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FINITE-TO-ONE OPEN MAPPINGS

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1. Introduction. The class of finite-to-one open mappings on manifolds contains some important subclasses. Any non-constant analytic function from a bounded region in its domain of definition is finite-to-one. Church [**2**] showed that any light strongly open C^n map $f: R^n \to R^n$ is discrete. A number of papers concerning discrete open mappings on manifolds have been published; see [**1–6**; **8–9**; **11–14**].

A result of Černavskiĭ [**1**] (see also [**13**]) shows that for any discrete strongly open mapping $f: M^n \to N^n$ of an n-manifold into an n-manifold, the branch set of f has dimension less than $n - 1$. *If* f is also a closed map, then $N(f)$ is finite and the set of points x for which $N(x, f) = N(f)$ is an open dense connected subset of M^n. In the following, if M^n and N^n are n-manifolds without boundary, if R is a region in M^n such that $\partial R = \partial(\bar{R})$, and if $f: \bar{R} \to N^n$ is a discrete open and closed mapping such that $f(R)$ is open in N^n, we prove that the set of points x in $\bar{R}$, for which $N(x, f) = N(f)$, contains a dense open subset of ∂R.

All references to cohomology theory may be found in [**10**]. The shift of dimension and use of reduced cohomology should be noted [**10**, p. 64], i.e., for a pair of spaces (X, A), A closed in X, the $(p + 1)$st cohomology group $H^{p+1}(X, A)$ corresponds to the group $H^p(X, A)$ in other developments.

The definition and necessary properties of the topological index of a point y with respect to a mapping f and a domain D, $\mu(y, f, D)$, and of the local degree of a point x with respect to a mapping f, $i(x, f)$, appear in [**13**]. For a detailed development of the topological index, see [**10**].

2. Notation and terminology. All topological spaces considered are assumed to be Hausdorff and all mappings on topological spaces are assumed to be continuous. For a space X and subsets A and B with $A \subset B \subset X$, we denote the boundary of A relative to B by $\partial_B A$ and simplify $\partial_X A$ to ∂A. Denote the complement of A with respect to B by $C_B A$ and simplify $C_X A$ to CA. A mapping $f: X \to Y$ is discrete (light) if each point inverse is discrete (totally disconnected) in the relative topology. The map f is open if the image of each open set of X is open in $f(X)$ and is strongly open if the image of each open set is open in Y. The branch set of f, B_f, is the set of points at which f fails to be a local homeomorphism. The multiplicity of f at x, $N(x, f)$, is the number of points in $f^{-1}f(x)$ if it is finite, and $+\infty$ otherwise. The multiplicity

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of f on X, $N(f)$, is the supremum of $N(x,f)$, $x \in X$. Let R^n represent a Euclidean n-space.

3. Preliminary results. For a finite-to-one open mapping f and a positive integer i, let $K_i(f)$ be the union of all points x in X for which $N(x,f) \leqq i$. For open mappings, $N(\ ,f)$ is lower semi-continuous, so that $K_i(f)$ is closed for each positive integer.

3.1. LEMMA. *Let f: $X \rightarrow Y$ be a discrete open mapping, where X and Y are locally compact spaces and F is a locally compact subset of X. Then for any open set U in X for which $U \cap F \neq \emptyset$, there exists an open subset V of X such that $V \subset U$, $F \cap V \neq \emptyset$, and $f|F \cap V$ is a homeomorphism of $F \cap V$ onto $f(F \cap V)$. Furthermore, $F \cap V$ is an inverse set of $f|V$.*

Proof. We can assume that $\bar{U}$ is compact so that $f|\bar{U}$ is finite-to-one and hence $f|U$ is finite-to-one. Write

$$U \cap F = \bigcup_{i=1}^{\infty} K_i(f|U) \cap F$$

and apply Baire's theorem to obtain an integer n for which the interior, T', of $K_n(f|U) \cap F$ relative to $U \cap F$ is not empty. Choose an open subset W' of U such that $W' \cap F = T'$. Choose $x_1 \in T'$ such that

$$N(x_1, f|W') = \max_{x \in T'} N(x, f|W').$$

Then $N(x_1,\ f|W') \leqq n$; thus suppose that $N(x_1,\ f|W') = k$ and that $(f|W')^{-1}(f|W')(x_1) = \{x_1, \ldots, x_k\}$. Choose pairwise disjoint open sets M_j of the x_j with $M_j \subseteq W'$, $j = 1, \ldots, k$. For

$$V = M_1 \cap (f|W')^{-1}(f(M_1) \cap f(M_2) \cap \ldots \cap f(M_k)) \quad \text{and} \quad T = V \cap F,$$

it follows that $N(x,\ f|V) = 1$ for all $x \in T$ and $f|T$ is a homeomorphism.

3.2. LEMMA. *Let f: $(A, A_0) \rightarrow (B, B_0)$ be a mapping of compact pairs such that $f(CA_0) \subset CB_0$, $f(\partial A_0) \subset \partial B_0$, and*

$$(f|\overline{CA_0})^p \colon H^p(\overline{CB_0}, \partial B_0) \rightarrow H^p(\overline{CA_0}, \partial A_0)$$

is an isomorphism. Then for $\partial A_0 \neq \emptyset$ or $p \neq 1$, f^p: $H^p(B, B_0) \rightarrow H^p(A, A_0)$ is an isomorphism and if $(f|\overline{CA_0})^p$ is onto, then so is f^p.

Proof. Consider the following diagram, where i_1^p and i_2^p are induced by inclusion.

$$\begin{array}{ccc} H^p(B, B_0) & \xrightarrow{\ i_2^p\ } & H^p(\overline{CB_0}, \partial B_0) \\ \Big\downarrow f^p & & \Big\downarrow (f|\overline{CA_0})^p \\ H^p(A, A_0) & \xrightarrow{\ i_1^p\ } & H^p(\overline{CA_0}, \partial A_0) \end{array}$$

For $p \neq 1$ or $\partial A_0 \neq \emptyset$, by strong excision [**10**, p. 86], i_1^p and i_2^p are onto isomorphisms and the diagram is commutative so that f^p is an isomorphism and if $(f|\overline{CA_0})^p$ is onto, then so is f^p.

3.3. Lemma. *Let* f: $(A, A_0) \to (B, B_0)$ *be an onto mapping of compact pairs with* $f(A_0) = B_0$. *If for every* $x \in \overline{CA_0}$, $N(x,f) = 1$, *then if* $\partial A_0 \neq \emptyset$ *or* $p \neq 1$, f^p: $H^p(B, B_0) \to H^p(A, A_0)$ *is an onto isomorphism.*

Proof. By hypothesis, $f(CA_0) = CB_0$ and $f|\overline{CA_0}$ is a homeomorphism of $\overline{CA_0}$ onto $\overline{CB_0}$ so that $f(\partial A_0) = \partial B_0$. Thus the mapping

$$(f|\overline{CA_0}): (\overline{CA_0}, \partial A_0) \to (\overline{CB_0}, \partial B_0)$$

induces a homomorphism $(f|\overline{CA_0})^p$: $H^p(\overline{CB_0}, \partial B_0) \to H^p(\overline{CA}, \partial A_0)$ which is an onto isomorphism. Thus, by 3.2, f^p is an onto isomorphism.

3.4. Theorem. *Let* U *and* V *be bounded domains in* R^n *such that* $\partial U = \partial(\bar{U})$, *and let* f: $\bar{U} \to \bar{V}$ *be a mapping with* $f(\partial U) = \partial V$ *and* $f(U) = V$. *Let* A *be a proper closed subset of* ∂U *such that* $\overline{C_{\partial U} A}$ *is an inverse set of* f *and* $N(x,f) = 1$ *for each* x *in* $\overline{C_{\partial U} A}$. *Then*

$$f^{n+1}: H^{n+1}(\bar{V}, \partial V) \to H^{n+1}(\bar{U}, \partial U)$$

is an onto isomorphism.

Proof. For $n > 1$, the mapping $f|\partial U$: $(\partial U, A) \to (\partial V, f(A))$ satisfies the hypothesis of 3.3 and for $n = 1$, A is either empty or a single point. Hence, $(f|\partial U)^n$: $H^n(\partial V, f(A)) \to H^n(\partial U, A)$ is an onto isomorphism. Consider the following diagram:

$$\begin{array}{ccccc} H^n(\partial U, A) & \xrightarrow{\delta_1} & H^{n+1}(\bar{U}, \partial U) & \xrightarrow{i} & H^{n+1}(\bar{U}, A) \\ \uparrow (f|\partial U)^n & & \uparrow f^{n+1} & & \\ H^n(\partial V, f(A)) & \xrightarrow{\delta_2} & H^{n+1}(\bar{V}, \partial V) & & \end{array}$$

where the top row is obtained from the exact sequence of the triple $(\bar{U}, \partial U, A)$ and the bottom row is obtained from the exact sequence of the triple $(\bar{V}, \partial V, f(A))$. Since $\bar{U} - A$ is non-empty, connected, and not open in R^n, it follows that $H^{n+1}(\bar{U}, A) = 0$, and consequently δ_1 is onto by exactness in the top row. Thus $\delta_1(f|\partial U)^n$ is onto so that f^{n+1} is necessarily onto. Since both $H^{n+1}(\bar{U}, \partial U)$ and $H^{n+1}(\bar{V}, \partial V)$ are isomorphic to the additive group of integers, it follows that f^{n+1} is an onto isomorphism.

3.5. Theorem. *Let* U *be an open subset of* R^n, *with* $\bar{U}$ *compact*, $\partial U = \partial(\bar{U})$, *and* A *a closed non-empty subset of* ∂U *with* $\overline{\text{int}_{\partial U} A} = A$. *Then there is no mapping* f: $\bar{U} \to R^n$ *such that*

(i) f *is discrete,*

(ii) $f|U$ *is strongly open,*

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(iii) $N(x, f) = 1$ *for all* $x \in A$, *and*
(iv) $(\mathrm{int}_{\partial U} A) \cap B_f \neq \emptyset$.

Proof. Suppose that there exists a mapping f with properties (i)–(iv). The mapping $f|(\bar{U} - f^{-1}f(\partial U))$ is an open and closed mapping so that components of $\bar{U} - f^{-1}f(\partial U)$ map onto components of $f(U) - (f(U) \cap f(\partial U))$. Let T be a component of $R^n - f^{-1}f(\overline{\partial U - A})$ which contains points of $A \cap B_f$. Such a T exists since $[A - (\overline{\partial U - A})] \cap B_f \neq \emptyset$ and $N(x, f) = 1$ for all x in A. The set T is open and $T \cap \partial U \neq \emptyset$ so that $T \cap U \neq \emptyset$. It follows that components of $\bar{U} - f^{-1}f(\partial U)$ which meet T are necessarily in $T \cap U$.

If the mapping $f|T \cap U$ is one-to-one, then $f|T \cap \bar{U}$ is one-to-one since $T \cap \partial U \subset A$ and, furthermore, $f|T \cap \bar{U}$ is also a strongly open mapping into $f(\bar{U})$. This implies that $B_f \cap (T \cap \bar{U}) = \emptyset$ which is contrary to the choice of T. Hence $N(f|T \cap U) > 1$.

Assuming that f is one-to-one on each component of $T \cap U$ implies that there are at least two components K_1 and K_2 of $T \cap U$ with $f(K_1) \cap f(K_2) \neq \emptyset$. Since K_1 and K_2 are also components of $\bar{U} - f^{-1}f(\partial U)$, it follows that $f(K_1) = f(K_2)$. For $i = 1, 2$, $\partial_T K_i \subset f^{-1}f(\partial U) \cap T \subset A$ and since $f(\bar{K}_1) = f(\bar{K}_2)$ and $N(x, f) = 1$ for $x \in A$, $\partial_T K_1 = \partial_T K_2$. The mapping $g = (f|K_2 \cup \partial_T K_2)^{-1}(f|K_1 \cup \partial_T K_1)$ is one-to-one from $K_1 \cup \partial_T K_1$ onto $K_2 \cup \partial_T K_2$. Being the composition of homeomorphisms, g is a homeomorphism which is the identity function on $\partial_T K_1$. By [**13**, 5.2], $K_1 \cup K_2 \cup \partial_T K_1 = T$; hence we have T, open in R^n, such that $T \subset \bar{U}$ and $T \cap \partial U \neq \emptyset$, which is contradictory.

It now follows that there must be a component K of $T \cap U$ with $N(f|K) > 1$ and, as before, $\emptyset \neq \partial K \cap T \subset A$. The set K is a component of $\bar{U} - f^{-1}f(\partial U)$; thus $\partial K \subset f^{-1}f(\partial U)$, and hence $f(K) \cap f(\partial K) = \emptyset$. Furthermore, $f(K)$ is open and $f(K) \cup f(\partial K) = f(\bar{K}) = f(K) \cup \partial f(K)$ so that $f(\partial K) = \partial f(K)$. Applying 3.4, one obtains $|\mu(y, f, K)| = 1$, for every $y \in f(K)$. By [**13**, 5.4], $\dim B_{f|K} \leqq n - 2$; therefore $K - B_{f|K}$ is open and connected and thus $i(x, f)$ is constant on $K - B_{f|K}$. However,

$$|\mu(y,f,K)| = \left| \sum_{x \in f^{-1}(y) \cap K} i(x,f) \right| \qquad \text{for every } y \in [f(K) - f(B_{f|K})].$$

We then have $N(x, f|K) = 1$ for every $x \in [K - f^{-1}f(B_{f|K})]$ and

$$\dim f^{-1}f(B_{f|K}) \leqq n - 2,$$

and so f is one-to-one on an open dense set in K. Since $f|K$ is open, it follows that f is one-to-one on K. This is contrary to the choice of K so that the theorem is valid.

4. Main theorems. In this section we will use the following.

Definition. Let X and Y be n-manifolds without boundary, A a subset of X, and let f be a map $f\colon A \to Y$. If D is open in X with $D \subseteq A$, then let $\gamma_{D,F} = \{x \in D \mid f(x) \notin \operatorname{int}_Y f(D)\}$.

4.1. THEOREM. *Let X and Y be n-manifolds without boundary, D a domain in X such that $\partial D = \partial(\bar{D})$, and let $f\colon \bar{D} \to Y$ be a discrete open mapping. Then $CB_f \cap \partial D$ is a dense open subset of the closure of $\partial D - (\bar{\gamma}_{D,F} \cap \partial D)$.*

Proof. Clearly, from Brouwer's Theorem on Invariance of Domain [**7**, pp. 95–97], $\gamma_{D,f} \subseteq B_f$, and hence $\bar{\gamma}_{D,f} \subseteq B_f$, since B_f is closed; thus $CB_f \cap \partial D \subseteq \partial D - (\bar{\gamma}_{D,f} \cap \partial D)$. If the theorem is false, then there is an open set $U \subseteq X$ such that $\emptyset \neq U \cap \partial D \subseteq \partial D - (\bar{\gamma}_{D,f} \cap \partial D)$ and $U \cap \partial D \subseteq B_f$. Further, we can assume that $U \cap \bar{\gamma}_{D,f} = \emptyset$. Applying 3.1, we can pick an open connected conditionally compact set $V \subseteq U$ such that $V \cap \partial D \neq \emptyset$ and for each $x \in \overline{V \cap \partial D}$, $N(x, f|\overline{V \cap D}) = 1$. Further, V may be chosen arbitrarily small, so that $\bar{V}$ and $f(\overline{V \cap D})$ lie in domains in X and Y, respectively, which are homeomorphic to R^n. Then $f|\overline{V \cap D}$ may be considered to be a mapping from $\overline{V \cap D}$ into R^n, with $\overline{V \cap D} \subseteq R^n$.

Let $A = \overline{V \cap \partial D}$. Then A is a closed subset of $\partial(\overline{V \cap D})$ and $\operatorname{int}_{\partial(\overline{V \cap D})} A = V \cap \partial D$ is dense in A. Further:

(i) $f|\overline{V \cap D}$ is discrete,
(ii) $f|\operatorname{int}(\overline{V \cap D})$ is a strongly open map since $\operatorname{int}(\overline{V \cap D}) \subseteq D - \gamma_{D,f}$,
(iii) $N(x, f|\overline{V \cap D}) = 1$, for every $x \in A$, and
(iv) $B_{f|\overline{V \cap D}} \supseteq A$.

But by 3.5, no such mapping can exist. Hence, the theorem follows.

As an immediate consequence of 4.1, we have the following.

4.2. COROLLARY. *Given $f\colon \bar{D} \to Y$ as above, if $f(D)$ is open in Y, then $CB_f \cap \partial D$ is a dense open set in ∂D.*

Given the hypothesis of 4.1, if $\bar{D}$ and $f(\bar{D})$ are n-manifolds with boundary, then it follows that $\partial D - (\bar{\gamma}_{D,f} \cap \partial D)$ is dense in ∂D. Hence, $CB_f \cap \partial D$ is dense in ∂D and $\dim B_f \cap \partial D \leqq n - 2$.

4.3. THEOREM. *Let X and Y be n-manifolds without boundary, D a domain in X such that $\partial D = \partial(\bar{D})$, and $f\colon \bar{D} \to Y$ an open, closed, discrete mapping such that $f(D)$ is open in Y. Then $\partial D - (f^{-1}f(B_f) \cap \partial D)$ is a dense open set in ∂D.*

Proof. By 4.2, $CB_f \cap \partial D$ is dense in ∂D. Hence, $f(\partial D) \subseteq \partial f(D)$, and so $f^{-1}f(B_f) \cap \partial D = f^{-1}f(\partial D \cap B_f)$. Also, D is an inverse set of f; hence by [**13**, 5.5], $N(f|D) < \infty$ and since f is open, $N(f) = N(f|D)$.

Assume that there is an open set W in $\bar{D}$ such that $\emptyset \neq (W \cap \partial D) \subseteq f^{-1}f(B_f)$. Then there is a point $x_1 \in W \cap \partial D$ such that

$$N(x_1, f) = \max_{x \in W \cap \partial D} N(x, f) = k < \infty \qquad \text{and} \qquad f^{-1}f(x_1) = \{x_1, \ldots, x_k\}.$$

Now there are pairwise disjoint open neighbourhoods, W_i, of the x_i, $i = 1, \ldots, k$, with $f(W_1) = \ldots = f(W_k)$ and $\bar{W}_1 \subseteq W$. For some j, $1 \leqq j \leqq k$, $x_j \in B_f \cap (\partial D \cap W_j)$. But we can choose $\bar{W}_j$ small enough that $\bar{W}_j$ and

$f(\bar{W}_j)$ are contained in domains of X and Y, respectively, which are homeomorphic to R^n and $f|\bar{W}_j$ induces a map with the properties in 3.5, which is a contradiction.

4.4 Maximum multiplicity theorem. *Let X and Y be n-manifolds without boundary, D a domain in X such that $\partial D = \partial(\bar{D})$, and $f\colon \bar{D} \to Y$ an open, closed, discrete mapping such that $f(D)$ is open in Y. Then $N(f) = N(f|\partial D)$ and $N(x, f|\partial D) = N(f|\partial D)$ for every $x \in \partial D \cap (\bar{D} - f^{-1}f(B_f))$, which is a dense open set of ∂D.*

Proof. As in the proof of 4.3, $f(\partial D) \cap f(D) = \emptyset$, and so $f|D$ is closed. By [**13**, 5.5], $N(x, f|D) = N(f|D) < \infty$ for all $x \in D - (f^{-1}f(B_f) \cap D)$ and $\dim(f^{-1}f(B_f) \cap D) \leqq n - 2$. Hence, $D - (f^{-1}f(B_f) \cap D)$ is connected; hence $\bar{D} - f^{-1}f(B_f)$ is connected. Since f is closed, $N(\ ,f)$ is upper semi-continuous on $\bar{D} - f^{-1}f(B_f)$. But $N(\ ,f)$ is lower semi-continuous on $\bar{D}$, since f is open, and hence $N(\ ,f)$ is constant on $\bar{D} - f^{-1}f(B_f)$ and $N(f) = N(x, f)$, for every $x \in \bar{D} - f^{-1}f(B_f)$. By 4.3, $\partial D \cap (\bar{D} - f^{-1}f(B_f))$ is dense in ∂D and since $f(D) \cap f(\partial D) = \emptyset$, $N(f|\partial D) \geqq N(x, f|\partial D) = N(x, f) = N(f) \geqq N(f|\partial D)$ for every $x \in \partial D \cap (\bar{D} - f^{-1}f(B_f))$. Hence, the theorem follows.

As an immediate consequence of 4.4, we have the following corollary.

4.5. Corollary. *Given $f\colon \bar{D} \to Y$ as in* (4.4), *if there exists a non-empty open subset, T, of ∂D such that $N(x, f) = 1$ for each $x \in T$, then f is a homeomorphism.*

As a final remark, it should be noted that Černavskiĭ's results and a simple construction can be used to obtain some of the results of this paper in the special case when X and Y are n-manifolds with non-empty boundary and $f\colon (X, \partial X) \to (Y, \partial Y)$ is a discrete open and closed mapping such that $f(\operatorname{int} X) \subset \operatorname{int} Y$. To this end, let X' be the n-manifold without boundary obtained by identifying two copies of X along ∂X, let Y' be the corresponding n-manifold without boundary obtained by identifying two copies of Y along ∂Y, and let g be the natural extension of f to a discrete open and closed map of X' into Y'. By Černavskiĭ's result, $\dim(B_g \cap \partial X) \leqq n - 2$, so that $\dim(B_f \cap \partial X) \leqq n - 2$.

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