $p\mathbb{Z}[\zeta_d + \zeta_d^{-1}]$ is a product of g distinct maximal ideals: $P_1 \cdots P_g$, and for each i,

$$\mathbb{Z}[\zeta_d + \zeta_d^{-1}]/P_i \cong \mathbb{F}_{p^f}.$$

So, by the Chinese Remainder Theorem,

$$O(d)/pO(d) \cong M_2(\mathbb{F}_{p^f}^{\,g}).$$

As in [24], p. 23, the residue degree f is the order of the Frobenius substitution:

$$\sigma_p \in \mathrm{Aut}\big(\mathbb{Q}(\zeta_d + \zeta_d^{-1})/\mathbb{Q}\big).$$

By [29], p. 14, the Frobenius substitution $\hat{\sigma}_p$ in $\mathrm{Aut}\big(\mathbb{Q}(\zeta_d)/\mathbb{Q}\big)$ takes ζ_d to ζ_d^p. So the restriction of $\hat{\sigma}_p$ to $\mathbb{Q}(\zeta_d + \zeta_d^{-1})$ satisfies the property that defines σ_p. Thus the restriction map:

$$\mathrm{Aut}\big(\mathbb{Q}(\zeta_d)\big) \longrightarrow \mathrm{Aut}\big(\mathbb{Q}(\zeta_d + \zeta_d^{-1})\big)$$

takes $\hat{\sigma}_p$ to σ_p; it is also surjective, with kernel of order 2, generated by complex conjugation. The order of $\hat{\sigma}_p$ is the order of $\bar{p}$ in $(\mathbb{Z}/d\mathbb{Z})^*$. So the order of σ_p is

$$\begin{cases} |\langle \bar{p} \rangle|, & \text{if } \overline{-1} \notin \langle \bar{p} \rangle, \\ \frac{1}{2}|\langle \bar{p} \rangle|, & \text{if } \overline{-1} \in \langle \bar{p} \rangle. \end{cases}$$

Thus this order f is the smallest positive integer with

$$p^f \equiv \pm 1 (\mathrm{mod}\, d).$$

So we have:

PROPOSITION 8.2. *Suppose G is dihedral of order $2m$ with rotation subgroup $H = \langle a \rangle$, and p and $\mathcal{D}$ are chosen as in (2.1). Then*

$$K_3\big(O(\mathcal{D})/pO(\mathcal{D})\big) \cong \bigoplus_{d \in \mathcal{D}} K_3\big(O(d)/pO(d)\big)$$

and for each $d > 2$ in $\mathcal{D}$,

$$\begin{aligned} K_3\big(O(d)/pO(d)\big) &\cong [K_3(\mathbb{F}_{p^f})]^{\varphi(d)/2f} \\ &\cong [\mathbb{Z}/(p^{2f}-1)\mathbb{Z}]^{\varphi(d)/2f}, \end{aligned}$$

where f is the least positive integer with $p^f \equiv \pm 1 \pmod{d}$, while:

$$\begin{aligned} K_3\big(O(2)/pO(2)\big) &\cong K_3\big(O(1)/pO(1)\big) \\ &\cong \begin{cases} K_3(\mathbb{F}_p \oplus \mathbb{F}_p) \cong [\mathbb{Z}/(p^2-1)\mathbb{Z}]^2, & \text{if } p \text{ is odd}, \\ K_3(\mathbb{F}_2[\mathbb{Z}/2\mathbb{Z}]) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}, & \text{if } p = 2. \end{cases} \end{aligned}$$ ∎

The last isomorphism:

$$K_3(\mathbb{F}_2[\mathbb{Z}/2\mathbb{Z}]) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$$

may be found in [1], Theorem 9.16, p. 175.

Suppose, on the other hand, that $G = \langle a \rangle$ is a cyclic of any finite order m, and p and $\mathcal{D}$ are chosen as in (2.1). Since $p \nmid \mathrm{lcm}(\mathcal{D})$,

$$O(\mathcal{D})/pO(\mathcal{D}) \cong \bigoplus_{d \in \mathcal{D}} \mathbb{Z}[\zeta_d]/p\mathbb{Z}[\zeta_d].$$

Since $p \nmid d$, p is unramified in $\mathbb{Z}[\zeta_d]$. So there are positive integers f and g with $fg = \varphi(d)$, $p\mathbb{Z}[\zeta_d] = P_1 \cdots P_g$ for distinct maximal ideals P_i, and for each i,

$$\mathbb{Z}[\zeta_d]/P_i \cong \mathbb{F}_{p^f}.$$

This f is the multiplicative order of $p \pmod{d}$. Thus:

PROPOSITION 8.3. *Suppose G is cyclic of order m generated by a, and p and $\mathcal{D}$ are chosen as in (2.1). Then*

$$K_3\big(O(\mathcal{D})/pO(\mathcal{D})\big) \cong \bigoplus_{d\in\mathcal{D}} K_3(\mathbb{Z}[\zeta_d]/p\mathbb{Z}[\zeta_d]),$$

and for each $d \in \mathcal{D}$,

$$\begin{aligned} K_3(\mathbb{Z}[\zeta_d]/p\mathbb{Z}[\zeta_d]) &\cong [K_3(\mathbb{F}_{p^f})]^{\varphi(d)/f} \\ &\cong [\mathbb{Z}/(p^{2f}-1)\mathbb{Z}]^{\varphi(d)/f} \end{aligned}$$

where f is the least positive integer with $p^f \equiv 1 (\mathrm{mod}\, d)$. ∎

9. **The bounds on $SK_2(\mathbb{Z}G)$ and $K_3(\mathbb{Z}G)$.** When the computations of Sections 4–8 are assembled as in Sections 1–3, we obtain lower and upper bounds for $SK_2(\mathbb{Z}G)$ and lower bounds for $K_3(\mathbb{Z}G)$. As in Corollary 2.3, for each ordering of the prime factors $p_1, \ldots, p_t$ of m, there is a filtration:

$$0 = F^0 \subseteq \cdots \subseteq F^t \subseteq SK_2(\mathbb{Z}G);$$

and, for each i, F^i/F^{i-1} is the direct product, over all positive divisors d of $p_1 \cdots p_{i-1}$, of the cokernels of maps:

$$K_3\big(O(p_i d\mathcal{D}_i)\big) \longrightarrow K_3\big(O(d\mathcal{D}_i)/p_i\big) \oplus B_2(d\mathcal{D}_i, p_i d\mathcal{D}_i) \tag{9.1}$$

from Mayer-Vietoris sequences of the square (2.1). Recall that $\mathcal{D}_i$ consists of all positive divisors e of $p_{i+1}\cdots p_t$. As we saw in Sections 2 and 4, the codomains of these maps are finite abelian groups; so their quotient F^i/F^{i-1} is too. By Proposition 4.4, $B_2(d\mathcal{D}_i, p_i d\mathcal{D}_i)$ is an elementary abelian p_i-group, and by Lemma 2.2 and Quillen's formula for K_n of a finite filed, $K_3\big(O(d\mathcal{D}_i)/p_i\big)$ has no p_i-torsion.

So the p_i-primary part of F^i/F^{i-1} comes exclusively from the birelative groups; it is a direct product, over all positive divisors d of $p_1\cdots p_{i-1}$, of the cokernels of the maps:

$$K_3\big(O(p_i d\mathcal{D}_i)\big) \longrightarrow B_2(d\mathcal{D}_i, p_i d\mathcal{D}_i).$$

The domain of this map is presently intractible if $i < t$ (so that $\mathcal{D}_i$ has two or more elements). But the p_t-primary part of F^t/F^{t-1} is the direct product, over all positive divisors d of m/p_t, of the cokernels of maps:

$$K_3\big(O(p_t d)\big) \longrightarrow B_2(d, p_t d),$$

and these cokernels can be estimated by using the computations of the domain and codomain, exemplified in Sections 5, 6 and 7.

On the other hand, if p is a prime and $p \neq p_i$, the p-primary part of the layer F^i/F^{i-1} has no contribution from the birelative groups; it is the direct product of the p-primary parts of the cokernels of K_3 of the top maps in commutative squares of ring homomorphisms:

$$\begin{array}{ccc} \mathcal{O}(p_i d\mathcal{D}_i) & \longrightarrow & \mathcal{O}(d\mathcal{D}_i)/p_i \\ \cap\!\mid & & \wr\!\| \\ \bigoplus_{e|p_{i+1}\cdots p_t}\mathcal{O}(p_i de) & \longrightarrow & \bigoplus_{e|p_{i+1}\cdots p_t}\mathcal{O}(de)/p_i \end{array} \tag{9.2}$$

as d ranges over the positive divisors of $p_1\cdots p_{i-1}$. A lower bound for these cokernels is obtained by using an upper bound for their images, and for these we may use the images of K_3 of the bottom maps in (9.2). So for $p \neq p_i$, the p-primary part of F^i/F^{i-1} maps onto the direct product of the p-primary parts of the cokernels of the maps:

$$K_3\big(\mathcal{O}(p_i c)\big) \to K_3\big(\mathcal{O}(c)/p_i\big) \tag{9.3}$$

as c $(= de)$ ranges over all positive divisors of m/p_i.

The direct product of the p-primary parts of the codomains of the maps (9.3) is an upper bound, mapping onto the p-primary part of F^i/F^{i-1}, for the product of the p-primary parts of these codomains is the p-primary part of

$$\bigoplus_{c|\frac{m}{p_i}}\Big[K_3\big(\mathcal{O}(c)/p_i\big) \oplus B_2(c, p_i c)\Big]; \tag{9.4}$$

and this (finite) group is isomorphic, by Proposition 4.6 and equation (8.1), to the direct product of the codomains in (9.1), which map onto F^i/F^{i-1}.

We organize these estimates of p-primary torsion as follows: If p divides m, order the prime factors of m so that $p = p_t$ is last. In the resulting filtration, estimate the rank of the p-primary part of F^t/F^{t-1} using birelative computations. Then obtain upper and lower bounds for the p-primary part of F^i/F^{i-1} for $i < t$ by using the codomains and cokernels of the maps (9.3). Note that these bounds are independent of the order chosen for the prime factors of m/p.

For those primes p not dividing m, there is no birelative contribution to p-primary torsion in F^t. The p-primary part of each F^i/F^{i-1} $(1 \le i \le t)$ is between those of the codomains and cokernels of maps (9.3), and these bounds are independent of the order chosen for the prime factors of m.

Before assembling our final theorems, we record one more way in which cyclic and dihedral groups are cooperative:

PROPOSITION 9.5. *Suppose a group G is a cyclic of square-free order m, or dihedral of square-free order $2m$. Suppose*

$$0 = F^0 \subseteq \cdots \subseteq F^t \subseteq SK_2(\mathbb{Z}G)$$

is the filtration from Corollary 2.3 associated with the prime factors $p_1, \ldots p_t$ of m. Then $F^t = SK_2(\mathbb{Z}G)$.

PROOF. In the cyclic case, F^t is the kernel of a homomorphism:

$$SK_2(\mathbb{Z}G) \to \bigoplus_{d|m} SK_2(\mathbb{Z}[\zeta_d]),$$

and by localization sequences, the groups $SK_2(\mathbb{Z}[\zeta_d])$ vanish.

If G is dihedral of square-free order $2m$, we can extend the filtration of Section 2:

$$\mathbb{Z}G = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_t = \bigoplus_{d|m} O(d) \tag{9.6}$$

by one step: $R_t \subseteq R_{t+1}$ as in Section 3. In this last step $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ is split into $\mathbb{Z} \oplus \mathbb{Z}$, with conductor 2.

Then $SK_2(\mathbb{Z}G)/F^{t+1}$ is isomorphic to subgroup of $SK_2(R_{t+1})$, which vanishes by [12], Theorem 1.1, because R_{t+1} is hereditary. And F^{t+1}/F^t is (by Proposition 1.1) the kernel of the map f in the Mayer-Vietoris sequence:

$$K_3(\mathbb{Z}) \oplus K_3(\mathbb{Z}) \longrightarrow K_3(\mathbb{F}_2) \oplus B_2(1,2) \longrightarrow K_2(\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]) \xrightarrow{f} K_2(\mathbb{Z}) \oplus K_2(\mathbb{Z}) \longrightarrow K_2(\mathbb{F}_2).$$

By computations of Dunwoody [5] and Silvester [19], Section 10, the domain and codomain of f are both $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; since $K_2(\mathbb{F}_2)$ vanishes, f is an isomorphism. ∎

NOTE. The last argument is a proof that the corkernel of $K_3(\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z})$ and $SK_2(\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}])$ both vanish. But these facts have been known for a long time (see [26]).

For finite cyclic groups there is a great gap between those upper and lower bounds for $SK_2(\mathbb{Z}G)$ which we can produce without a better understanding of the maps in the Mayer-Vietoris sequences. The reader can use as upper bounds, the groups in (9.4), computed in Theorem 6.1 and Proposition 8.3. The difficulty in obtaining lower bounds is due to the substantial free part of $K_3(\mathbb{Z}[\zeta_{2d}])$. However, birelative groups provide the following lower bound:

THEOREM 9.7. *Suppose m is an even square-free integer greater than 2. If G is a cyclic group of order m, then $SK_2(\mathbb{Z}G)$ has a quotient which is an elementary abelian 2-group of* rank *at least:*

$$\frac{(m/2)+1}{2} - 2^{t-1}$$

where t is the number of prime factors of m.

PROOF. Apply Corollary 2.3 with $s = 1$ and $p_t = 2$. Then by Proposition 9.5, $SK_2(\mathbb{Z}G) = F^t$; and F^t/F^{t-1} has 2-primary part which is the direct product, over all positive divisors d of $m/2$, of the cokernels of the maps:

$$K_3(\mathbb{Z}[\zeta_{2d}]) \longrightarrow B_2(d, 2d).$$

By Theorem 6.1,

$$B_2(d, 2d) \cong (\mathbb{Z}/2\mathbb{Z})^{\varphi(d)}.$$

If $d = 1$, the cokernel in question vanishes (see the end of the proof of Proposition 9.5). On the other hand, since d is odd, if $d > 1$, formula (7.2) shows that $K_3(\mathbb{Z}[\zeta_{2d}])$

is generated by $\varphi(d)/2 + 1$ elements. So the 2-primary part of F^t/F^{t-1} is elementary abelian of rank at least:

$$\sum_{\substack{d \mid \frac{m}{2} \\ d>1}} \varphi(d) - \left(\frac{\varphi(d)}{2} + 1\right) = \sum_{\substack{d \mid \frac{m}{2} \\ d>1}} \left(\frac{\varphi(d)}{2} - 1\right)$$
$$= \frac{(m/2) - 1}{2} - (2^{t-1} - 1)$$
$$= \frac{(m/2) + 1}{2} - 2^{t-1}. \qquad \blacksquare$$

To describe the bounds for $SK_2(\mathbb{Z}G)$ when G is a square-free order dihedral group, we introduce some simplifying notation. If p is prime, $c > 2$, and $p \nmid c$, there is a least positive integer f with $p^f \equiv \pm 1 \pmod{c}$. Define

$$\sigma(p, c) = (p^{2f} - 1)^{\varphi(c)/2f}.$$

The exponent,

$$g(p, c) = \varphi(c)/2f$$

is the number of primes in $\mathbb{Z}[\zeta_c + \zeta_c^{-1}]$ lying over p. By Proposition 8.2, $\sigma(p, c)$ is the order of K_3 of a certain finite ring.

Suppose $p_1, \ldots, p_t$ are distinct odd primes with product m. For each prime p, let $\mu(p)$ denote the largest integer μ for which p^μ divides

$$\prod_{i=1}^{t} \prod_{\substack{c \mid \frac{m}{p_i} \\ c>1}} \sigma(p_i, c),$$

and let $\nu(p)$ denote the largest integer ν for which p^ν divides

$$\prod_{i=1}^{t} (p_i^2 - 1).$$

THEOREM 9.8. *Suppose G is a dihedral group of square-free order $2m$, and p is a prime factor of m. Then $SK_2(\mathbb{Z}G)$ has a quotient $SK_2(\mathbb{Z}G)/F$ which is an elementary abelian p-group of rank at least:*

$$\frac{(m/p) + 1}{2} - 2^{t-1},$$

where t is the number of prime factors of m. The group F has p-primary part of order p^u, where

$$\mu(p) + 2\nu(p) \geq u \geq \mu(p) + \nu(p) - \begin{cases} (t-1)2^{t-2}, & \text{if } p \neq 3, \\ (t-1)2^{t-1}, & \text{if } p = 3. \end{cases}$$

If q is a prime not dividing m, then $SK_2(\mathbb{Z}G)$ has q-primary part of order q^v, where

$$\mu(q) + 2\nu(q) \geq v$$
$$\geq \mu(q) + \nu(q) - \begin{cases} 0, & \text{if } q \neq 2, 3 \\ t2^{t-1}, & \text{if } q = 3 \\ \left(\sum_{i=1}^{t} \sum_{\substack{c \mid \frac{m}{p_i} \\ c>1}} g(q, c)\right) + t\left(3(2^{t-1}) + 1\right), & \text{if } q = 2. \end{cases}$$

NOTE. The happy coincidence of the formulas for birelative contributions in Theorems 9.7 and 9.8 appears to be just a coincidence. The expression:

$$\frac{n+1}{2} - 2^k,$$

where n is the product of k distinct odd primes, has nonnegative value and is zero only for $n = 3$.

PROOF. Apply Corollary 2.3 with $s = 2$ and $p_t = p$. By Proposition 9.5, $SK_2(\mathbb{Z}G) = F^t$; and F^t / F^{t-1} has p-primary part which is the direct product, over all positive divisors d of m/p, of the cokernels of:

$$K_3\big(O(pd)\big) \longrightarrow B_2(d, pd).$$

By Theorem 5.2,

$$B_2(d, pd) \cong \begin{cases} (\mathbb{Z}/p\mathbb{Z})^{\varphi(d)/2}, & \text{if } d > 1, \\ 0, & \text{if } d = 1. \end{cases}$$

By Proposition 7.3, $K_3(O(pd)\big)$ has only cyclic p-torsion. So the p-primary part of F^t / F^{t-1} is elementary abelian of rank at least:

$$\sum_{\substack{d \mid \frac{m}{p} \\ d>1}} \Big[\frac{\varphi(d)}{2} - 1\Big] = \frac{(m/p) - 1}{2} - (2^{t-1} - 1)$$
$$= \frac{(m/p) + 1}{2} - 2^{t-1}.$$

Take F to be the kernel of the composite of $F^t \longrightarrow F^t / F^{t-1}$ followed by projection to the p-primary part; so the p-primary parts of F and F^{t-1} are the same. For upper and lower bounds of the p-primary parts of each F^i / F^{i-1}, where $i < t$, we use the direct products of the p-primary parts of the codomains and cokernels, respectively, of the maps (9.3), as c runs through the positive divisors of m/p_i.

Since $p \mid m$, $p \neq 2$. There is nothing about the groups $K_3\big(O(p_i c)\big)$ and $K_3\big(O(c)/p_i\big)$, computed in Propositions 7.3 and 8.2, that would prevent the maps (9.3) from being injective on p-primary parts. So for our lower bounds, we must assume this injectivity. Suppose p^n is the largest power of p dividing $p_i^2 - 1$. If $p \neq 3$, the p-part of the order of $K_3\big(O(p_i c)\big)$ is:

$$\begin{cases} 1, & \text{if } c \neq 1 \text{ and } p \nmid c, \\ p, & \text{if } p \mid c, \\ p^n, & \text{if } c = 1, \end{cases}$$

and the number of divisors c of m/p_i which are divisible by p is the number of positive divisors of $m/p_i p$, namely 2^{t-2}. On the other hand, for $p = 3$, the p-part of the order of $K_3\big(O(p_i c)\big)$ is:

$$\begin{cases} 3, & \text{if } c \neq 1, \\ 3^{1+n}, & \text{if } c = 1, \end{cases}$$

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and the number of positive divisors c of m/p_i is 2^{t-1}. So the upper and lower bounds for the p-part of the order of F^i/F^{i-1} differ by a factor of p raised to the power:

$$\begin{cases} n + 2^{t-2}, & \text{if } p \neq 3, \\ n + 2^{t-1}, & \text{if } p = 3. \end{cases}$$

Multiplying these factors over all i with $1 \leq i < t$, we obtain the asserted bounds for the p-part of the order of F.

Now assume q is a prime not dividing m. If $q \neq 2$, the bounds on the q-part of the order of $SK_2(\mathbb{Z}G)$ are obtained just as they were for the order of F above, except that we multiply over all i with $1 \leq i \leq t$.

But if $q = 2$, it is not appropriate to assume injectivity on q-primary torsion of the maps (9.3). By Proposition 7.3, if 2^n is the largest power of 2 dividing $p_i^2 - 1$, the 2-primary part of $K_3\big(O(p_i c)\big)$ is:

$$\text{(9.9)} \qquad \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{(\varphi(p_i c)/2)-1} \oplus \mathbb{Z}/16\mathbb{Z}, & \text{if } c \neq 1, \\ (\mathbb{Z}/2\mathbb{Z})^{(\varphi(p_i c)/2)-1} \oplus \mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/2^n\mathbb{Z}, & \text{if } c = 1. \end{cases}$$

And by Proposition 8.2, if $f(c)$ denotes the least positive integer f with $p_i^f \equiv \pm 1 \pmod{c}$, then the 2-primary part of $K_3\big(O(c)/p_i\big)$ is that of:

$$\begin{cases} [\mathbb{Z}/(p_i^{2f(c)} - 1)\mathbb{Z}]^{\varphi(c)/2f(c)}, & \text{if } c \neq 1, \\ [\mathbb{Z}/(p_i^2 - 1)\mathbb{Z}]^2, & \text{if } c = 1. \end{cases}$$

If $c \neq 1$, the image (in $K_3\big(O(c)/p_i\big)$) of the first term in (9.9) is contained in, and may be as large as:

$$[\mathbb{Z}/2\mathbb{Z}]^{\varphi(c)/2f(c)} = [\mathbb{Z}/2\mathbb{Z}]^{g(p_i,c)},$$

which contains all elements of order 2 in $K_3\big(O(c)/p_i\big)$. The image of the second term in (9.9) is cyclic of order at most the minimum of 16 and the 2-part of $p_i^{2f(c)} - 1$ (the latter is a multiple of 8, and is divisible by 16 if and only if either $p_i \equiv \pm 1 \pmod 8$ or $f(c)$ is even). So for $c \neq 1$, the image of (9.3) has 2-primary part of order at most 2 raised to the power:

$$g(p_i, c) + 4 - 1.$$

If $c = 1$, the image of the first term in (9.9) is at most:

$$\begin{cases} 0, & \text{if } p_i = 3, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } p_i = 5, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } p_i \geq 7. \end{cases}$$

The image of the second term is cyclic of order at most the minimum of 16 and the 2-part of $p_i^2 - 1$ (which is a multiple of 8, and divisible by 16 if and only if $p_i \equiv \pm 1 \pmod 8$). The image of the third term is cyclic of order at most the 2-part of $p_i^2 - 1$. Taking intersections of these images into account, the image of (9.3) for $c = 1$ has 2-primary part of order at most 2 raised to the power:

$$2 + 4 + n - 2 = 4 + n$$

where 2^n is the 2-part of $p_i^2 - 1$.

Summing these exponents over all positive divisors c of m/p_i, and again over all i for $1 \leq i \leq t$, we obtain:

$$\sum_{i=1}^{t} \sum_{c \mid \frac{m}{p_i}} g(p_i, c) + t\big(3(2^{t-1}) + 1\big) + \nu(2),$$

as an upper bound for the power to which 2 divides the direct product of the images in the maps (9.3). The direct product of the codomains of those maps has 2-primary part of order 2 raised to the power $\mu(2) + 2\nu(2)$. ∎

NOTE. As a perusal of the above proof shows, the lower bound for 2-primary torsion in $SK_2(\mathbb{Z}G)$, for G dihedral of square-free order $2m$, may be sharpened for some m, by considering the primes p_i dividing m for which $p_i < 7$ or

$$16 \not| (p_i^{2f(c)} - 1).$$

If $G \longrightarrow G'$ is a split surjective homomorphism of finite groups, then for all $n \geq 0$, $SK_n(\mathbb{Z}G')$ is a quotient of $SK_n(\mathbb{Z}G)$. In case G' is dihedral of square-free order and $n = 2$, we obtain similar results even if $G \longrightarrow G'$ is not quite split:

THEOREM 9.10. *Suppose D_m is a dihedral group of square-free order $2m$ and G is a finite group with a normal subgroup N of order relatively prime to m, for which $G/N \cong D_m$. Then $SK_2(\mathbb{Z}D_m)$ is a subquotient of $SK_2(\mathbb{Z}G)$.*

PROOF. By Proposition 9.5, in the filtration

$$\mathbb{Z}D_m = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_t = \bigoplus_{d \mid m} O(d)$$

of Section 2,

$$SK_2(\mathbb{Z}D_m) = \text{kernel } K_2(R_0 \longrightarrow R_t).$$

Now

$$\mathbb{Q}G = (1-e)\mathbb{Q}G \oplus e\mathbb{Q}G$$

where e is the central idempotent

$$e = \frac{1}{|N|} \sum_{n \in N} n.$$

By means of the isomorphism $G/N \cong D_m$, we can identify $e\mathbb{Z}G$ with $\mathbb{Z}D_m$. Let $\mathcal{A}$ denote $(1-e)\mathbb{Z}G$. Then

$$\mathbb{Z}G \subseteq \mathcal{A} \oplus R_0 \subseteq \mathcal{A} \oplus R_1 \subseteq \cdots \subseteq \mathcal{A} \oplus R_t$$

is a filtration with conductors $|N|, p_1, \ldots, p_t$ satisfying the hypotheses of Proposition 1.1. Hence

$$\begin{aligned} SK_2(\mathbb{Z}D_m) &= \ker K_2(R_0 \longrightarrow R_t) \\ &\cong \ker K_2(\mathcal{A} \oplus R_0 \longrightarrow \mathcal{A} \oplus R_t) \\ &\cong \frac{\ker K_2(\mathbb{Z}G \longrightarrow \mathcal{A} \oplus R_t)}{\ker K_2(\mathbb{Z}G \longrightarrow \mathcal{A} \oplus R_0)} \end{aligned}$$

which is a subquotient of $SK_2(\mathbb{Z}G)$. ■

EXAMPLE. If Q_m is the dicyclic group of order $4m$ (presented by generators a, b subject to relations $a^m = b^2$, $b^4 = 1$, $ba = a^{-1}b$), and if m is odd and square-free, then $SK_2(\mathbb{Z}D_m)$ is a subquotient of $SK_2(\mathbb{Z}Q_m)$.

If p is an odd prime, the Mayer-Vietoris sequence of the square (2.1) for $G = D_p$, with $\mathcal{D} = \{1\}$, reduces to a sequence:

$$K_3(\mathbb{Z}D_p) \xrightarrow{f} K_3\big(O(p)\big) \xrightarrow{g} K_3\big(O(1)/p\big) \xrightarrow{h} SK_2(\mathbb{Z}D_p) \longrightarrow 1$$

which is exact except at $K_3\big(O(p)\big)$, where we only know that $\ker(g) \subseteq \mathrm{image}(f)$. The vanishing of the groups $SK_2\big(O(p)\big)$ and $SK_2\big(O(1)\big)$ have been discussed above in the proof of Proposition 9.5.

For primes p dividing the odd square-free integer m, the unknown size of the image of g accounts for the contribution $\nu(p)$ to the gap between upper and lower bounds in Theorem 9.8. This gap could be narrowed if we know the actual size of $SK_2(\mathbb{Z}D_p)$ for each prime p dividing m.

On the other hand, since the kernel of g is contained in the image of f, we can use this kernel to determine some lower bounds for $K_3(\mathbb{Z}D_p)$. By Proposition 7.3,

$$K_3\big(O(p)\big) \cong (\mathbb{Z}/2\mathbb{Z})^{(p-3)/2} \oplus \mathbb{Z}/2\,\mathrm{lcm}(24, 2p)\mathbb{Z} \oplus \mathbb{Z}/(p^2-1)\mathbb{Z}$$

and by Proposition 8.2,

$$K_3\big(O(1)/p\big) \cong [\mathbb{Z}/(p^2-1)\mathbb{Z}]^2.$$

So the kernel of g has p-primary part $\mathbb{Z}/p\mathbb{Z}$, and has 2-primary part containing a copy of

$$(\mathbb{Z}/2\mathbb{Z})^{(p-3)/2}.$$

Since there are split surjective homomorphisms: $\mathbb{Z}D_m \longrightarrow \mathbb{Z}D_p$ for every prime factor p of the odd square-free integer m, we obtain:

THEOREM 9.11. *If G is a dihedral group of square-free order $2m$, then $K_3(\mathbb{Z}G)$ maps onto $\mathbb{Z}/m\mathbb{Z}$, and onto $(\mathbb{Z}/2\mathbb{Z})^{(p-3)/2}$ for every prime factor p of m.* ■

REFERENCES

1. J. E. Aisbett, E. Lluis-Puebla and V. Snaith, *On $K_*(\mathbb{Z}/n)$ and $K_*(F_q[t]/(t^2))$*, Memoirs Amer. Math. Soc. (57) **329**(1985), 200 pp.
2. S. Chaladus, *Lower bounds of the order of $K_2(\mathbb{Z}G)$ for a cyclic p-group G*, Bull. Acad. Pol. Sci., Ser. Sci. Math. **27**(1979), 665–669.
3. ———, *Funktor K_2 dla wybranych pierscieni*, Part II, Prace Naukowe **5**(1982), 17–25.
4. R. K. Dennis, M. E. Keating and M. R. Stein, *Lower bounds for the order of $K_2(\mathbb{Z}G)$ and $Wh_2(G)$*, Math. Ann. **223**(1976), 97–103.
5. M. J. Dunwoody, *$K_2(\mathbb{Z}\Pi)$ for Π a group of order two or three*, J. London Math. Soc. (2) **11**(1975), 481–490.
6. D. R. Grayson, *The K-theory of hereditary categories*, J. Pure Appl. Algebra **11**(1977), 67–74.

7. D. Guin-Waléry and J.-L. Loday, *Obstruction à l'excision en K-théorie algébrique*, in Algebraic K-Theory, Lect. Notes Math, **854** Springer-Verlag, 1981, 179–216.

8. M. Karoubi, *Localisation de formes quadratiques I*, Ann. Sci. Ec. Norm. Sup. (4) **7**(1974), 359–404.

9. M. E. Keating, *On the K-theory of tiled orders*, J. Algebra **43**(1976), 193–197.

10. F. Keune, *Doubly relative K-theory and the relative K_3*, J. Pure Appl. Algebra **20**(1981), 39–53.

11. M. Kolster, *K_2 of rings of algebraic integers*, preprint, 1990.

12. A. O. Kuku, *Higher algebriac K-theory of group-rings and orders in algebras over number fields*, Comm. Algebra (8) **10**(1982), 805–816.

13. R. C. Laubenbacher, *Generalized Mayer-Vietoris sequences in algebraic K-theory*, J. Pure Appl. Algebra **51**(1988), 175–192.

14. ———, *On the K-theory of $\mathbb{Z}G$, G a group of square-free order*, in Algebraic K-Theory: Connections with Geometry and Topology, Kluwer Academic Publ., 1989, 189–208.

15. M. Levine, *The indecomposable K_3 of fields*, Ann. Sci. Ec. Norm. Sup. (4) **22**(1989), 255-344.

16. B. Magurn, *SK_1 of dihedral groups*, J. Algebra **51** (2) (1978), 399–415.

17. ———, *Whitehead groups of some hyperelementary groups*, J. London Math. Soc. (2) **21**(1980), 176–188.

18. A. S. Merkurjev, A. A. Suslin, *On the K_3 of a field*, LOMI Preprints E-2-87, USSR Acad. Sci. Steklov Math. Inst., Leningrad Dept. (1987), 34 pp.

19. J. Milnor, *Introduction to Algebraic K-Theory*, Ann. Math. Stud. **72** Princeton U. Press, 1971, 184 pp.

20. R. Oliver, *Lower bounds for $K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p\pi)$ and $K_2(\mathbb{Z}\pi)$*, J. Algebra **94**(1985), 425–487.

21. ———, *K_2 of p-adic group rings of abelian p-groups*, Math. Z. **195**(1987), 505–558.

22. D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96**(1972), 552–586.

23. I. Reiner, *Maximal Orders*, Academic Press, 1975, 395 pp.

24. J.-P. Serre, *Local Fields*, (transl. by M. J. Greenberg), Grad. Texts Math. **67** Springer-Verlag, 1979, 241 pp.

25. C. Soulé, *K-theorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. **55** (1979), 251–295.

26. M. R. Stein, *Excision and K_2 of group rings*, J. Pure Appl. Algebra **18**(1980), 213–224.

27. R. G. Swan, *Induced representations and projective modules*, Ann. of Math. (3) **71**(1960), 552–578.

28. J. B. Wagoner, *Algebraic invariants for pseudo-isotopies*, Proc. Liverpool Singularities Symp. II (1969/70), in Lect. Notes Math **209** Springer-Verlag, 1971, 164–190.

29. L. C. Washington, *Introduction to Cyclotomic Fields*, Grad. Texts Math. **83** Springer-Verlag, 1982, 389 pp.

30. C. A. Weibel, *K-theory and analytic isomorphisms*, Invent. Math. **61**(1980), 177–197.

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