

On Willmore's Inequality for Submanifolds

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Abstract. Let M be an m dimensional submanifold in the Euclidean space \mathbf{R}^n and H be the mean curvature of M . We obtain some low geometric estimates of the total square mean curvature $\int_M H^2 d\sigma$. The low bounds are geometric invariants involving the volume of M , the total scalar curvature of M , the Euler characteristic and the circumscribed ball of M .

1 Introduction

Let M be an m -dimensional submanifold, which is assumed to be C^2 smooth in the Euclidean space \mathbf{R}^n , and let H be the mean curvature of M . If $d\sigma$ denotes the volume density of M , we wish to find a low bound for the total square mean curvature $\int_M H^2 d\sigma$. The answer is still open for many cases. There has been some progress for the related mean curvature integrals since the last century. Reference can be easily found in the geometric literature, for example, [2, 3, 8, 12].

We call the total square mean curvature $\int_M H^2 d\sigma$ the *Willmore functional* based on the following well-known result of Willmore.

Proposition 1 (Willmore) *Let M be a compact surface in \mathbf{R}^3 and H be the mean curvature of M . Then $\int_M H^2 d\sigma \geq 4\pi$, where $d\sigma$ is the volume element of M , with equality if and only if M is a standard sphere.*

If $\dim(M) \neq 2$, then the Willmore functional is not a Riemannian invariant, so by applying a homothetic transformation, the value may approach zero. So there is no lower bound in this case. However if we assume that $\text{Vol}(M)$, the volume of M , is positive, then the Willmore functional should have a lower bound.

The following result is due to B-Y Chen [2].

Proposition 2 (B-Y Chen) *Let M be a closed submanifold of dimension m in the Euclidean space \mathbf{R}^n and H be the mean curvature of M . Then $\int_M |H|^m d\sigma \geq O_m$, with equality if and only if M is imbedded as an m -sphere of \mathbf{R}^n .*

Here O_m is the area of the m -dimensional unit sphere, and its value is given by

$$O_m = \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)},$$

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where Γ denotes the gamma function.

B-Y Chen also achieved the following inequality: $\int_M H^2 d\sigma \geq \frac{\lambda_p}{m} \text{Vol}(M)$, where p is the lower order of the immersion (in the case of Chen finite type theory), λ_p is the p -th nonnegative eigenvalue of Laplacian, and the equality holds when and only when the immersion is of 1-type with order p .

We can estimate a Willmore functional for a 3-dimensional convex hypersurface M in \mathbf{R}^4 by applying the theory of kinematic measure and Minkowski geometry. The integral $\int_M H^2 d\sigma$ obtained is bounded below by the surface area of M , the volume of the convex body K that M bounds, and the Minkowski quermassintegrals of the convex body K (see [19]). Also, if we assume that M is a convex hypersurface which is C^2 smooth and has a positive volume $\text{Vol}(M)$ in \mathbf{R}^n , then the Willmore functional of M is bounded below by the 3rd-order Minkowski quermassintegrals of the convex body K that M bounds, with equality when M is a standard $(n - 1)$ -sphere in \mathbf{R}^n [20].

In this paper, we obtain some lower bounds of the Willmore functional for the submanifold M of dimensions $\frac{n+1}{2}$ in the Euclidean space \mathbf{R}^n . These lower bounds are geometric invariants involving the volume of M , the total scalar curvature of M , the Euler characteristic and the circumscribed ball of M . We cannot obtain a Willmore functional lower estimate for the submanifold in [19, 20] since the results in [19, 20] are restricted to convex hypersurfaces.

2 Preliminaries

For an m dimensional submanifold M in \mathbf{R}^n , if we pick any pair of independent tangent vectors in $T_p(M)$, say u and v , then for every unit vector $w = \lambda u + \mu v$, there is a unique geodesic in M starting at p with tangent vector w . The set of all such geodesics, as w describes the unit circle in the plan spanned by u and v , sweep a surface whose Gauss curvature at p is the sectional curvature $K_\Pi = K[u, v]$ of the plane Π spanned by u and v .

Let e_1, \dots, e_m be an orthonormal basis of the tangent space $T_p(M)$ of M at p . Then the quantity $S = 2 \sum_{1 \leq i < j \leq m} K[e_i, e_j]$ is independent of the choice of basis and is called the scalar curvature of M at p .

For a hypersurface Σ in \mathbf{R}^n , we may choose the e_1, \dots, e_{n-1} to be the principal curvature directions at p . Then the scalar curvature S of Σ may be expressed in terms of the principal curvatures $\kappa_1, \dots, \kappa_{n-1}$ by $S = 2 \sum_{1 \leq i < j \leq n-1} \kappa_i \kappa_j$. One considers the Gauss map $G: p \rightarrow N(p)$, whose differential $dG_p: x'(t) \rightarrow N'(t)$, ($x(0) = p$) satisfies Rodrigues' equations $dG_p(e_i) = -\kappa_i e_i$, $i = 1, \dots, n - 1$. We have the mean curvature

$$H = \frac{1}{n - 1} (\kappa_1 + \dots + \kappa_{n-1}) = \frac{1}{n - 1} \text{trace}(dG_p)$$

and the Gauss-Kronecker curvature $\kappa_1 \cdots \kappa_{n-1} = (-1)^{n-1} \det(dG_p)$. The j th-order mean curvature is the j th-order elementary symmetric function of the principal curvatures. We denote by H_j the j th-order mean curvature, normalized such that

$$\prod_{j=1}^{n-1} (1 + t\kappa_j) = \sum_{j=0}^{n-1} \binom{n-1}{j} H_j t^j.$$

Thus, $H_1 = H$, the mean curvature, and H_{n-1} is the Gauss–Kronecker curvature. For $n = 3$, that is all, but in higher dimensions there are intermediate ones. Among them, H_2 plays a special role in differential geometry.

Therefore we have $S = (n - 1)(n - 2)H_2$.

The j th-order integral of mean curvature $M_j(\Sigma)$ is defined by

$$M_j(\Sigma) = \int_{\Sigma} H_j d\sigma = \binom{n-1}{j}^{-1} \int_{\Sigma} \{\kappa_{i_1}, \dots, \kappa_{i_j}\} d\sigma, \quad j = 1, \dots, n-1,$$

where $\{\kappa_{i_1}, \dots, \kappa_{i_j}\}$ denotes the j -th elementary symmetric function of the principal curvatures. We let $M_0(\Sigma) = F$, the area of Σ , for completeness.

If Σ is a convex hypersurface bounding a convex body K in \mathbf{R}^n , we have the relations $M_j(\Sigma) = nW_{j+1}(K)$ between integrals of mean curvatures of Σ ($\equiv \partial K$) and j th-order Minkowski quermassintegrals W_j of K , $j = 0, 1, \dots, n-1$ (see [9–11]).

Note that Minkowski quermassintegrals W_i are well defined for any convex figure, whereas $M_j(\partial K)$ makes sense only if ∂K is of class C^2 .

We have the total scalar curvature of a convex hypersurface Σ bounding a convex body K in \mathbf{R}^n : $\tilde{S} = \int_{\Sigma} S d\sigma = n(n-1)(n-2)W_3(K)$.

Let G be the group of rigid motions of Euclidean space \mathbf{R}^n . Let dg be the normalized kinematic density (the Harr measure). Let M^p, N^q be two compact submanifolds in \mathbf{R}^n , which are assumed to be in a general position, that is, $\dim(M^p \cap gN^q) = p + q - n \geq 0$ for almost all $g \in G$. Let $I(M^p \cap gN^q)$ be an invariant (intrinsic or extrinsic). For example, one would like to let $I(M^p \cap gN^q)$ be $\text{Vol}(M^p \cap gN^q)$, the volume of $M^p \cap gN^q$, or let $I(M^p \cap gN^q)$ be the mean curvature of $M^p \cap gN^q$. The following integral is called the kinematic formula in integral geometry

$$\int_G I(M^p \cap gN^q) dg = \sum_{j=1}^n C_{j p q n} \text{Inv}_j(M^p) \text{Inv}_{n-j}(N^q),$$

where each $\text{Inv}_j(*)$ is an invariant and C 's are constants depending on indices.

Refer to [4–7, 9–11, 15, 17] for more concrete kinematic formulas.

If we can estimate the kinematic formula $\int_G I(M^p \cap gN^q) dg$ from below (or from above) in terms of geometric invariants of M^p and N^q , then we obtain a geometric inequality of the form

$$\begin{aligned} \sum_{j=1}^n C_{j p q n} \text{Inv}_j(M^p) \text{Inv}_{n-j}(N^q) &\geq f(A_M^1, \dots, A_M^r; A_N^1, \dots, A_N^r), \\ &(\leq) f(A_M^1, \dots, A_M^r; A_N^1, \dots, A_N^r), \end{aligned}$$

where each of $A_M^\alpha, A_N^\alpha (\alpha = 1, \dots, r)$ is an integral geometric invariant.

In a special case, let $M^p \equiv N^q \equiv M$. One can immediately obtain an inequality about the integral geometric invariant of M :

$$\begin{aligned} F(A_M^1, \dots, A_M^r) &\leq 0, \\ &(\geq) 0. \end{aligned}$$

This is a geometric inequality about the submanifold M . See [13–16, 21] for more detail.

3 Main Theorems

Main Theorem A Let M be a submanifold of dimension $\frac{n+1}{2}$ in the Euclidean space \mathbf{R}^n and let H be the mean curvature of M . Denote by \tilde{S} the total scalar curvature of M and R the radius of the circumscribed ball of M . Then

$$\int_M H^2 d\sigma \geq \frac{1}{3(n+1)^2} \left(8\tilde{S}(M) + \frac{n+5}{R^2} \text{Vol}(M) \right).$$

Main Theorem B Let M be a submanifold of dimension $\frac{n+1}{2}$ in the Euclidean space \mathbf{R}^n and let H be the mean curvature of M . Denote by $\chi(M)$ the Euler characteristic of M and R the radius of the minimum circumscribed ball of M . If $\frac{n+1}{2}$ is even, then we have

$$\int_M H^2 d\sigma \geq \frac{1}{3(n+1)^2} \left(\frac{2^{\frac{n+7}{2}} \pi^{\frac{n+1}{4}}}{(n-1)(n-5) \dots 2} \chi(M) + \frac{n+5}{R^2} \text{Vol}(M) \right).$$

Let us first prove the following.

Theorem 1 Let M^p and N^q be two submanifolds of dimensions p and q , respectively, in the Euclidean space \mathbf{R}^n . Let \tilde{H} and \tilde{S} be, respectively, the total square mean curvature and the total scalar curvature. Denote by R the radius of the smallest circumscribed balls of M and N . If $p + q - n = 1$, then we have

$$\begin{aligned} & \frac{1}{R^2} \text{Vol}(M) \text{Vol}(N) \\ & \leq \frac{2\pi}{(p-1)p(p+2)} \frac{O_{q-1}}{O_{q+1}} [3(p-1)p^2\tilde{H}_M - 2(n-q)\tilde{S}_M] \text{Vol}(N) \\ & \quad + \frac{2\pi}{(q-1)q(q+2)} \frac{O_{p-1}}{O_{p+1}} [3(q-1)q^2\tilde{H}_N - 2(n-p)\tilde{S}_N] \text{Vol}(M). \end{aligned}$$

Proof For submanifolds M^p, N^q in \mathbf{R}^n , let H_g be the mean curvature of the intersection $M^p \cap gN^q$. We have the corrected C-S. Chen's kinematic formula (see [4, 7, 15, 17, 18]):

$$\begin{aligned} & \int_G \left(\int_{M^p \cap gN^q} H_g^2 d\sigma \right) dg \\ & = C_0 [(p-1)p^2(p+q-n+2)\tilde{H}(M) - 2(n-q)\tilde{S}(M)] \text{Vol}(N) \\ & \quad + C_2 [(q-1)q^2(p+q-n+2)\tilde{H}(N) - 2(n-p)\tilde{S}(N)] \text{Vol}(M), \end{aligned}$$

where

$$\begin{aligned} C_0 &= \frac{1}{(p+q-n)(p-1)p(p+2)} \frac{O_{p-1}}{O_{p+q-n-1}} \frac{O_n \dots O_1 O_{q-1} O_{p+q-n+1} O_{p+q-n}}{O_{p-1} O_p O_{q+1} O_q}, \\ C_2 &= \frac{1}{(p+q-n)(q-1)q(q+2)} \frac{O_{q-1}}{O_{p+q-n-1}} \frac{O_n \dots O_1 O_{p-1} O_{p+q-n+1} O_{p+q-n}}{O_{q-1} O_q O_{p+1} O_p}. \end{aligned}$$

When $\dim(M^p \cap gN^q) = p + q - n = 1$, let κ be the curvature of $M^p \cap gN^q$. Then we have

$$\begin{aligned} \int_G \left(\int_{M^p \cap gN^q} \kappa^2 ds \right) dg &= \frac{C'_0}{(p-1)p(p+2)} [3(p-1)p^2\tilde{H}(M) - 2(n-q)\tilde{S}(M)] \text{Vol}(N) \\ &\quad + \frac{C'_2}{(q-1)q(q+2)} [3(q-1)q^2\tilde{H}(N) - 2(n-p)\tilde{S}(N)] \text{Vol}(M), \end{aligned}$$

where

$$C'_0 = \frac{O_n \cdots O_1 O_{q-1} O_2 O_1}{O_0 O_p O_{q+1} O_q}, \quad C'_2 = \frac{O_n \cdots O_1 O_{p-1} O_2 O_1}{O_0 O_q O_{p+1} O_p}.$$

One known inequality (see [1]) for the curve $\Gamma_g = M^p \cap gN^q$ in \mathbf{R}^n becomes

$$L_g = \int_{\Gamma_g} ds \leq R \int_{\Gamma_g} \kappa ds,$$

where R is the smaller radius of the minimum circumscribed balls of M^p and N^q .

By Hölder's inequality we have

$$L_g^2 \leq R^2 \left(\int_{\Gamma_g} \kappa ds \right)^2 \leq R^2 L_g \int_{\Gamma_g} \kappa^2 ds,$$

and therefore

$$\frac{L_g}{R^2} \leq \int_{\Gamma_g} \kappa^2 ds.$$

Hence

$$\frac{1}{R^2} \int_G L_g dg \leq \int_G \left(\int_{\Gamma_g} \kappa^2 ds \right) dg.$$

By Santaló's formula [9, 10]

$$\int_G \text{Vol}(M^p \cap gN^q) dg = \frac{O_n \cdots O_1 O_{p+q-n}}{O_p O_q} \text{Vol}(M^p) \text{Vol}(N^q),$$

we have

$$\begin{aligned} \frac{1}{R^2} \text{Vol}(M) \text{Vol}(N) &\leq \\ &\frac{O_2}{(p-1)p(p+2)} \frac{O_{q-1}}{O_0 O_{q+1}} [3(p-1)p^2\tilde{H}(M) - 2(n-q)\tilde{S}(M)] \text{Vol}(N) \\ &\quad + \frac{O_2}{(q-1)q(q+2)} \frac{O_{p-1}}{O_0 O_{p+1}} [3(q-1)q^2\tilde{H}(N) - 2(n-p)\tilde{S}(N)] \text{Vol}(M). \end{aligned}$$

This completes the proof of Theorem 1. ■

Let $M^p \equiv N^q \equiv M$, i.e., $p = q = \frac{n+1}{2}$. Then by Theorem 1 we have

$$\int_M H^2 d\sigma \geq \frac{8}{3(n+1)^2} \tilde{S}(M) + \frac{n+5}{12(n+1)\pi R^2} \frac{O_{\frac{n+3}{2}}}{O_{\frac{n-1}{2}}} \text{Vol}(M).$$

Noticing that

$$\frac{O_{\frac{n+3}{2}}}{O_{\frac{n-1}{2}}} = \frac{4\pi}{n+1},$$

we obtain the Main Theorem A.

If $\frac{n+1}{2}$ is even, then by Theorem 1 and the Gauss–Bonnet formula [6], we have

$$\int_M H^2 d\sigma \geq \frac{8}{3(n+1)^2} \frac{(4\pi)^{\frac{n+1}{4}}}{(n-1)(n-5)\dots 2} \chi(M) + \frac{n+5}{3(n+1)^2 R^2} \text{Vol}(M),$$

where $\chi(M)$ is the Euler characteristic of M . This completes the proof of Main Theorem B.

When $n = 3$, we immediately obtain the following.

Theorem 2 *Let Σ be a compact surface of C^2 smooth in \mathbf{R}^3 . Denote by $\chi(\Sigma)$ the Euler characteristic, H the mean curvature, A the surface area and R the radius of the minimum circumscribed ball of Σ . Then we have*

$$\int_{\Sigma} H^2 d\sigma \geq \frac{\pi}{3} \chi(\Sigma) + \frac{A}{6R^2}.$$

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