INDUCTIVE AND PROJECTIVE LIMITS OF NORMED SPACES

by JOHN S. PYM

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Let $\{U_i, u_{ij}\}$ be an inductive system of normed linear spaces U_i and continuous linear maps $u_{ij}\colon U_j \to U_i$. (We write $j \prec i$ if $u_{ij}\colon U_j \to U_i$.) An inductive limit of the system with respect to a class $(\mathfrak{U}, \mathfrak{M})$ of spaces A in $\mathfrak{U}$ and maps f in $\mathfrak{M}$ is a space $U_{\mathfrak{U}}$ in $\mathfrak{U}$ and a system $u_i\colon U_i \to U_{\mathfrak{U}}$ of maps in $\mathfrak{M}$ such that (i) $u_i \circ u_{ij} = u_j$ whenever $j \prec i$, and that (ii) if A is any space in $\mathfrak{U}$ and $f_i\colon U_i \to A$ is any system of maps in $\mathfrak{M}$ for which $f_i \circ u_{ij} = f_j$ $(j \prec i)$, then there is a unique map $f\colon U_{\mathfrak{U}} \to A$ in $\mathfrak{M}$ such that $f_i = f \circ u_i$ for each i. If $\mathfrak{U}$ is the class of all vector spaces and $\mathfrak{M}$ is the class of linear maps, we obtain the algebraic inductive limit, which we denote simply by U. The usual choice is to take $\mathfrak{U}$ to be the class of locally convex spaces and $\mathfrak{M}$ the class of continuous linear maps; the inductive limit U_L then always exists [**1**, § 16 C]. If $\mathfrak{M}$ is again the continuous linear mappings but $\mathfrak{U}$ contains only normed spaces, the corresponding inductive limit U_N may not always exist. However, if in addition we require that $\mathfrak{M}$ contains just contractions (norm-decreasing linear mappings), then an inductive limit U_C will exist if every u_{ij} is a contraction [**2**]. We shall give a condition under which these limits coincide (as far as possible), and consider the corresponding condition for projective limits.

THEOREM 1. *If U_N exists, it is isomorphic (as a locally convex space) with U_L.*

Proof. As U_N is a locally convex space, there is a unique map $f\colon U_L \to U_N$ such that the composite maps $U_i \to U_L \xrightarrow{f} U_N$ are the canonical maps into U_N. We find a continuous linear inverse for f. Let p be any continuous seminorm on U_L, and let $U_p = U_L/p^{-1}(0)$ be the normed quotient space. The continuous maps $U_i \to U_L \to U_p$ provide a continuous map $g_p\colon U_N \to U_p$, and the maps g_p yield a continuous linear map g of U_N into the projective limit of the spaces U_p, viz. U_L itself [**1**, § 16 D(*h*)]. It is easy to see that the maps $U_i \to U_N \xrightarrow{g} U_L$ are the canonical maps into U_L, and we deduce that both $f \circ g$ and $g \circ f$ are identity maps.

We shall say that $\{U_i, u_{ij}\}$ is *countably directed* (resp. *directed*) if, for any countable (resp. finite) set $\{i_1, i_2, \ldots\}$, there is a j such that $i_n \prec j$ for every n. If the system is directed and each u_{ij} is an injection, then the canonical maps $u_i\colon U_i \to U$ are injections. It is shown in [**2**] that if, in addition, u_{ij} is a contraction for each i, j, then $\|x\| = \inf_i \|u_i^{-1}(x)\|_i$ (where $\|\cdot\|_i$ denotes the norm in U_i) defines a seminorm in U, and U_C is the quotient of U by the subspace $\{x\colon \|x\| = 0\}$.

THEOREM 2. *Let $\{U_i, u_{ij}\}$ be countably directed, and let each u_{ij} be a contraction and an injection. Then, for each x, there is an i such that $\|x\| = \|u_i^{-1}(x)\|_i$. Further, $\|\cdot\|$ is a norm on U, and U_N and U_C both exist and are isomorphic to $(U, \|\cdot\|)$. If each U_i is a Banach space, so also is $(U, \|\cdot\|)$.*

Proof. For $x \in U$, take (i_n) such that $\|u_{i_n}^{-1}(x)\|_{i_n} \to \|x\|$. For any j with $i_n \prec j$ for all n, we have $\|x\| \leqq \|u_j^{-1}(x)\|_j = \|u_{ji_n}(u_{i_n}^{-1}(x))\|_j \leqq \|u_{i_n}^{-1}(x)\|_{i_n} \to \|x\|$; whence $\|u_j^{-1}(x)\|_j = \|x\|$. We can say more: if $\{x_n\}$ is any countable subset of U, there is a j such that $\|u_j^{-1}(x_n)\| = \|x_n\|$ for each n (for let j_n satisfy $\|u_{j_n}^{-1}(x_n)\|_{j_n} = \|x_n\|$, and take j with $j_n \prec j$ for all n). Therefore, if

$\| x \| = 0$, there is a j such that $\| u_j^{-1}(x) \|_j = 0$; whence $u_j^{-1}(x) = 0$, and $x = 0$. This and other elementary arguments show that $\| \cdot \|$ is a norm on U. Let f_i: $U_i \to A$ be any system of continuous linear maps into a normed space A for which $f_i \circ u_{ij} = f_j$ whenever $j \prec i$, and let f: $U \to A$ be the canonical linear map for which $f \circ u_i = f_i$ (all i). Put

$$K = \sup \{ \| f(x) \| : \| x \| \leqq 1 \},$$

and choose (x_n) with $\| x_n \| \leqq 1$ such that $\| f(x_n) \| \to K$. We can find a j such that $\| u_j^{-1}(x_n) \|_j \leqq 1$ for every n, and $\| f_j(u_j^{-1}(x_n)) \| = \| f(x_n) \| \to K$; therefore $K \leqq \| f_j \|$. We conclude that f is continuous, and that if each f_j is a contraction, so is f; thus $(U, \| \cdot \|)$ coincides with U_N and U_C. Finally, if (x_n) is a Cauchy sequence in $(U, \| \cdot \|)$, we can find a j such that

$$\| u_j^{-1}(x_n - x_m) \|_j = \| x_n - x_m \| \to 0$$

as $m, n \to \infty$. If U_j is complete, there is a y such that $u_j^{-1}(x_n) \to y$; whence

$$x_n = u_j(u_j^{-1}(x_n)) \to y.$$

The completeness of each U_j therefore implies the completeness of $(U, \| \cdot \|)$.

The notation we use for projective limits is parallel to that for inductive limits. The following result is obtained by arguments similar to those of Theorem 2.

THEOREM 3. *Let $\{V_i, v_{ij}\}$ be a countably directed projective system of normed spaces with each v_{ij} a contraction. For $x \in V$, the algebraic projective limit, write $\| x \| = \sup_i \| v_i(x) \|_i$. Then there is an i such that $\| x \| = \| v_i(x) \|_i$. Further, $\| \cdot \|$ is a norm on V, and V_N and V_C both exist and are isomorphic with $(V, \| \cdot \|)$. If each V_i is a Banach space, so also is $(V, \| \cdot \|)$.*

We remark that if $\{U_i, u_{ij}\}$ satisfies the hypotheses of Theorem 2, if V_i is the normed dual of U_i, and if v_{ij} is the adjoint of u_{ij}, then $\{V_i, v_{ij}\}$ satisfies the hypotheses of Theorem 3. It is shown in Theorem 2 of [**2**] that V_C is the normed dual of U_C.

Examples. (i) Let X be locally compact, and let $M(X)$ be the usual space of bounded Radon measures. For $\mu \geqq 0$, $\mu \in M(X)$, write $L^1(\mu) = \{ \nu \in M(X) : \nu \ll \mu \}$. The system of spaces $L^1(\mu)$ with the inclusion maps satisfies the conditions of Theorem 2 (countably directed because, given a sequence (μ_n), we can find a sequence (a_n) of real numbers, with $a_n > 0$ for each n, and $\sum a_n \mu_n \in M(X)$). It is easy to see that $M(X)$ is the (normed) inductive limit of these subspaces.

The dual of each $L^1(\mu)$ may be identified with $L^\infty(\mu)$. The normed dual of $M(X)$ is thus the normed projective limit of the spaces $L^\infty(\mu)$.

(ii) Even in this special situation, V_N may not be isomorphic with V_L (cf. Theorem 1). Thus take X to be uncountable and discrete. Let m be the (unbounded) measure which assigns mass 1 to each point. Then $M(X) = L^1(m)$, and $M(X)^* = L^\infty(m)$ is the space $B(X)$ of all bounded functions on X. The canonical projection from $B(X)$ to $L^\infty(\mu)$ maps the bounded function f on X to its restriction to the (necessarily countable) support of μ. The locally convex space projective limit of the spaces $L^\infty(\mu)$ therefore consists of $B(X)$ with a topology defined by neighbourhoods $N(S, \varepsilon) = \{ f : |f(x)| < \varepsilon \text{ for all } x \in S \}$ $(\varepsilon > 0,\ S$ a countable subset of $X)$. But, of course, $M(X)^*$ is $B(X)$ with the uniform norm.

(iii) The limit U_N may be distinct from U_C. Thus let B be any Banach space, and let U_n $(n = 1, 2, \ldots)$ have the same underlying space as B, but with norm defined by $\|x\|_n = \|x\|/n$; the maps $U_n \to U_m$ are the identities if $n < m$. Then U_N may be taken to be B; but $U_C = \{0\}$.

REFERENCES

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