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Abstract

We connect two developments that aim to extend Voevodsky's theory of motives over a field in such a way as to encompass non-$\mathbf{A}^1$-invariant phenomena. One is theory of *reciprocity sheaves* introduced by Kahn, Saito and Yamazaki. The other is theory of the triangulated category $\mathbf{logDM}^{\mathrm{eff}}$ of *logarithmic motives* launched by Binda, Park and Østvær. We prove that the Nisnevich cohomology of reciprocity sheaves is representable in $\mathbf{logDM}^{\mathrm{eff}}$.

Contents

Introduction

We fix once and for all a perfect base field k. The main purpose of this paper is to connect two developments that aim to extend Voevodsky's theory of motives over k in such a way as to encompass non-$\mathbf{A}^1$-invariant phenomena. One is the theory of *reciprocity sheaves* introduced by Kahn, Saito and Yamazaki [KSY16, KSY22] and developed in [Sai20, BRS22]. Voevodsky's theory is based on the category $\mathbf{PST}$ of *presheaves with transfers*, defined as the category of additive presheaves of abelian groups on the category $\mathbf{Cor}$ of finite correspondences: $\mathbf{Cor}$ has the same objects as the category $\mathbf{Sm}$ of separated smooth schemes of finite type over k, and morphisms in $\mathbf{Cor}$ are finite correspondences. Let $\mathbf{NST} \subset \mathbf{PST}$ be the full subcategory of Nisnevich sheaves, that is, those objects $F \in \mathbf{PST}$ whose restrictions F_X to the small étale site $X_{\text{ét}}$ over X are Nisnevich sheaves for all $X \in \mathbf{Sm}$. Voevodsky proved that $\mathbf{NST}$ is a Grothendieck abelian

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category and defined the triangulated category $\mathbf{DM}^{\mathrm{eff}}$ of effective motives as the localization of the derived category $D(\mathbf{NST})$ of complexes in $\mathbf{NST}$ with respect to an $\mathbf{A}^1$-*weak equivalence* (see [MVW06, Definition 14.1]). It is equipped with a functor $M : \mathbf{Sm} \to \mathbf{DM}^{\mathrm{eff}}$ associating the motive $M(X)$ of $X \in \mathbf{Sm}$.

Let $\mathbf{HI}_{\mathrm{Nis}} \subset \mathbf{NST}$ be the full subcategory consisting of $\mathbf{A}^1$-invariant objects, namely such $F \in \mathbf{NST}$ that the projection $\pi_X : X \times \mathbf{A}^1 \to X$ induces an isomorphism $\pi_X^* : F(X) \simeq F(X \times \mathbf{A}^1)$ for any $X \in \mathbf{Sm}$. We say that $F \in \mathbf{HI}_{\mathrm{Nis}}$ is strictly $\mathbf{A}^1$-invariant if π_X induces isomorphisms

$$\pi_X^* : H^i_{\mathrm{Nis}}(X, F_X) \simeq H^i_{\mathrm{Nis}}(X \times \mathbf{A}^1, F_{X\times\mathbf{A}^1}) \quad \text{for all } i \geq 0.$$

The following theorem plays a fundamental role in Voevodsky's theory.

Theorem 0.1 (Voevodsky [Voe00]). *Any $F \in \mathbf{HI}_{\mathrm{Nis}}$ is strictly $\mathbf{A}^1$-invariant and we have a natural isomorphism*

$$H^i_{\mathrm{Nis}}(X, F_X) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}}(M(X), L^{\mathbf{A}^1}F[i]) \quad \text{for } X \in \mathbf{Sm}, \tag{0.1.1}$$

where $L^{\mathbf{A}^1} : D(\mathbf{NST}) \to \mathbf{DM}^{\mathrm{eff}}$ is the localization functor.

Notice that there are interesting and important objects of $\mathbf{NST}$ which do not belong to $\mathbf{HI}_{\mathrm{Nis}}$. Such examples are given by the sheaves Ω^i of (absolute or relative) differential forms; the p-typical de Rham–Witt sheaves $W_m\Omega^i$ of Bloch, Deligne and Illusie; smooth commutative k-group schemes with a unipotent part (seen as objects of $\mathbf{NST}$); and the complexes $R\varepsilon_*\mathbb{Z}/p^r(n)$ with $\mathrm{ch}(k) = p > 0$, where $\mathbb{Z}/p^r(n)$ is the étale motivic complex of weight n with $\mathbb{Z}/p^r$ coefficients and ε is the change of site functor from the étale to the Nisnevich topology. For such examples, (0.1.1) fails to hold since $\pi_X : X \times \mathbf{A}^1 \to X$ induces an isomorphism $M(X \times \mathbf{A}^1) \simeq M(X)$ in $\mathbf{DM}^{\mathrm{eff}}$ but the maps induced on the cohomology of those sheaves are not isomorphisms.

The category $\mathbf{RSC}_{\mathrm{Nis}}$ of reciprocity sheaves is a full abelian subcategory of $\mathbf{NST}$ that contains $\mathbf{HI}_{\mathrm{Nis}}$ as well as the non-$\mathbf{A}^1$-invariant objects mentioned above. Heuristically, its objects satisfy the property that for any $X \in \mathbf{Sm}$, each section $a \in F(X)$ 'has bounded ramification at infinity' and the objects of $\mathbf{HI}_{\mathrm{Nis}}$ are special reciprocity sheaves with the property that every section $a \in F(X)$ has 'tame' ramification at infinity.[1] Slightly more exotic examples of reciprocity sheaves are given by the sheaves Conn^1 (for $\mathrm{ch}(k) = 0$), whose sections over X are rank 1-connections, or Lisse^1_ℓ (in case $\mathrm{ch}(k) = p > 0$), whose sections on X are the lisse $\overline{\mathbb{Q}}_\ell$-sheaves of rank 1. Since $\mathbf{RSC}_{\mathrm{Nis}}$ is an abelian category equipped with a lax symmetric monoidal structure by [RSY22], many more interesting examples can be manufactured by taking kernels, quotients and tensor products (see [BRS22, § 11.1] for more examples).

The main purpose of this paper is to establish formula (0.1.1) for all $F \in \mathbf{RSC}_{\mathrm{Nis}}$ in a new category which enlarges $\mathbf{DM}^{\mathrm{eff}}$ (see (0.2)). It is the triangulated category $\mathbf{logDM}^{\mathrm{eff}}$ of *logarithmic motives* introduced by Binda, Park and Østvær in [BPØ22]. Let $\mathrm{l}\mathbf{Sm}$ be the category of log smooth and separated fs log schemes of finite type over k, and $\mathrm{l}\mathbf{Cor}$ be the category with the same objects as $\mathrm{l}\mathbf{Sm}$ and whose morphisms are log finite correspondences (see [BPØ22, Definition 2.1.1]). Let $\mathbf{PSh}^{\mathrm{ltr}}$ be the category of additive presheaves of abelian groups on $\mathrm{l}\mathbf{Cor}$ and $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}} \subset \mathbf{PSh}^{\mathrm{ltr}}$ be the full subcategory consisting of those $\mathcal{F}$ whose restrictions to $\mathrm{l}\mathbf{Sm}$ are dividing Nisnevich sheaves (see [BPØ22, Definition 3.1.4]). It is shown in [BPØ22, Theorem 1.2.1] that $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$ is a Grothendieck abelian category, and $\mathbf{logDM}^{\mathrm{eff}}$ is defined as the localization of the derived category $D(\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}})$ of complexes in $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$ with respect to a $\overline{\square}$-*weak equivalence*, where $\overline{\square}$ is $\mathbf{P}^1$ with the log structure associated to the effective divisor $\infty \hookrightarrow \mathbf{P}^1$

[1] This heuristic viewpoint is manifested in [RS21a, Theorem 2].

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(see [BPØ22, Definition 5.2.1]).[2] It is equipped with a functor $M : \mathrm{l}\mathbf{Sm} \to \mathbf{logDM}^{\mathrm{eff}}$ associating the logarithmic motive $M(\mathfrak{X})$ of $\mathfrak{X} \in \mathrm{l}\mathbf{Sm}$. Thanks to [BM12, Theorem 1,1], the standard t-structure on $D(\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}})$ induces a t-structure on $\mathbf{logDM}^{\mathrm{eff}}$ called the homotopy t-structure, and its heart is identified with the abelian full subcategory $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \subset \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$ consisting of strictly $\overline{\square}$-invariant objects in the sense of [BPØ22, Definition 5.2.2].[3] We can now state the main result of this paper.

THEOREM 0.2 (Theorems 6.1 and 6.3). *There exists an exact and fully faithful functor*

$$\mathcal{L}og : \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \ : \ F \to F^{\log} = \mathcal{L}og(F). \tag{0.2.1}$$

For $X \in \mathbf{Sm}$ we have a natural isomorphism

$$H^i_{\mathrm{Nis}}(X, F_X) \simeq \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(M(X, \mathrm{triv}), L^{\overline{\square}} F^{\log}[i]), \tag{0.2.2}$$

where $L^{\overline{\square}} : D(\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}) \to \mathbf{logDM}^{\mathrm{eff}}$ is the localization functor and (X, triv) is the log scheme with the trivial log structure.

We remark (see Remark 5.6) that, for $F = \Omega^i$, $F^{\log}(\mathfrak{X})$ for $\mathfrak{X} \in \mathrm{l}\mathbf{Sm}$ whose underlying scheme is smooth agrees with the sheaf of logarithmic differential forms of $\mathfrak{X}$ at least assuming $\mathrm{ch}(k) = 0$.[4]

We now explain the organization of the paper.

In § 1 we discuss some preliminaries and fix notation. We recall the definitions and basic properties of *modulus (pre)sheaves with transfers* from [KMSY21a, KMSY21b, KSY22, Sai20]. These are a generalization of Voevodsky's (pre)sheaves with transfers to a version with modulus. The category $\underline{\mathbf{M}}\mathbf{Cor}$ of *modulus correspondences* is introduced. Its objects are pairs $\mathcal{X} = (\overline{X}, D)$, where $\overline{X}$ is a separated scheme of finite type over k equipped with an effective Cartier divisor D such that the *interior* $\overline{X} - D = X$ is smooth. The morphisms are finite correspondences on the interiors satisfying admissibility and a properness condition. Let $\underline{\mathbf{M}}\mathbf{PST}$ be the category of additive presheaves of abelian groups on $\underline{\mathbf{M}}\mathbf{Cor}$. A full subcategory $\underline{\mathbf{M}}\mathbf{NST} \subset \underline{\mathbf{M}}\mathbf{PST}$ of *Nisnevich sheaves* is defined and there is a functor (see § 1(20))

$$\underline{\omega}^{\mathbf{CI}} : \mathbf{RSC}_{\mathrm{Nis}} \to \underline{\mathbf{M}}\mathbf{NST}\,.$$

For every $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and $X \in \mathbf{Sm}$, it provides an exhaustive filtration on the group $F(X)$ of sections over X which measures the depth of ramification along a boundary of a partial compactification of X: for $(\overline{X}, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ with $\overline{X} - D = X$, we get the subgroups $\tilde{F}(\overline{X}, D) \subset F(X)$ with $\tilde{F} = \underline{\omega}^{\mathbf{CI}} F$ such that $\tilde{F}(\overline{X}, D_1) \subset \tilde{F}(\overline{X}, D_2)$ if $D_1 \leq D_2$.

In § 2 we prove as a key technical input an analogue of the Zariski–Nagata purity theorem [SGA2, X 3.4] for $\tilde{F}(\overline{X}, D)$ as above. This asserts the exactness of the sequence

$$0 \to \tilde{F}(\overline{X}, D) \to F(X) \to \bigoplus_{\xi \in D^{(0)}} \frac{F(\overline{X}^h_{|\xi} - \xi)}{\tilde{F}(\overline{X}^h_{|\xi}, \xi)},$$

where $\overline{X} \in \mathbf{Sm}$ and D is a reduced simple normal crossing divisor, and where $D^{(0)}$ is the set of the irreducible components of D and $\overline{X}^h_{|\xi}$ is the henselization of $\overline{X}$ at ξ. In [RS21b] this result is generalized to the case where D may not be reduced under the assumption that $\overline{X}$ admits a smooth compactification.

[2] In fact it is defined in [BPØ22, Definition 5.2.1] as the localization of the homotopy category of complexes in $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$ with respect to a $\overline{\square}$-local descent model structure.
[3] It is a logarithmic analogue of Voevodsky's strict $\mathbf{A}^1$-invariance.
[4] The assumption is necessary to use [RS21a, Corollary 6.8] proved in the case $\mathrm{ch}(k) = 0$. We expect that it can be dispensed with by using a forthcoming work of K. Rülling extending [RS21a, Corollary 6.8] to the case $\mathrm{ch}(k) > 0$.

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In § 3 we review *higher local symbols* for reciprocity sheaves constructed in [RS21c]. These are an effective tool with which one can decide when a given element of $F(X)$ with $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and $X \in \mathbf{Sm}$ belongs to $\tilde{F}(\overline{X}, D)$ as above. The construction of the pairing depends on pushforward maps for the cohomology of reciprocity sheaves constructed in [BRS22] (which means that Theorem 0.2 depends on the result of [BRS22]).

In § 4 we prove the following result. Let $\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}^{\mathrm{fin}}$ be the subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ whose objects are pairs (X, D) such that $X \in \mathbf{Sm}$ and the reduced divisor D_{red} underlying D is a simple normal crossing divisor on X and whose morphisms are modulus correspondences satisfying a finiteness conditions instead of the properness condition (see § 1(5)). Then, for $F \in \mathbf{RSC}_{\mathrm{Nis}}$, the association

$$\tilde{F}^{\log} : (X, D) \to \underline{\omega}^{\mathbf{CI}} F(X, D_{\mathrm{red}})$$

gives a presheaf on $\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}^{\mathrm{fin}}$, which gives rise to a cohomology theory $H^i_{\log}(-, \tilde{F}^{\log})$ on $\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}^{\mathrm{fin}}$, called *the ith logarithmic cohomology with coefficient* F (see Definition 4.4). The higher local symbols for F play a fundamental role in the proof of the result.

In § 5 we prove the invariance of logarithmic cohomology under blowups. Let $\Lambda_{\mathrm{ls}}^{\mathrm{fin}}$ be the subcategory of $\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}^{\mathrm{fin}}$ whose objects are the same as $\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}^{\mathrm{fin}}$ and whose morphisms are those $\rho : (Y, E) \to (X, D)$ where $E = \rho^* D$ and ρ are induced by blowups of X in smooth centers $Z \subset D$ which are normal crossing to D (see the beginning of the section). Then, for $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and $\rho : \mathcal{Y} \to \mathcal{X}$ in $\Lambda_{\mathrm{ls}}^{\mathrm{fin}}$, we have

$$\rho^* : H^i_{\log}(\mathcal{X}, F) \cong H^i_{\log}(\mathcal{Y}, F) \quad \forall i \geq 0.$$

In § 6 we prove Theorem 0.2, which is a formal consequence of the theorems in §§ 4 and 5.

1. Preliminaries

We fix once and for all a perfect base field k. In this section we recall the definitions and basic properties of modulus sheaves with transfers from [KMSY21a, Sai20].

(1) Denote by **Sch** the category of separated schemes of finite type over k and by **Sm** the full subcategory of smooth schemes. For $X, Y \in \mathbf{Sm}$, an integral closed subscheme of $X \times Y$ that is finite and surjective over a connected component of X is called a *prime correspondence from* X *to* Y. The category **Cor** of finite correspondences has the same objects as **Sm**, and for $X, Y \in \mathbf{Sm}$, $\mathbf{Cor}(X, Y)$ is the free abelian group on the set of all prime correspondences from X to Y (see [Voe00]). We consider **Sm** as a subcategory of **Cor** by regarding a morphism in **Sm** as its graph in **Cor**.

Let **PST** be the category of additive presheaves of abelian groups on **Cor** whose objects are called *presheaves with transfers*. Let $\mathbf{NST} \subseteq \mathbf{PST}$ be the category of Nisnevich sheaves with transfers and let

$$a^V_{\mathrm{Nis}} : \mathbf{PST} \to \mathbf{NST}$$

be Voevodsky's Nisnevich sheafification functor, which is an exact left adjoint to the inclusion $\mathbf{NST} \to \mathbf{PST}$. Let $\mathbf{HI} \subseteq \mathbf{PST}$ be the category of $\mathbf{A}^1$-invariant presheaves and put $\mathbf{HI}_{\mathrm{Nis}} = \mathbf{HI} \cap \mathbf{NST} \subseteq \mathbf{NST}$.

(2) Let $\mathbf{Sm}^{\mathrm{pro}}$ be the category of k-schemes X which are essentially smooth over k, that is, X is a limit $\varprojlim_{i \in I} X_i$ over a filtered set I, where X_i is smooth over k and all transition maps are étale. Note that $\operatorname{Spec} K \in \mathbf{Sm}^{\mathrm{pro}}$ for a function field K over k thanks to the assumption that k is perfect. We define $\mathbf{Cor}^{\mathrm{pro}}$ whose objects are the same as $\mathbf{Sm}^{\mathrm{pro}}$ and whose morphisms are defined as [RS21a, Definition 2,2]. We extend $F \in \mathbf{PST}$ to a presheaf on $\mathbf{Cor}^{\mathrm{pro}}$ by $F(X) := \varinjlim_{i \in I} F(X_i)$ for X as above.

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(3) We recall the definition of the category $\underline{\mathbf{M}}\mathbf{Cor}$ from [KMSY21a, Definition 1.3.1]. A pair $\mathcal{X} = (X, D)$ consisting of $X \in \mathbf{Sch}$ and an effective Cartier divisor D on X is called a *modulus pair* if $X - D \in \mathbf{Sm}$. Let $\mathcal{X} = (X, D_X)$, $\mathcal{Y} = (Y, D_Y)$ be modulus pairs and $\Gamma \in \mathbf{Cor}(X - D_X, Y - D_Y)$ be a prime correspondence. Let $\overline{\Gamma} \subseteq X \times Y$ be the closure of Γ, and let $\overline{\Gamma}^N \to X \times Y$ be the normalization. We say that Γ is *admissible* (respectively, *left proper*) if $(D_X)_{\overline{\Gamma}^N} \geq (D_Y)_{\overline{\Gamma}^N}$ (respectively, if $\overline{\Gamma}$ is proper over X). Let $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y})$ be the subgroup of $\mathbf{Cor}(X - D_X, Y - D_Y)$ generated by all admissible left proper prime correspondences. The category $\underline{\mathbf{M}}\mathbf{Cor}$ has modulus pairs as objects and $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y})$ as the group of morphisms from $\mathcal{X}$ to $\mathcal{Y}$.

(4) Let $\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}} \subset \underline{\mathbf{M}}\mathbf{Cor}$ be the full subcategory of $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ with $X \in \mathbf{Sm}$ and $|D|$ a normal crossing divisor on X.

(5) Let $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}} \subset \underline{\mathbf{M}}\mathbf{Cor}$ be the full subcategory of the same objects such that $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(\mathcal{X}, \mathcal{Y})$ are generated by all admissible *finite* prime correspondences, where finite prime correspondences are defined by replacing the left properness in (3) by finiteness. We also define $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}_{\mathrm{ls}} = \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}} \cap \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$.

(6) There is a canonical pair of adjoint functors $\lambda \dashv \underline{\omega}$:

$$\lambda : \mathbf{Cor} \to \underline{\mathbf{M}}\mathbf{Cor} \quad X \mapsto (X, \emptyset),$$

$$\underline{\omega} : \underline{\mathbf{M}}\mathbf{Cor} \to \mathbf{Cor} \quad (X, D) \mapsto X - D.$$

(7) There is a full subcategory $\mathbf{MCor} \subset \underline{\mathbf{M}}\mathbf{Cor}$ consisting of *proper modulus pairs*, where a modulus pair (X, D) is *proper* if X is proper. Let $\tau : \mathbf{MCor} \hookrightarrow \underline{\mathbf{M}}\mathbf{Cor}$ be the inclusion functor and $\omega = \underline{\omega}\tau$.

(8) Let $\mathbf{MPST}$ (respectively, $\underline{\mathbf{M}}\mathbf{PST}$) be the category of additive presheaves of abelian groups on $\mathbf{MCor}$ (respectively, $\underline{\mathbf{M}}\mathbf{Cor}$) whose objects are called *modulus presheaves with transfers*. For $\mathcal{X} \in \mathbf{MCor}$, let $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) = \underline{\mathbf{M}}\mathbf{Cor}(-, \mathcal{X})$ be the representable object of $\underline{\mathbf{M}}\mathbf{PST}$. We sometimes write $\mathcal{X}$ for $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X})$ for simplicity.

(9) In the same manner as (2), the category $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$ is defined and $F \in \underline{\mathbf{M}}\mathbf{PST}$ is extended to a presheaf on $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$ (see [RS21a, § 3.7]).

(10) The adjunction $\lambda \dashv \underline{\omega}$ induces a string of four adjoint functors ($\lambda_! = \underline{\omega}^!, \lambda^* = \underline{\omega}_!, \lambda_* = \underline{\omega}^*, \underline{\omega}_*$) (see [KMSY21a, Proposition 2.3.1]):

$$\underline{\mathbf{M}}\mathbf{PST} \; \begin{matrix} \xleftarrow{\underline{\omega}^!} \\ \xrightarrow{\underline{\omega}_!} \\ \xleftarrow{\underline{\omega}^*} \\ \xrightarrow{\underline{\omega}_*} \end{matrix} \; \mathbf{PST}$$

where $\underline{\omega}_!, \underline{\omega}_*$ are localizations and $\underline{\omega}^!$ and $\underline{\omega}^*$ are fully faithful.

(11) The functor τ yields a string of three adjoint functors $(\tau_!, \tau^*, \tau_*)$:

$$\mathbf{MPST} \; \begin{matrix} \xrightarrow{\tau_!} \\ \xleftarrow{\tau^*} \\ \xrightarrow{\tau_*} \end{matrix} \; \underline{\mathbf{M}}\mathbf{PST}$$

where $\tau_!, \tau_*$ are fully faithful and τ^* is a localization; $\tau_!$ has a pro-left adjoint $\tau^!$, hence is exact (see [KMSY21a, Proposition 2.4.1]). We will denote by $\underline{\mathbf{M}}\mathbf{PST}^{\tau}$ the essential image of $\tau_!$ in $\underline{\mathbf{M}}\mathbf{PST}$.

(12) The modulus pair $\overline{\square} := (\mathbf{P}^1, \infty)$ has an interval structure induced by that of $\mathbf{A}^1$ (see [KSY22, Lemma 2.1.3]). We say that $F \in \mathbf{MPST}$ is $\overline{\square}$-invariant if $p^* : F(\mathcal{X}) \to F(\mathcal{X} \otimes \overline{\square})$ is an isomorphism for any $\mathcal{X} \in \mathbf{MCor}$, where $p : \mathcal{X} \otimes \overline{\square} \to \mathcal{X}$ is the projection. Let $\mathbf{CI}$ be

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the full subcategory of $\mathbf{MPST}$ consisting of all $\overline{\square}$-invariant objects and $\mathbf{CI}^\tau \subset \underline{\mathbf{M}}\mathbf{PST}$ be the essential image of $\mathbf{CI}$ under $\tau_!$.

(13) Recall from [KSY22, Theorem 2.1.8] that $\mathbf{CI}$ is a Serre subcategory of $\mathbf{MPST}$, and that the inclusion functor $i^{\overline{\square}} : \mathbf{CI} \to \mathbf{MPST}$ has a left adjoint $h_0^{\overline{\square}}$ and a right adjoint $h^0_{\overline{\square}}$ given for $F \in \mathbf{MPST}$ and $\mathcal{X} \in \mathbf{MCor}$ by

$$h_0^{\overline{\square}}(F)(\mathcal{X}) = \mathrm{Coker}(i_0^* - i_1^* : F(\mathcal{X} \otimes \overline{\square}) \to F(\mathcal{X})),$$
$$h^0_{\overline{\square}}(F)(\mathcal{X}) = \mathrm{Hom}(h_0^{\overline{\square}}(\mathcal{X}), F).$$

For $\mathcal{X} \in \mathbf{MCor}$, we write $h_0^{\overline{\square}}(\mathcal{X}) = h_0^{\overline{\square}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X})) \in \mathbf{CI}$, and by abuse of notation we also write $h_0^{\overline{\square}}(\mathcal{X})$ for $\tau_! h_0^{\overline{\square}}(\mathcal{X}) \in \mathbf{CI}^\tau$.

(14) For $F \in \underline{\mathbf{M}}\mathbf{PST}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$, write $F_\mathcal{X}$ for the presheaf on the small étale site $X_{\text{ét}}$ over X given by $U \to F(\mathcal{X}_U)$ for $U \to X$ étale, where $\mathcal{X}_U = (U, D_{|U}) \in \underline{\mathbf{M}}\mathbf{Cor}$. We say that F is a Nisnevich sheaf if $F_\mathcal{X}$ is also one for all $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$ (see [KMSY21a, § 3]). We write $\underline{\mathbf{M}}\mathbf{NST} \subset \underline{\mathbf{M}}\mathbf{PST}$ for the full subcategory of Nisnevich sheaves and put

$$\mathbf{MNST}^\tau = \underline{\mathbf{M}}\mathbf{NST} \cap \mathbf{MPST}^\tau, \quad \mathbf{CI}^\tau_{\mathrm{Nis}} = \mathbf{CI}^\tau \cap \mathbf{MNST}^\tau.$$

By [KMSY21a, Proposition 3.5.3] and [KMSY21b, Theorem 2], the inclusion functor $i_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{PST}$ has an exact left adjoint $\underline{a}_{\mathrm{Nis}}$ such that $\underline{a}_{\mathrm{Nis}}(\mathbf{MPST}^\tau) \subset \mathbf{MNST}^\tau$. The functor $\underline{a}_{\mathrm{Nis}}$ has the following description. For $F \in \underline{\mathbf{M}}\mathbf{PST}$ and $\mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$, let $F_{\mathcal{Y},\mathrm{Nis}}$ be the usual Nisnevich sheafification of $F_\mathcal{Y}$. Then, for $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$, we have

$$\underline{a}_{\mathrm{Nis}} F(X, D) = \varinjlim_{f: Y \to X} F_{(Y, f^*D),\mathrm{Nis}}(Y)$$

where the colimit is taken over all proper maps $f : Y \to X$ that induce isomorphisms $Y - |f^*D| \xrightarrow{\sim} X - |D|$.

(15) By [KMSY21b, Proposition 6.2.1], $\underline{\omega}^*$ and $\underline{\omega}_!$ from (10) respect $\underline{\mathbf{M}}\mathbf{NST}$ and $\mathbf{NST}$ and induce a pair of adjoint functors (which for simplicity we write $\underline{\omega}_!$ and $\underline{\omega}^*$). Moreover, we have

$$\underline{\omega}_! \underline{a}_{\mathrm{Nis}} = a^V_{\mathrm{Nis}} \underline{\omega}_!.$$

By [KSY22, Lemma 2.3.1] and [KMSY21b, Proposition 6.2.1a)], for $F \in \mathbf{PST}$, we have $F \in \mathbf{HI}$ (respectively, $F \in \mathbf{HI}_{\mathrm{Nis}}$) if and only if $\underline{\omega}^* F \in \mathbf{CI}^\tau$ (respectively, $\underline{\omega}^* F \in \mathbf{CI}^\tau_{\mathrm{Nis}}$).

(16) We say that $F \in \underline{\mathbf{M}}\mathbf{PST}$ is *semipure* if the unit map

$$u : F \to \underline{\omega}^* \underline{\omega}_! F$$

is injective. For $F \in \underline{\mathbf{M}}\mathbf{PST}$ (respectively, $F \in \underline{\mathbf{M}}\mathbf{NST}$), let $F^{\mathrm{sp}} \in \underline{\mathbf{M}}\mathbf{PST}$ (respectively, $F^{\mathrm{sp}} \in \underline{\mathbf{M}}\mathbf{NST}$) be the image of $F \to \underline{\omega}^* \underline{\omega}_! F$ (called the semipurification of F. See [Sai20, Lemma 1.30]). For $F \in \underline{\mathbf{M}}\mathbf{PST}$ we have

$$\underline{a}_{\mathrm{Nis}}(F^{\mathrm{sp}}) \simeq (\underline{a}_{\mathrm{Nis}} F)^{\mathrm{sp}}.$$

This follows from the fact that $\underline{a}_{\mathrm{Nis}}$ is exact and commutes with $\underline{\omega}^* \underline{\omega}_!$. For $F \in \mathbf{MPST}^\tau$ we have $F^{\mathrm{sp}} \in \mathbf{MPST}^\tau$ since τ is exact and $\underline{\omega}^* \underline{\omega}_! \tau_! = \tau_! \omega^* \omega_!$.

(17) Let $\mathbf{CI}^{\tau,\mathrm{sp}} \subset \mathbf{CI}^\tau$ be the full subcategory of semipure objects and consider the full subcategory

$$\mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}} = \mathbf{CI}^{\tau,\mathrm{sp}} \cap \mathbf{MNST}^\tau \subset \mathbf{CI}^\tau_{\mathrm{Nis}}.$$

By [Sai20, Theorems 0.1 and 0.4], we have $\underline{a}_{\mathrm{Nis}}(\mathbf{CI}^{\tau,\mathrm{sp}}) \subset \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$.

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(18) We write $\mathbf{RSC} \subseteq \mathbf{PST}$ for the essential image of $\mathbf{CI}$ under $\omega_!$ (which is the same as the essential image of $\mathbf{CI}^{\tau,\mathrm{sp}}$ under $\underline{\omega}_!$ since $\omega_! = \underline{\omega}_! \tau_!$ and $\underline{\omega}_! F = \underline{\omega}_! F^{\mathrm{sp}}$). Put $\mathbf{RSC}_{\mathrm{Nis}} = \mathbf{RSC} \cap \mathbf{NST}$. The objects of $\mathbf{RSC}$ (respectively, $\mathbf{RSC}_{\mathrm{Nis}}$) are called reciprocity presheaves (respectively, sheaves). By [Sai20, Theorem 0.1], we have

$$a^V_{\mathrm{Nis}}(\mathbf{RSC}) \subset \mathbf{RSC}_{\mathrm{Nis}}. \tag{1.0.1}$$

We have $\mathbf{HI} \subseteq \mathbf{RSC}$ which also contains smooth commutative group schemes (which may have non-trivial unipotent part), the sheaf Ω^i of Kähler differentials, and the de Rham–Witt sheaves $W_n\Omega^i$ (see [KSY16, KSY22]).

(19) $\mathbf{NST}$ is a Grothendieck abelian category by [Voe00, Lemma 3.1.6] and we can make $\mathbf{RSC}_{\mathrm{Nis}}$ its full subabelian category as follows. We define the kernel (respectively, cokernel) of a map $\phi : F \to G$ in $\mathbf{RSC}_{\mathrm{Nis}}$ to be that of ϕ as a map in $\mathbf{NST}$. Here we need (1.0.1) to ensure that the cokernel of ϕ in $\mathbf{NST}$ stays in $\mathbf{RSC}_{\mathrm{Nis}}$. By definition, a sequence $0 \to F \to G \to H \to 0$ is exact in $\mathbf{RSC}_{\mathrm{Nis}}$ if and only if it is exact in $\mathbf{NST}$.

(20) By [KSY22, Proposition 2.3.7] we have a pair of adjoint functors

$$\mathbf{CI} \underset{\overset{\omega_!}{\longrightarrow}}{\overset{\omega^{\mathbf{CI}}}{\longleftarrow}} \mathbf{RSC}, \tag{1.0.2}$$

where $\omega^{\mathbf{CI}} = h^0_{\square}\omega^*$ and is fully faithful. It induces a pair of adjoint functors

$$\mathbf{CI}^{\tau} \underset{\overset{\underline{\omega}_!}{\longrightarrow}}{\overset{\underline{\omega}^{\mathbf{CI}}}{\longleftarrow}} \mathbf{RSC}, \tag{1.0.3}$$

where $\underline{\omega}^{\mathbf{CI}} = \tau_! h^0_{\square}\omega^*$ and is fully faithful. Indeed, let $F = \tau_! \hat{F}$ for $\hat{F} \in \mathbf{CI}$ and $G \in \mathbf{RSC}$. In view of (13) and the exactness and full faithfulness of $\tau_!$, we have

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{CI}^{\tau}}(F, \tau_! h^0_{\square}\omega^* G) \simeq \mathrm{Hom}_{\mathbf{CI}}(\hat{F}, h^0_{\square}\omega^* G)\\ &\simeq \mathrm{Hom}_{\mathbf{MPST}}(\hat{F}, \omega^* G) \simeq \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\tau_!\hat{F}, \underline{\omega}^* G) \simeq \mathrm{Hom}_{\mathbf{RSC}}(\underline{\omega}_! F, G).\end{aligned}$$

In view of (15), (1.0.3) induces a pair of adjoint functors

$$\mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}} \underset{\overset{\underline{\omega}_!}{\longrightarrow}}{\overset{\underline{\omega}^{\mathbf{CI}}}{\longleftarrow}} \mathbf{RSC}_{\mathrm{Nis}}. \tag{1.0.4}$$

2. Purity with reduced modulus

For $F \in \underline{\mathbf{M}}\mathbf{PST}$, we put

$$F_{-1} = \mathrm{Ker}\left(\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}((\mathbf{P}^1 - 0, \infty), F) \xrightarrow{i_1^*} F\right),$$

$$F^{(1)}_{-1} \quad = \mathrm{Ker}\left(\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}((\mathbf{P}^1, 0+\infty), F) \xrightarrow{i_1^*} F\right)$$

Note that if $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$, then $F_{-1}, F^{(1)}_{-1} \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ and

$$\begin{aligned}F^{(1)}_{-1}(\mathcal{X}) &= \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\mathbf{P}^1, 0+\infty)^0, \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F)),\\ F_{-1}(\mathcal{X}) &= \varinjlim_n \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\mathbf{P}^1, n\cdot 0+\infty)^0, \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F))\end{aligned} \tag{2.0.1}$$

for $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$, where

$$h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\mathbf{P}^1, n\cdot 0+\infty)^0 = \mathrm{Coker}\left(\mathbb{Z} = \mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}\, k, \emptyset) \xrightarrow{i_1} h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\mathbf{P}^1, n\cdot 0+\infty)\right).$$

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Definition 2.1. For $e_1, \dots, e_r \in \{0,1\}$, put

$$\tau^{(e_1,\dots,e_r)}F = \tau^{(e_r)} \cdots \tau^{(e_1)}F,$$

where

$$\tau^{(0)}F = F_{-1} \quad \text{and} \quad \tau^{(1)}F = F_{-1}/F_{-1}^{(1)},$$

where the quotient is taken in $\underline{\mathbf{M}}\mathbf{PST}$.

The existence of retractions in the following lemma was suggested by A. Merici. It implies $\tau^{(e_1,\dots,e_r)}F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ if $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$.

Lemma 2.2. *For $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$, the inclusion $F_{-1}^{(1)} \to F_{-1}$ admits a retraction $s_F : F_{-1} \to F_{-1}^{(1)}$ such that for any map $\phi : F \to G$ in $\mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$, the following diagram is commutative:*

$$\begin{array}{ccc} F_{-1} & \xrightarrow{s_F} & F_{-1}^{(1)} \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ G_{-1} & \xrightarrow{s_F} & G_{-1}^{(1)} \end{array}$$

In particular, $\tau^{(1)}F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ if $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$.

Proof. In view of (2.0.1), this follows from [BRS22, Lemma 2.4]. □

Theorem 2.3. *Let $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$. Let $K\{t_1,\dots,t_n\}$ be the henselization of $K[t_1,\dots,t_n]$ at $(t_1,\dots,t_n)$ and $\mathcal{X} = \operatorname{Spec} K\{t_1,\dots,t_n\}$ and $D = \{t_1^{e_1}\cdots t_n^{e_n} = 0\} \subset \mathcal{X}$ with $e_1,\dots,e_n \in \{0,1\}$. For a subset $I \subset [1,n]$ let $i_{\mathcal{H}} : \mathcal{H} \hookrightarrow \mathcal{X}$ be the closed immersion defined by $\{t_i = 0\}_{i\in I}$ and $D_{\mathcal{H}} = \{\prod_{j\in[1,n]-I} t_j^{e_j} = 0\} \subset \mathcal{H}$. Then*

$$R^\nu i_{\mathcal{H}}^! F_{(\mathcal{X},D)} = 0 \quad \text{for } \nu \neq q := |I|, \tag{2.3.1}$$

and there is an isomorphism

$$(\tau^{(e_I)}F)_{(\mathcal{H},D_{\mathcal{H}})} \simeq R^q i_{\mathcal{H}}^! F_{(\mathcal{X},D)} \quad \text{with } e_I = (e_i)_{i\in I} \in \mathbb{Z}^q_{\geq 0}. \tag{2.3.2}$$

Proof. The proof is divided into two steps.

Step 1: we prove (2.3.1) and (2.3.2) for $q = |I| = 1$. For $\nu = 0$, (2.3.1) follows from the semipurity of F and [Sai20, Theorem 3.1]. Thus, it suffices to show (2.3.1) only for $\nu > 1$. Let $J = \{j \in [1,n] \mid e_j \neq 0\}$ and $r = |J|$. If $\dim(\mathcal{X}) = 0$, the assertion is trivial. If $r = 0$, the assertion follows from [Sai20, Corollary 8.6(3)]. Assume $r > 0$ and $\dim(\mathcal{X}) \geq 1$, and proceed by the double induction on r and $\dim(\mathcal{X})$. Without loss of generality, we may assume

(♠) $e_1 \neq 0$, and $\mathcal{H} = \{t_1 = 0\}$ if $\mathcal{H} \subset |D|$.

Let $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}$ be the closed immersion defined by $\{t_1 = 0\}$ and $D_{\mathcal{Z}} = \{t_2^{e_2}\cdots t_r^{e_r} = 0\} \subset \mathcal{Z}$ and $D' = \{t_2^{e_2}\cdots t_r^{e_r} = 0\} \subset \mathcal{X}$. By [Sai20, Lemma 7.1], we have an exact sequence sheaves on $\mathcal{X}_{\mathrm{Nis}}$,

$$0 \to F_{(\mathcal{X},D')} \to F_{(\mathcal{X},D)} \to \iota_*(F_{-1}^{(e_1)})_{(\mathcal{Z},D_{\mathcal{Z}})} \to 0,$$

which gives rise to a long exact sequence of sheaves on $\mathcal{H}_{\mathrm{Nis}}$,

$$\cdots \to R^\nu i_{\mathcal{H}}^! F_{(\mathcal{X},D')} \to R^\nu i_{\mathcal{H}}^! F_{(\mathcal{X},D)} \to R^\nu i_{\mathcal{H}}^! \iota_*(F_{-1}^{(e_1)})_{(\mathcal{Z},D_{\mathcal{Z}})} \to \cdots. \tag{2.3.3}$$

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By the induction hypothesis, $R^\nu i_{\mathcal{H}}^! F_{(\mathcal{X},D')} = 0$ for $\nu > 1$. If $\mathcal{H} \neq \mathcal{Z}$, we have a Cartesian diagram of closed immersions

$$\begin{array}{ccc} \mathcal{H}\cap\mathcal{Z} & \xrightarrow{\iota'} & \mathcal{H} \\ {\scriptstyle i_{\mathcal{H}\cap\mathcal{Z}}}\downarrow & & \downarrow{\scriptstyle i_{\mathcal{H}}} \\ \mathcal{Z} & \xrightarrow{\iota} & \mathcal{X} \end{array}$$

and we have an isomorphism

$$R^\nu i_{\mathcal{H}}^! \iota_*(F_{-1}^{(e_1)})_{(\mathcal{Z},D_{\mathcal{Z}})} \simeq \iota'_* R^\nu i_{\mathcal{H}\cap\mathcal{Z}}^!(F_{-1}^{(e_1)})_{(\mathcal{Z},D_{\mathcal{Z}})}.$$

By the induction hypothesis, $R^\nu i_{\mathcal{H}\cap\mathcal{Z}}^!(F_{-1}^{(e_1)})_{(\mathcal{Z},D_{\mathcal{Z}})} = 0$ for $\nu > 1$, noting that $F_{-1}^{(e_1)} \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,\mathrm{sp}}$ by Lemma 2.2. So the desired vanishing follows from (2.3.3). Moreover, the assumptions (♠) and $\mathcal{H} \neq \mathcal{Z}$ imply that $\mathcal{H} \not\subset |D|$. Then (2.3.2) (with $q = 1$) follows from [Sai20, Lemma 7.1(2)].

If $\mathcal{Z} = \mathcal{H}$, we have

$$R^\nu i_{\mathcal{H}}^! \iota_*(F_{-1}^{(e_1)})_{(\mathcal{Z},D_{\mathcal{Z}})} = R^\nu \iota^! \iota_*(F_{-1}^{(e_1)})_{(\mathcal{Z},D_{\mathcal{Z}})},$$

which vanishes for $\nu > 0$. Hence, (2.3.3) gives the desired vanishing together with an exact sequence

$$0 \to (F_{-1}^{(e_1)})_{(\mathcal{H},D_{\mathcal{H}})} \xrightarrow{\delta} R^1 i_{\mathcal{H}}^! F_{(\mathcal{X},D')} \to R^1 i_{\mathcal{H}}^! F_{(\mathcal{X},D)} \to 0.$$

By [Sai20, Lemma 7.1(2)] we have an isomorphism

$$(F_{-1})_{(\mathcal{H},D_{\mathcal{H}})} \simeq R^1 i_{\mathcal{H}}^! F_{(\mathcal{X},D')}$$

through which δ is identified with the map induced by the canonical map $F_{-1}^{(e_1)} \to F_{-1}$. This proves the desired isomorphism (2.3.2) for $\mathcal{Z} = \mathcal{H}$ and completes step 1.

Step 2: we prove the theorem by induction on q assuming $q > 0$. Let $I = \{i_1, \ldots, i_q\} \subset [1, n]$ and $\mathcal{Y} \subset \mathcal{X}$ be the closed subscheme defined by $\{t_{i_1} = 0\}$. Let $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{X}$ and $i_{\mathcal{H},\mathcal{Y}} : \mathcal{H} \to \mathcal{Y}$ be the induced closed immersions. By step 1 we have $R^\nu i_{\mathcal{Y}}^! F_{(\mathcal{X},D)} = 0$ for $\nu \neq 1$ and we have an isomorphism

$$(\tau^{(e_{i_1})} F)_{(\mathcal{Y},D_{\mathcal{Y}})} \simeq R^1 i_{\mathcal{Y}}^! F_{(\mathcal{X},D)} \quad \text{with } D_{\mathcal{Y}} = \{t_1^{e_1} \cdots \overset{\vee}{t_{i_1}^{e_{i_1}}} \cdots t_n^{e_n} = 0\} \subset \mathcal{Y}.$$

Note $\tau^{(e_{i_1})} F \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,\mathrm{sp}}$ by Lemma 2.2. Thus, by the induction hypothesis, we have $R^\nu i_{\mathcal{H},\mathcal{Y}}^! \tau^{(e_{i_1})} F_{(\mathcal{Y},D_{\mathcal{Y}})} = 0$ for $\nu \neq q-1$. By the spectral sequence

$$E_2^{a,b} = R^b i_{\mathcal{H},\mathcal{Y}}^! R^a i_{\mathcal{Y}}^! F_{(\mathcal{X},D)} \Rightarrow R^{a+b} i_{\mathcal{H}}^! F_{(\mathcal{X},D)},$$

we get the desired vanishing (2.3.1) and an isomorphism

$$\begin{aligned} R^q i_{\mathcal{H}}^! F_{(\mathcal{X},D)} &\simeq R^{q-1} i_{\mathcal{H},\mathcal{Y}}^! R^1 i_{\mathcal{Y}}^! F_{(\mathcal{X},D)} \simeq R^{q-1} i_{\mathcal{H},\mathcal{Y}}^! (\tau^{(e_{i_1})} F)_{(\mathcal{Y},D_{\mathcal{Y}})} \\ &\simeq (\tau^{(e_{i_2},\ldots,e_{i_q})}(\tau^{(e_{i_1})} F))_{(\mathcal{H},D_{\mathcal{H}})} \simeq (\tau^{(e_{i_1},e_{i_2},\ldots,e_{i_q})} F)_{(\mathcal{H},D_{\mathcal{H}})}, \end{aligned}$$

where the third isomorphism holds by the induction hypothesis. This completes the proof of the theorem. □

We say that $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ is reduced if so is D. The following Corollaries 2.4 and 2.5 are immediate consequences of Theorem 2.3.

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Corollary 2.4. *Take* $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ *and* $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ *reduced. Let* $x \in X^{(n)}$ *with* $K = k(x)$ *and let* $\mathcal{X} = X^h_{|x}$ *be the henselization of* X *at* x*. Then*

$$H^i_x(X_{\mathrm{Nis}}, F_{(X,D)}) = 0 \quad \text{for } i \neq n.$$

Choosing an isomorphism

$$\varepsilon : \mathcal{X} \simeq \operatorname{Spec} K\{t_1, \ldots, t_n\}$$

such that $D_{|\mathcal{X}} = \{t_1^{e_1} \cdots t_n^{e_n} = 0\} \subset \mathcal{X}$ *with* $e_1, \ldots, e_n \in \{0, 1\}$*, there exists an isomorphism depending on* ε*:*

$$\theta_\varepsilon : \tau^{(e_1,e_2,\ldots,e_n)} F(x) \simeq H^n_x(X_{\mathrm{Nis}}, F_{(X,D)}).$$

Corollary 2.5. *For* $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ *and* $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ *reduced, the following sequence is exact:*

$$0 \to F(X, D) \to F(X - D, \emptyset) \to \bigoplus_{\xi \in D^{(0)}} \frac{F(X^h_{|\xi} - \xi, \emptyset)}{F(X^h_{|\xi}, \xi)}.$$

The idea of deducing the following corollary from the above is due to A. Merici.

Corollary 2.6. *Let* $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ *be reduced.*

(1) *Assume given an exact sequence in* $\underline{\mathbf{M}}\mathbf{NST}$*,*

$$0 \to H \xrightarrow{\phi} G \xrightarrow{\psi} F, \tag{2.6.1}$$

such that $F, G, H \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ *and that* $\underline{\omega}_!\psi$ *is surjective in* $\mathbf{NST}$*. If* X *is henselian local, then*

$$0 \to H(\mathcal{X}) \to G(\mathcal{X}) \to F(\mathcal{X}) \to 0$$

is exact.

(2) *Let* $\gamma : F \to G$ *be a map in* $\mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ *such that* $\underline{\omega}_!\gamma$ *is an isomorphism. Then* $F(\mathcal{X}) \to G(\mathcal{X})$ *is an isomorphism.*

(3) *For* $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$*, the unit map* $u : F \to \underline{\omega}^{\mathbf{CI}}\underline{\omega}_! F$ *induces an isomorphism* $F(\mathcal{X}) \cong \underline{\omega}^{\mathbf{CI}}\underline{\omega}_! F(\mathcal{X})$*.*

Proof. To show (1), it suffices to show the surjectivity of $G(\mathcal{X}) \to F(\mathcal{X})$. Let $\eta \in X$ be the generic point and consider the following commutative diagram of the Cousin complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H(\mathcal{X}) & \longrightarrow & H(\eta) & \longrightarrow & \bigoplus_{x \in X^{(1)}} H^1_x(X, H_{\mathcal{X}}) & \longrightarrow & \bigoplus_{y \in X^{(2)}} H^2_y(X, H_{\mathcal{X}}) \\
 & & \downarrow & & \downarrow{\scriptstyle \phi(\eta)} & & \downarrow{\scriptstyle H^1_x(\phi)} & & \downarrow{\scriptstyle H^2_y(\phi)} \\
0 & \longrightarrow & G(\mathcal{X}) & \longrightarrow & G(\eta) & \longrightarrow & \bigoplus_{x \in X^{(1)}} H^1_x(X, G_{\mathcal{X}}) & \longrightarrow & \bigoplus_{y \in X^{(2)}} H^2_y(X, G_{\mathcal{X}}) \\
 & & \downarrow & & \downarrow{\scriptstyle \psi(\eta)} & & \downarrow{\scriptstyle H^1_x(\psi)} & & \downarrow{\scriptstyle H^2_y(\psi)} \\
0 & \longrightarrow & F(\mathcal{X}) & \longrightarrow & F(\eta) & \longrightarrow & \bigoplus_{x \in X^{(1)}} H^1_x(X, F_{\mathcal{X}}) & \longrightarrow & \bigoplus_{y \in X^{(2)}} H^2_y(X, F_{\mathcal{X}})
\end{array}$$

By Corollary 2.4, the horizontal sequences are exact. By the assumption, $\psi(\eta)$ is surjective. By a diagram chase we are reduced to showing the following claim.

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CLAIM 2.6.1.

(i) For $x \in X^{(1)}$, the sequence

$$H^1_x(X, H_{\mathcal{X}}) \to H^1_x(X, G_{\mathcal{X}}) \to H^1_x(X, F_{\mathcal{X}})$$

is exact.

(ii) For $y \in X^{(2)}$, $H^2_y(\phi)$ is injective.

To show (i), by Corollary 2.4, it suffices to show the exactness of $\tau^{(e)}H \to \tau^{(e)}G \to \tau^{(e)}F$ for $e \in \{0, 1\}$. The case $e = 0$ follows from the left exactness of the endofunctor $\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\mathcal{X}, -)$ on $\underline{\mathbf{M}}\mathbf{NST}$ for any $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$. We have a commutative diagram

$$\begin{array}{ccccc}
\tau^{(1)}H & \xrightarrow{\phi} & \tau^{(1)}G & \xrightarrow{\psi} & \tau^{(1)}F \\
{\scriptstyle p_H}\big\uparrow\big\downarrow{\scriptstyle s_H} & & {\scriptstyle p_G}\big\uparrow\big\downarrow{\scriptstyle s_G} & & {\scriptstyle p_F}\big\uparrow\big\downarrow{\scriptstyle s_F} \\
\tau^{(0)}H & \xrightarrow{\phi} & \tau^{(0)}G & \xrightarrow{\psi} & \tau^{(0)}F
\end{array}$$

where p_* are the projections and s_* is a right inverse of p_* coming from the retractions from Lemma 2.2. We have

$$\phi \circ p_H = p_G \circ \phi, \quad \psi \circ p_G = p_F \circ \psi, \quad \phi \circ s_H = s_G \circ \phi, \quad \psi \circ s_G = s_F \circ \psi.$$

By a diagram chase, the case $e = 1$ follows from the case $e = 0$.

To show (ii), by Corollary 2.4, it suffices to show the injectivity of $\tau^{(\underline{e})}H \to \tau^{(\underline{e})}G$ for $\underline{e} \in \{(0, 0), (0, 1), (1, 0), (1.1)\}$. The case $\underline{e} = (0, 0)$ follows from the same left exactness as above, and the other cases from this case thanks to Lemma 2.2.

To show (2), we may assume $\mathcal{X}$ is henselian local. Then it follows from (1).

Finally, (3) follows from (2) since $\underline{\omega}_! u$ is an isomorphism. This completes the proof of the corollary. □

3. Review of higher local symbols

In this section we recall from [RS21c] the higher local symbols for reciprocity sheaves, which are a fundamental tool to prove Theorem 4.2, one of the main theorems of this paper. First we introduce some basic notation. In this section X is a reduced noetherian separated scheme of dimension $d < \infty$ such that $X_{(0)} = X^{(d)}$.

Let K be a field. For an integer $r \geq 0$, let $K^{\mathrm{M}}_r(K)$ be the Milnor K-group of K. Let A be a local domain with the function field K. For an ideal $I \subset A$, let $\overline{K}^{\mathrm{M}}_r(A, I) \subset K^{\mathrm{M}}_r(K)$ denote the subgroup generated by symbols

$$\{1 + a, b_1, \ldots, b_{r-1}\} \quad \text{with } a \in I,\ b_i \in A^{\times}.$$

Let A be a noetherian excellent one-dimensional local domain with function field K and residue field F. Let $\tilde{A}$ be the normalization of A and S be the set of the maximal ideals of $\tilde{A}$. For $\mathfrak{m} \in S$, denote $\kappa(\mathfrak{m}) = \tilde{A}/\mathfrak{m}$. Then we define

$$\partial_A := \sum_{\mathfrak{m} \in S} \mathrm{Nm}_{\kappa(\mathfrak{m})/F} \circ \partial_{\mathfrak{m}} : K^{\mathrm{M}}_r(K) \to K^{\mathrm{M}}_{r-1}(F), \tag{3.0.1}$$

where $\partial_{\mathfrak{m}} : K^{\mathrm{M}}_r(K) \to K^{\mathrm{M}}_{r-1}(\kappa(\mathfrak{m}))$ denotes the tame symbol for the discrete valuation ring $\tilde{A}_{\mathfrak{m}}$, the localization of $\tilde{A}$ at $\mathfrak{m}$, and $\mathrm{Nm}_{\kappa(\mathfrak{m})/F}$ is the norm map.

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For $x, y \in X$, we write

$$y < x :\Longleftrightarrow \overline{\{y\}} \subsetneq \overline{\{x\}}, \quad \text{that is, } y \in \overline{\{x\}} \text{ and } y \neq x.$$

A *chain* on X is a sequence

$$\underline{x} = (x_0, \dots, x_n) \quad \text{with } x_0 < x_1 < \cdots < x_n. \tag{3.0.2}$$

The chain $\underline{x}$ is a *maximal Paršin chain* (or *maximal chain*) if $n = d$ and $x_i \in X_{(i)}$. Note that the assumptions on X imply $x_i \in \overline{\{x_{i+1}\}}^{(1)}$. We denote

$$\mathrm{mc}(X) = \{\text{maximal chains on } X\}.$$

A *maximal chain with break at* $r \in \{0, \dots, d\}$ is a chain (3.0.2) with $n = d-1$ and $x_i \in X_{(i)}$, for $i < r$, and $x_i \in X_{(i+1)}$, for $i \geq r$. We denote

$$\mathrm{mc}_r(X) = \{\text{maximal chain with break at } r \text{ on } X\}.$$

For $\underline{x} = (x_0, \dots, x_{d-1}) \in \mathrm{mc}_r(X)$, we denote by $b(\underline{x})$ the set of $y \in X_{(r)}$ such that

$$\underline{x}(y) := (x_0, \dots, x_{r-1}, y, x_r, \dots, x_{d-1}) \in \mathrm{mc}(X). \tag{3.0.3}$$

In the rest of this section we fix $F = \underline{\omega}^{\mathbf{CI}} G \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ with $G \in \mathbf{RSC}_{\mathrm{Nis}}$ (cf. (1.0.4)). We also fix a function field K over the base field k. Let X be an integral scheme of finite type over K and assume $d = \dim(X) \geq 1$. Recall from [RS21c, § 5] that we have a collection of bilinear pairings (cf. the convention from § 1(9))

$$\left\{(-,-)_{X/K,\underline{x}} : F(K(X)) \otimes K^{\mathrm{M}}_d(K(X)) \to F(K)\right\}_{\underline{x} \in \mathrm{mc}(X)}. \tag{3.0.4}$$

The following properties hold for all $a \in F(K(X))$ (see Remark 3.1 below).

(HS1) Let $X \hookrightarrow X'$ be an open immersion where X' is an integral K-scheme of dimension d. Then, for all $\beta \in K^{\mathrm{M}}_d(K(X))$,

$$(a, \beta)_{X/K,\underline{x}} = (a, \beta)_{X'/K,\underline{x}}.$$

(HS2) Let $\underline{x} = (x_0, \dots, x_{d-1}, x_d) \in \mathrm{mc}(X)$ and $Z \subset X$ be the closure of $z = x_{d-1}$, and set $\underline{x}' = (x_0, \dots, x_{d-1}) \in \mathrm{mc}(Z)$. Assume $a \in F(\mathcal{O}_{X,z})$ and let $a(z) \in F(K(Z))$ be the restriction of a. Then

$$(a, \beta)_{X/K,\underline{x}} = (a(z), \partial_z \beta)_{Z/K,\underline{x}'} \quad \text{for } \beta \in K^{\mathrm{M}}_d(K(X)),$$

where $\partial_z : K^{\mathrm{M}}_d(K(X)) \to K^{\mathrm{M}}_{d-1}(K(Z))$ is the map (3.0.1) for $A = \mathcal{O}_{X,z}$.

(HS3) Let $D \subset X$ be an effective Cartier divisor with $I_D \subset \mathcal{O}_X$ its ideal sheaf. Assume that $X \backslash D$ is regular so that $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$ and that $a \in F(X, D)$. For $\underline{x} = (x_0, \dots, x_{d-1}, x_d) \in \mathrm{mc}(X)$, we have

$$(a, \beta)_{X/K,\underline{x}} = 0 \quad \text{for } \beta \in \overline{K}^{\mathrm{M}}_d(\mathcal{O}_{X,x_{d-1}}, I_D \mathcal{O}_{X,x_{d-1}}).$$

(HS4) Let $\underline{x}' \in \mathrm{mc}_r(X)$ with $0 \leq r \leq d-1$. For $\beta \in K^{\mathrm{M}}_d(K(X))$,

$$(a, \beta)_{X/K,\underline{x}'(y)} = 0 \quad \text{for almost all } y \in (\underline{x}').$$

Assume that either $r \geq 1$ or $r = 0$, X is quasi-projective, and the closure of x_1 in X is projective over K, where $\underline{x}' = (x_1, \dots, x_d)$. Then

$$\sum_{y \in b(\underline{x}')} (a, \beta)_{X/K,\underline{x}'(y)} = 0.$$

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Remark 3.1. Properties (HS1)–(HS4) are slight variants of the (stronger) properties (HS1)–(HS4) in [RS21c, Proposition 5.3], where the Milnor K-group $K_d^{\mathrm{M}}(K^h_{X,\underline{x}})$ of the iterated henselization $K^h_{X,\underline{x}}$ of $K(X)$ along the chain $\underline{x}$ is used instead of $K_d^{\mathrm{M}}(K(X))$. The version stated here follows easily using the natural maps $\iota_{\underline{x}} : K(X) \to K^h_{X,\underline{x}}$ and the commutative diagram in the situation of (HS2),

$$\begin{array}{ccc} K_d^{\mathrm{M}}(K^h_{X,\underline{x}}) & \xrightarrow{\partial_{\underline{x}}} & K_{d-1}^{\mathrm{M}}(K^h_{Z,\underline{x}'}) \\ \uparrow{\scriptstyle \iota_{\underline{x}}} & & \uparrow{\scriptstyle \iota_{\underline{x}'}} \\ K_d^{\mathrm{M}}(K(X)) & \xrightarrow{\partial_z} & K_{d-1}^{\mathrm{M}}(K(Z)) \end{array}$$

and the commutative diagram in the situation of (HS4),

$$\begin{array}{ccc} & & K_{d-1}^{\mathrm{M}}(K^h_{X,\underline{x}'}) \\ & \nearrow^{\iota_{\underline{x}'}} & \downarrow{\scriptstyle \iota_y} \\ K_d^{\mathrm{M}}(K(X)) & \xrightarrow[\iota_{\underline{x}'(y)}]{} & K_{d-1}^{\mathrm{M}}(K^h_{X,\underline{x}'(y)}) \end{array}$$

where $\partial_{\underline{x}}$ (respectively, ι_y) is defined in [RS21c, (4.1.1)] (respectively, [RS21c, (3.2.3)]). We also note that $\overline{K}_d^{\mathrm{M}}(\mathcal{O}_{X,x_{d-1}}, I_D\mathcal{O}_{X,x_{d-1}})$ in (HS2) coincides with the Zariski stalk at x_{d-1} of the sheaf $\overline{V}_{d,X|D}$ defined in [RS21c, 4.4].

For a scheme Z over k, write $Z_K = Z \otimes_k K$. If Z_K is integral, we denote by $K(Z)$ the function field of Z_K. We quote the following result from [RS21c, Proposition 7.3]. It is a key tool in the proof of Theorem 4.2.

Proposition 3.2. *Let $X \in$ **Sm** and assume D is a* reduced *simple normal crossing divisor on X with $I_D \subset \mathcal{O}_X$ its ideal sheaf. Let $U \subset X$ be an open subset containing all the generic points of D. Let $a \in F(X \backslash D)$. Assume that, for all function fields K/k and for all $\underline{x} = (x_0, \dots, x_{d-1}, x_d) \in \mathrm{mc}(U_K)$ with $x_{d-1} \in D_K^{(0)}$, we have*

$$(a, \beta)_{X_K/K,\underline{x}} = 0 \quad \text{for all } \beta \in \overline{K}^{\mathrm{M}}(\mathcal{O}_{X,x_{d-1}}, I_D\mathcal{O}_{X,x_{d-1}}).$$

Then $a \in F(X, D)$.

4. Logarithmic cohomology of reciprocity sheaves

For $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$, we write $\mathcal{X}_{\mathrm{red}} = (X, D_{\mathrm{red}}) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$. We say that $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ is reduced if $\mathcal{X} = \mathcal{X}_{\mathrm{red}}$.

Definition 4.1. Let $F \in \underline{\mathbf{M}}\mathbf{PST}$.

(1) We say that F is *log-semipure* if for any $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$, the map $F(\mathcal{X}_{\mathrm{red}}) \to F(\mathcal{X})$ is injective. Note that if F is semipure, F is log-semipure (cf. § 1(16)).
(2) We say that F is *logarithmic* if it is log-semipure and satisfies the condition that for $\mathcal{X}, \mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ with $\mathcal{X}$ reduced and $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(\mathcal{Y}, \mathcal{X})$, the image of $\alpha^* : F(\mathcal{X}) \to F(\mathcal{Y})$ is contained in $F(\mathcal{Y}_{\mathrm{red}}) \subset F(\mathcal{Y})$.

Let $\underline{\mathbf{M}}\mathbf{PST}_{\log}$ be the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}$ consisting of logarithmic objects and put $\underline{\mathbf{M}}\mathbf{NST}_{\log} = \underline{\mathbf{M}}\mathbf{NST} \cap \underline{\mathbf{M}}\mathbf{PST}_{\log}$.

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Theorem 4.2. *Any $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ is logarithmic, that is, $\mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}} \subset \underline{\mathbf{M}}\mathbf{NST}_{\log}$.*

We need a preliminary lemma for the proof of the theorem.

Lemma 4.3. *Let $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$. Let $\mathbf{A}^n_K = \operatorname{Spec} K[x_1,\dots,x_n]$ be the affine space over a function field K over k and $V = \operatorname{Spec} K\{x_1,\dots,x_n\}$ be the henselization of $\mathbf{A}^n_K$ at the origin and $\mathcal{L}_i = \{x_i = 0\} \subset V$ for $i \in [1,n]$. For an integer $0 < r \le n$, the natural map*

$$K\{x_{r+1},\dots,x_n\}[x_1,\dots,x_r] \to K\{x_1,\dots,x_n\}$$

induces a map in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$ (cf. § 1(9)):

$$\rho_r : (V, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \to (\mathbf{A}^r_S, \{x_1 \cdots x_r = 0\}) \simeq (\mathbf{A}^1, 0)^{\otimes r} \otimes (S, \emptyset),$$

where $S = \operatorname{Spec} K\{x_{r+1},\dots,x_n\}$. It induces

$$\rho_r^* : F(\mathbf{A}^r_S, \{x_1 \cdots x_r = 0\}) \to F(V, \mathcal{L}_1 + \cdots + \mathcal{L}_r). \tag{4.3.1}$$

Then $F(V, \mathcal{L}_1 + \cdots + \mathcal{L}_r)$ is generated by the image of ρ_r^* and

$$F(V, \mathcal{L}_1 + \cdots + \overset{\vee}{\mathcal{L}_i} + \cdots \mathcal{L}_r) \quad \text{for } i = 1,\dots,r.$$

Proof. For $\mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$, let $F^{\mathcal{Y}} \in \underline{\mathbf{M}}\mathbf{PST}$ be defined by $F^{\mathcal{Y}}(\mathcal{Z}) = F(\mathcal{Y} \otimes \mathcal{Z})$. Clearly, we have $F^{\mathcal{Y}} \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ for $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$. We prove the lemma by induction on r. The case $r = 1$ holds since by [Sai20, Lemmas 7.1 and 5.9], ρ_1 induces an isomorphism

$$F^{(\mathbf{A}^1,0)}(S)/F^{(\mathbf{A}^1,\emptyset)}(S) \xrightarrow{\simeq} F(V, \mathcal{L}_1)/F(V).$$

By definition $\mathcal{L}_1 = \operatorname{Spec} K\{x_2,\dots,x_n\}$ and we have a map in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$,

$$(V, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \to (\mathbf{A}^1, 0) \otimes (\mathcal{L}_1, \mathcal{L}_1 \cap (\mathcal{L}_2 + \cdots + \mathcal{L}_r)),$$

induced by the natural map $K\{x_2,\dots,x_n\}[x_1] \to K\{x_1,\dots,x_n\}$. By [Sai20, Lemmas 7.1 and 5.9], it induces an isomorphism

$$F^{(\mathbf{A}^1,0)}(\mathcal{L}_1, E)/F^{(\mathbf{A}^1,\emptyset)}(\mathcal{L}_1, E) \xrightarrow{\simeq} F(V, \mathcal{L}_1 + \cdots + \mathcal{L}_r)/F(V, \mathcal{L}_2 + \cdots + \mathcal{L}_r)$$

with $E = \mathcal{L}_1 \cap (\mathcal{L}_2 + \cdots + \mathcal{L}_r)$. By the induction hypothesis, $F^{(\mathbf{A}^1,0)}(\mathcal{L}_1, E)$ is generated by $F^{(\mathbf{A}^1,0)}(\mathcal{L}_1, E_j)$ with $E_j = \mathcal{L}_1 \cap (\mathcal{L}_2 \cdots + \overset{\vee}{\mathcal{L}_j} + \cdots \mathcal{L}_r)$ for $j = 2,\dots,r$ together with the image of the map

$$(F^{(\mathbf{A}^1,0)})^{(\mathbf{A}^1,0)^{\otimes r-1}}(S) = F^{(\mathbf{A}^1,0)^{\otimes r}}(S) \to F^{(\mathbf{A}^1,0)}(\mathcal{L}_1, E)$$

induced by

$$(\mathcal{L}_1, E) \to (\mathbf{A}^{r-1}_S, \{x_2 \cdots x_r = 0\}) \simeq (\mathbf{A}^1, 0)^{\otimes r-1} \otimes (S, \emptyset)$$

coming from the map $K\{x_{r+1},\dots,x_n\}[x_2,\dots,x_r] \to K\{x_2,\dots,x_d\}$. This proves the lemma. □

Proof of Theorem 4.2. By Corollary 2.6(3), we may assume $F = \underline{\omega}^{\mathbf{CI}} G$ for $G \in \mathbf{RSC}_{\mathrm{Nis}}$. Take $\mathcal{X} = (X, D), \mathcal{Y} = (Y, E) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ with $\mathcal{X}$ reduced, and let $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(\mathcal{Y}, \mathcal{X})$ be an elementary correspondence. We need to show that $\alpha^*(F(\mathcal{X})) \subset F(\mathcal{Y}_{\mathrm{red}})$. The question is Nisnevich local over X and Y. Hence, we may assume $(X, D) = (V, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$ in the notation of Lemma 4.3. If $r = 0$, we have $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}((Y, \emptyset), (X, \emptyset))$ by the assumption $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(\mathcal{Y}, \mathcal{X})$ so that

$$\alpha^*(F(\mathcal{X})) = \alpha^*(F(X, \emptyset)) \subset F(Y, \emptyset) \subset F(\mathcal{Y}_{\mathrm{red}}).$$

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Assume $r > 0$ and proceed by induction on r. By Lemma 4.3, we may then assume

$$(X, D) = \mathcal{M} := (\mathbf{A}^1, 0)^{\otimes r} \otimes (S, \emptyset) \quad \text{for } S \in \mathbf{Sm}^{\mathrm{pro}}.$$

On the other hand, by Corollary 2.5, we have an exact sequence

$$0 \to F(Y, E_{\mathrm{red}}) \to F(Y - E_{\mathrm{red}}, \emptyset) \to \bigoplus_{\xi \in E^{(0)}} \frac{F(Y^h_{|\xi} - \xi, \emptyset)}{F(Y^h_{|\xi}, \xi)}.$$

Hence, we may replace Y with its Nisnevich neighborhood of a generic point ξ of E. Using the assumption that k is perfect, we may then assume the following condition (♠). Recall that α is by definition an integral closed subscheme of $(Y - E) \times (X - D)$ finite surjective over $Y - E$, and its closure $\overline{\alpha}$ in $Y \times X$ is finite surjective over Y.

(♠) Let Y' be the normalization of $\overline{\alpha}$ and $E' := E \times_Y Y'$. Then X, Y, E and E' are irreducible, and α, Y', E_{red} and E'_{red} are essentially smooth over k.

Let $g : Y' \to Y$ and $f : Y' \to X$ be the induced maps. We have $E' = g^*E \geq f^*D$ as Cartier divisors on Y' by the modulus condition for α. Hence, these maps induce

$$F(X, D) \xrightarrow{f^*} F(Y', E') \xrightarrow{g_*} F(Y, E).$$

We claim that $\alpha^* : F(X, D) \to F(Y, E)$ agrees with this map. Indeed, this follows from the equality

$$\Gamma_f \circ {}^t\Gamma_g = \alpha \in \mathbf{Cor}(Y - E, X - D),$$

where ${}^t\Gamma_g \in \mathbf{Cor}(Y - E, Y' - E')$ is the transpose of the graph of g and $\Gamma_f \in \mathbf{Cor}(Y' - E', X - D)$ is the graph of f. By definition this follows from the equality

$${}^t\Gamma_g \times_{Y' - E'} \Gamma_f = \alpha \subset (Y - E) \times (X - D)$$

which one can check easily, noting that $Y' \to \overline{\alpha}$ is an isomorphism over α since α is regular by (♠). Then we get a commutative diagram

$$\begin{array}{ccccc} & & F(Y', E'_{\mathrm{red}}) & & \\ & & \downarrow \hookrightarrow & & \\ & & F(Y', E_{\mathrm{red}} \times_Y Y') & \xrightarrow{g_*} & F(Y, E_{\mathrm{red}}) \\ & & \downarrow \hookrightarrow & & \downarrow \hookrightarrow \\ F(X, D) & \xrightarrow{f^*} & F(Y', E') & \xrightarrow{g_*} & F(Y, E) \end{array}$$

where the top inclusion comes from the inequality $E_{\mathrm{red}} \times_Y Y' \geq E'_{\mathrm{red}}$ as Cartier divisors on Y' thanks to the semipurity of F (cf. § 1(16)). Hence, it suffices to show $f^*(F(X, D)) \subset F(Y', E'_{\mathrm{red}})$. By replacing (Y, E) with (Y', E'), we may now assume that α is induced by a morphism $f : Y \to X = \mathbf{A}^r \times S$. Then α factors in $\underline{\mathbf{M}}\mathbf{Cor}$ as

$$(Y, E) \xrightarrow{i} (\mathbf{A}^1, 0)^{\otimes r} \otimes (Y, \emptyset) \to (\mathbf{A}^1, 0)^{\otimes r} \otimes (S, \emptyset),$$

where the first map is induced by the map

$$i = (pr_{\mathbf{A}^r} \circ f, id_Y) : Y \to \mathbf{A}^r \times Y,$$

and the second is induced by

$$id_{\mathbf{A}^r} \times (pr_S \circ f) : \mathbf{A}^r \times Y \to \mathbf{A}^r \times S.$$

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Note that i is a section of the projection $\mathbf{A}^r \times Y \to Y$. Thus, we are reduced to showing $i^*(F((\mathbf{A}^1,0)^{\otimes r} \otimes (Y,\emptyset))) \subset F(Y, E_{\mathrm{red}})$. By Proposition 3.2 this follows from the following claim. ☐

Claim 4.3.1. Take $a \in F((\mathbf{A}^1,0)^{\otimes r} \otimes (Y,\emptyset))$. There exists an open neighborhood $U \subset Y$ of the generic point of E such that for every function field K over k and every $\delta = (\delta_0, \dots, \delta_{e-1}, \delta_e) \in \mathrm{mc}(U_K)$ with $\xi := \delta_{e-1} \in E_K^{(0)}$ and $e = \dim(Y)$, we have

$$(i^*(a)_K, \gamma)_{Y_K/K,\delta} = 0 \quad \forall \gamma \in \overline{K}^{\mathrm{M}}_e(\mathcal{O}_{Y_K,\xi}, \mathfrak{m}_\xi)$$

for the pairing from (3.0.4):

$$(-,-)_{Y_K/K,\delta} : F(K(Y)) \otimes K_d^{\mathrm{M}}(K(Y)) \to F(K).$$

Proof. After replacing Y by an open neighborhood of the generic point of E, we may assume that $Y = \mathrm{Spec}(A)$ is affine and $E_{\mathrm{red}} = \mathrm{Spec}(A/(\pi))$ for $\pi \in A$ and, moreover, that writing

$$\mathbf{A}^r \times Y = \mathrm{Spec}\, A[x_1, \dots, x_r], \quad (\mathbf{A}^1,0)^{\otimes r} \otimes (Y,\emptyset) = (\mathbf{A}^r_Y, \{x_1 \cdots x_r = 0\}),$$

we have

$$i(Y) = \bigcap_{1 \le i \le r} \{x_i - u_i \pi^{m_i} = 0\} \quad \text{with } m_i \in \mathbb{Z}_{\ge 0},\ u_i \in A^\times.$$

Let $\delta = (\delta_0, \dots, \delta_e)$ be as in the claim and put $\delta' = (\delta_0, \dots, \delta_{e-1}) \in \mathrm{mc}((E_{\mathrm{red}})_K)$. Put $\tilde{X}_K = \mathbf{A}^r \times Y_K$ and $F = \{\pi = 0\} \subset \tilde{X}_K$. Note $d := \dim(\tilde{X}_K) = e + r$. Let z_j for $e \le j \le d-1$ be the generic point of

$$Z_j = \bigcap_{1 \le i \le d-j} \{x_i - u_i \pi^{m_i} = 0\} \subset \tilde{X}_K$$

which lies over δ_e,[5] and w_j for $e-1 \le j \le d-2$ be the generic point of

$$W_j = F \cap Z_{j+1} = \{\pi = x_1 = \cdots = x_{d-j-1} = 0\}$$

which is contained in the closure of z_{j+1}. Note $\dim(Z_j) = \dim(W_j) = j$ and the section i induces isomorphisms

$$Y_K \simeq Z_e \quad \text{and} \quad (E_{\mathrm{red}})_K \simeq W_{e-1}. \tag{4.3.2}$$

Let $\sigma = (i(\delta'), w_e, \dots, w_{d-2}, \eta_1, \nu) \in \mathrm{mc}(\tilde{X}_K)$, where ν is the generic point of $\tilde{X}_K$ lying over δ_e, η_1 is the generic point of $D_1 = \{x_1 = 0\} \subset \tilde{X}_K$ contained in the closure of ν, and $i(\delta') \in \mathrm{mc}(W_{e-1})$ is the image of δ' under (4.3.2). Take any $\gamma \in \overline{K}^{\mathrm{M}}_e(\mathcal{O}_{Y_K,\xi}, \mathfrak{m}_\xi)$ and put

$$\beta = \left\{ \iota(\gamma), \frac{u_1\pi^{m_1} - x_1}{u_1 \pi^{m_1}}, \dots, \frac{u_r \pi^{m_r} - x_r}{u_r \pi^{m_r}} \right\} \in K_d^{\mathrm{M}}(\mathcal{O}_{\tilde{X}_K,\nu}), \tag{4.3.3}$$

where $\iota : K_e^{\mathrm{M}}(\mathcal{O}_{Y_K,\delta_e}) \to K_e^{\mathrm{M}}(\mathcal{O}_{\tilde{X}_K,\nu})$ is induced by the projection $\tilde{X}_K \to Y_K$. For $a \in F((\mathbf{A}^1,0)^{\otimes r} \otimes (Y,\emptyset))$ and its restriction $a_K \in F((\mathbf{A}^1,0)^{\otimes r} \otimes (Y_K,\emptyset))$, we have

$$\begin{aligned} 0 = (a_K, \beta)_{\tilde{X}_K/K,\sigma} &= - \sum_{\substack{\tau \in \tilde{X}_K^{(1)} - \{\eta_1\} \\ \tau > w_{d-2}}} (a_K, \beta)_{\tilde{X}_K/K,(i(\delta'), w_e, \dots, w_{d-2}, \tau, \nu)} \\ &= -(a_K, \beta)_{\tilde{X}_K/K,(i(\delta'), w_e, \dots, w_{d-2}, z_{d-1}, \nu)} \\ &= \pm((a_K)_{|Z_{d-1}}, \beta_1)_{Z_{d-1}/K,(i(\delta'), w_e, \dots, w_{d-2}, z_{d-1})}, \end{aligned}$$

[5] Although Y is assumed to be irreducible, Y_K may not be so and possibly a finite product of schemes essentially smooth over k, noting that k is perfect.

$$\beta_1 = \left\{ \iota_1(\gamma), \frac{u_2\pi^{m_2} - x_2}{u_2\pi^{m_2}}, \ldots, \frac{u_r\pi^{m_r} - x_r}{u_r\pi^{m_r}} \right\} \in K^{\mathrm{M}}_{d-1}(\mathcal{O}_{Z_{d-1},z_{d-1}}),$$

where $\iota_1 : K^{\mathrm{M}}_e(\mathcal{O}_{Y_K,\delta_e}) \to K^{\mathrm{M}}_e(\mathcal{O}_{Z_{d-1},z_{d-1}})$ is induced by the dominant map $Z_{d-1} \to Y_K$ induced by the projection $\tilde{X}_K \to Y_K$. The first equality follows from § 3 (HS3) applied to $D_1 \subset \tilde{X}_K$, noting that β lies in $\overline{K}^{\mathrm{M}}_d(\mathcal{O}_{\tilde{X}_K,\eta_1}, \mathfrak{m}_{\eta_1})$ since $(u_1\pi^{m_1} - x_1)/u_1\pi^{m_1} \in 1 + x_1\mathcal{O}_{\tilde{X}_K,\eta_1}$. The second follows from (HS4). The third equality holds since z_{d-1} is the unique $\tau \in \tilde{X}^{(1)}_K - \{\eta_1\}$ such that $\tau > w_{d-2}$ and $(a_K, \beta)_{\tilde{X}_K/K,(i(\delta'),w_e,\ldots,w_{d-2},\tau,\nu)}$ may not vanish, which follows from (HS2), noting that $\iota(\gamma)_{|F} = 0$. The final equality follows from (HS2). When $r = 1$, the last term in the above formula is equal to $((a_K)_{|Y_K}, \gamma)_{Y_K/K,\delta}$ by (4.3.2), so that the proof is complete. When $r > 1$, we further get

$$\begin{aligned} 0 &= ((a_K)_{|Z_{d-1}}, \beta_1)_{Z_{d-1}/K,(i(\delta'),w_e,\ldots,w_{d-2},z_{d-1})} \\ &= - \sum_{\substack{\tau \in Z^{(1)}_{d-1} - \{w_{d-2}\} \\ \tau > w_{d-3}}} ((a_K)_{|Z_{d-1}}, \beta_1)_{Z_{d-1}/K,(i(\delta'),w_e,\ldots,w_{d-3},\tau,z_{d-1})} \\ &= -((a_K)_{|Z_{d-1}}, \beta_1)_{Z_{d-1}/K,(i(\delta'),w_e,\ldots,w_{d-3},z_{d-2},z_{d-1})} \\ &= \pm((a_K)_{|Z_{d-2}}, \beta_2)_{Z_{d-2}/K,(i(\delta'),w_e,\ldots,w_{d-3},z_{d-2})}, \end{aligned}$$

$$\beta_2 = \left\{ \iota_2(\gamma), \frac{u_3\pi^{m_3} - x_3}{u_3\pi^{m_3}}, \ldots, \frac{u_r\pi^{m_r} - x_r}{u_r\pi^{m_r}} \right\} \in K^{\mathrm{M}}_{d-1}(\mathcal{O}_{Z_{d-2},z_{d-2}}),$$

where $\iota_2 : K^{\mathrm{M}}_e(\mathcal{O}_{Y_K,\delta_e}) \to K^{\mathrm{M}}_e(\mathcal{O}_{Z_{d-2},z_{d-2}})$ is induced by the dominant map $Z_{d-2} \to Y_K$ induced by the projection $\tilde{X}_K \to Y_K$. The above equalities hold by the same arguments as above, except that for the third equality there are a priori two $\tau \in Z^{(1)}_{d-1} - \{w_{d-2}\}$ with $\tau > w_{d-3}$ for which $((a_K)_{|Z_{d-1}}, \beta_1)_{Z_{d-1}/K,(i(\delta'),w_e,\ldots,w_{d-3},\tau,z_{d-1})}$ may not vanish. One is z_{d-2} and the other is the generic point η_2 of $Z_{d-1} \cap D_2$ with $D_2 = \{x_2 = 0\} \subset \tilde{X}_K$ which is contained in the closure of z_{d-1}. But $((a_K)_{|Z_{d-1}}, \beta_1)_{Z_{d-1}/K,(i(\delta'),w_e,\ldots,w_{d-3},\eta_2,z_{d-1})} = 0$. Indeed, $(a_K)_{|Z_{d-1}} \in F(\mathrm{Spec}(\mathcal{O}_{Z_{d-1},\eta_2}), \eta_2)$ since Z_{d-1} and D_2 intersect transversally in $\tilde{X}_K$. Hence, the vanishing follows from (HS3) applied to $Z_{d-1} \cap D_2 \subset Z_{d-1}$, noting that $\big((u_2\pi^{m_2} - x_2)/u_2\pi^{m_2}\big)_{|Z_{d-1}} \in 1 + x_2\mathcal{O}_{Z_{d-1},\eta_2}$ so that $\beta_1 \in K^{\mathrm{M}}_d(\mathcal{O}_{Z_{d-1},\eta_2}, \mathfrak{m}_{\eta_2})$. Repeating the same arguments, we finally get

$$0 = ((a_K)_{|Z_e}, \iota_r(\gamma))_{Z_e/K,(i(\delta'),z_e)} = ((a_K)_{|Y_K}, \gamma)_{Y_K/K,\delta},$$

where $\iota_r : K^{\mathrm{M}}_e(\mathcal{O}_{Y_K,\delta_e}) \to K^{\mathrm{M}}_e(\mathcal{O}_{Z_e,z_e})$ is induced by the isomorphism $Z_e \to Y_K$ induced by the projection $\tilde{X}_K \to Y_K$ and the second equality follows from (4.3.2). This completes the proof of the claim and Theorem 4.2. □

Definition 4.4. For $F \in \underline{\mathbf{M}}\mathbf{NST}_{\log}$ and an integer $i \geq 0$, consider the association

$$H^i_{\log}(-, F) : \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}_{\mathrm{ls}} \to \mathbf{Ab}\ ;\ (X, D) \to H^i(X_{\mathrm{Nis}}, F_{(X,D_{\mathrm{red}})}).$$

By the definition this gives a presheaf on $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}_{\mathrm{ls}}$, which we call *the ith logarithmic cohomology with coefficient F*.

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5. Invariance of logarithmic cohomology under blowups

Retain the notation of § 4.

Definition 5.1. Let $\Lambda_{\mathrm{ls}}^{\mathrm{fin}}$ be the class of morphisms $\rho:(Y,E)\to(X,D)$ in $\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}^{\mathrm{fin}}$ satisfying the following conditions.

(a) ρ is induced by a proper morphism $\rho: Y\to X$ inducing an isomorphism $Y\backslash E\xrightarrow{\simeq} X\backslash D$ and $E=\rho^*D$.
(b) Zariski locally on X, $\rho: Y\to X$ is the blowup of X in a smooth center $Z\subset D$ which is normal crossing to D.

Here, a smooth Z contained in D is normal crossing to D if, letting $D_1,\dots,D_n$ be the irreducible components of D, there exists a subset $I\subset\{1,\dots,n\}$ such that $Z\subset\bigcap_{i\in I}D_i$ and Z is not contained in D_j for any $j\notin I$ and intersects $\sum_{j\notin I}D_j$ transversally. Note that the condition is equivalent to that called strict normal crossing in [BPØ22, Definition 7.2.1].

Theorem 5.2. *For $F\in\mathbf{CI}_{\mathrm{Nis}}^{\tau,\mathrm{sp}}$ and $\rho:\mathcal{Y}\to\mathcal{X}$ in $\Lambda_{\mathrm{ls}}^{\mathrm{fin}}$, we have*

$$\rho^*: H^i_{\log}(\mathcal{X},F)\cong H^i_{\log}(\mathcal{Y},F)\quad \forall i\geq 0. \tag{5.2.1}$$

Proof. Write $\mathcal{Y}=(Y,E)$ and $\mathcal{X}=(X,D)$. First we prove the theorem for $i=0$. We may assume that D is reduced and $E=\rho^*D$. By [KMSY21a, Proposition 1.9.2 b)], ρ is invertible in $\underline{\mathbf{M}}\mathbf{Cor}$, so that $\rho^*: F(\mathcal{X})\cong F(\mathcal{Y})$. Since this factors through $F(Y,E_{\mathrm{red}})$ by Theorem 4.2, we get (5.2.1) for $i=0$.

To show (5.2.1) for $i>0$, it suffices to prove $R^i\rho_*F_{(Y,E_{\mathrm{red}})}=0$. The problem is Nisnevich local, so we may assume that ρ is induced by a blowup $\rho: Y\to X$ in a smooth center $Z\subset D$ normal crossing to D. By [KS21, Corollary 9], Nisnevich locally around a point of Z, (X,D) is isomorphic to

$$(\mathbf{A}^c, L_1+\cdots+L_r)\otimes\mathcal{W}\quad\text{with } \mathcal{W}=(W,W^\infty)\in\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}},$$

where $\mathbf{A}^c=\operatorname{Spec}k[t_1,\dots,t_c]$ with $c=\operatorname{codim}_z(Z,X)$ and $L_i=V(t_i)$ for $i=1,\dots,r$ with $1\leq r\leq c$, and Z corresponds to $0\times W$. Hence, the theorem follows from the following proposition. □

Proposition 5.3. *Let $F\in\mathbf{CI}_{\mathrm{Nis}}^{\tau,\mathrm{sp}}$ and $\mathcal{W}=(W,W^\infty)\in\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$. Let $\mathbf{A}^n=\operatorname{Spec}k[t_1,\dots,t_n]$ and put $L_i=V(t_i)$ for $1\leq i\leq n$. Let $\rho: Y\to\mathbf{A}^n$ be the blowup at the origin $0\in\mathbf{A}^n$ and $\tilde{L}_i\subset Y$ be the strict transforms of L_i for $1\leq i\leq n$ and $E=\rho^{-1}(0)\subset Y$. For any $1\leq r\leq n$, we have*

$$R^i\rho_{W*}F_{(Y,\tilde{L}_1+\cdots+\tilde{L}_r+E)\otimes\mathcal{W}}=0\quad \text{for } i\geq 1, \tag{5.3.1}$$

where $\rho_W:=\rho\times\mathrm{id}_W: Y\times W\to\mathbf{A}^2\times W$.

Lemma 5.4. *Proposition 5.3 holds for $n=2$.*

Proof. The case $r=1$ is proved in [BRS22, Lemma 2.13] and we show the case $r=2$.[6] Put $D=L_1+L_2$. By the case $i=0$ of Theorem 5.2, we get

$$F_{(\mathbf{A}^2,D)\otimes\mathcal{W}}\cong\rho_{W*}F_{(Y,\tilde{L}_1+\tilde{L}_2+E)\otimes\mathcal{W}}. \tag{5.4.1}$$

Set

$$\mathcal{F}:=F_{(Y,\tilde{L}_1+\tilde{L}_2+E)\otimes\mathcal{W}},$$

[6] The following argument is adopted from [BRS22, Lemma 2.13], but the present case is easier.

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and $\mathbf{A}^2_W = \mathbf{A}^2 \times W$ with the projection $p : A^2_W \to W$. Since $R^i\rho_{W*}\mathcal{F}$ for $i \geq 1$ is supported in $0 \times W$, we have

$$\begin{aligned} R^i\rho_{W*}\mathcal{F} = 0 &\iff p_*R^i\rho_{W*}\mathcal{F} = 0 \\ &\iff (p_*R^i\rho_{W*}\mathcal{F})_w = 0 \quad \forall w \in W \\ &\iff H^0(\mathbf{A}^2_{W_w}, R^i\rho_{W*}\mathcal{F}) = 0 \quad \forall w \in W, \end{aligned}$$

where W_w is the henselization of W at w. Hence, it suffices to show $H^0(\mathbf{A}^2_W, R^i\rho_{W*}\mathcal{F}) = 0$, assuming W is henselian local. Then we have

$$H^j(\mathbf{A}^2_W, R^i\rho_{W*}\mathcal{F}) = 0, \quad \forall i, j \geq 1.$$

By (5.4.1) and [BRS22, Lemma 2.10],

$$H^i(\mathbf{A}^2_W, \rho_{W*}\mathcal{F}) = H^i(\mathbf{A}^2_W, F_{(\mathbf{A}^2,D)\otimes\mathcal{W}}) = 0.$$

Thus, the Leray spectral sequence yields

$$H^0(\mathbf{A}^2_W, R^i\rho_{W*}\mathcal{F}) = H^i(Y \times W, \mathcal{F}), \quad i \geq 0,$$

and we have to show that this group vanishes for $i \geq 1$. We can write

$$\mathbf{A}^2 = \operatorname{Spec} k[x,y] \quad \text{and} \quad L_1 = V(x),\ L_2 = V(y) \subset \mathbf{A}^2.$$

Then we have

$$Y = \operatorname{Proj} k[x,y][S,T]/(xT - yS) \subset \mathbf{A}^2 \times \mathbf{P}^1.$$

Denote by

$$\pi_0 : Y \hookrightarrow \mathbf{A}^2 \times \mathbf{P}^1 \to \mathbf{P}^1 = \operatorname{Proj} k[S,T]$$

the morphism induced by projection, and let $\pi : Y \times W \to \mathbf{P}^1_W$ be its base change. Then π_0 induces an isomorphism $E \simeq \mathbf{P}^1$, and we have

$$\tilde{L}_1 = \pi_0^{-1}(0), \quad \tilde{L}_2 = \pi_0^{-1}(\infty). \tag{5.4.2}$$

Set $s = S/T = x/y$ and write

$$\mathbf{P}^1\backslash\{\infty\} = \mathbf{A}^1_s := \operatorname{Spec} k[s], \quad \mathbf{P}^1\backslash\{0\} = \operatorname{Spec} k[\tfrac{1}{s}].$$

Set $U := \mathbf{A}^1_s \times W$, $V := (\mathbf{P}^1\backslash\{0\}) \times W$ and

$$\mathcal{U} := (\mathbf{A}^1_s, 0) \otimes \mathcal{W}, \quad \mathcal{V} := (\mathbf{P}^1\backslash\{0\}, \infty) \otimes \mathcal{W}.$$

We have

$$\pi^{-1}(U) = \mathbf{A}^1_y \times U, \quad \pi^{-1}(V) = \mathbf{A}^1_x \times V,$$

and the restriction of π to these open subsets is given by projection. Furthermore, $E \times W \subset Y$ is defined by $y = 0$ on $\pi^{-1}(U)$ and by $x = 0$ on $\pi^{-1}(V)$. In view of (5.4.2), we have

$$\mathcal{F}_{|\pi^{-1}(U)} = F_{(\mathbf{A}^1_y,0)\otimes\mathcal{U}}, \quad \mathcal{F}_{|\pi^{-1}(V)} = F_{(\mathbf{A}^1_x,0)\otimes\mathcal{V}}. \tag{5.4.3}$$

Thus, [BRS22, Lemma 2.10] yields

$$R^j\pi_*\mathcal{F} = 0 \quad \text{for } j \geq 1,$$

and it remains to show

$$H^i(\mathbf{P}^1_W, \pi_*\mathcal{F}) = 0 \quad \text{for } i \geq 1, \tag{5.4.4}$$

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where $\mathbf{P}^1_W = \mathbf{P}^1 \times W$. For this consider the map

$$a_0 : Y \to \mathbf{A}^1_x \times \mathbf{P}^1$$

which is the closed immersion $Y \hookrightarrow \mathbf{A}^2 \times \mathbf{P}^1$ followed by the projection $\mathbf{A}^2 \to \mathbf{A}^1_x$. Let $a : Y \times W \to \mathbf{A}^1_x \times \mathbf{P}^1 \times W$ be its base change. In view of (5.4.2), the map a induces a morphism in $\underline{\mathbf{M}}\mathbf{Cor}$,

$$\alpha : (Y, \tilde{L}_1 + \tilde{L}_2 + E) \otimes \mathcal{W} \to (\mathbf{A}^1_x, 0) \otimes (\mathbf{P}^1, \infty) \otimes \mathcal{W},$$

which is an isomorphism over $(\mathbf{A}^1_x, 0) \otimes (\mathbf{P}^1 \backslash \{0\}, \infty) \otimes \mathcal{W}$. Setting

$$F_1 := \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathbf{A}^1_x, 0), F) \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}},$$

it induces a map of Nisnevich sheaves on $\mathbf{P}^1_W$,

$$\pi_*(\alpha^*) : F_{1,(\mathbf{P}^1,\infty)\otimes\mathcal{W}} \to \pi_*\mathcal{F},$$

which becomes an isomorphism over $(\mathbf{P}^1 - \{0\}) \times W$. Hence, (5.4.4) follows from

$$H^i(\mathbf{P}^1_W, F_{1,(\mathbf{P}^1,\infty)\otimes\mathcal{W}}) = 0 \quad \text{for } i \geq 1,$$

which follows from [Sai20, Theorem 0.6]. □

LEMMA 5.5. *Let $N > 2$ be an integer and assume that Proposition 5.3 holds for $n < N$. Let $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ and $Z \subset X$ be a smooth integral closed subscheme with $2 \leq \mathrm{codim}(Z, X) =: c < N$. Assume*

$$D = D_1 + \cdots + D_r + D' \quad \textit{with } r \leq c,$$

where $D_1, \ldots, D_r$ are distinct and reduced irreducible components of D containing Z, and D' is an effective divisor on X such that none of the component of D' contains Z and Z is transversal to $|D'|$. Let $\rho : Y \to X$ be the blowup of X in Z, let $\tilde{D}_i$ and $\tilde{D}' \subset Y$ be the strict transforms of D_i and D' respectively, and let $E_Z = \rho^{-1}(Z)$. Then, for all $\mathcal{W} = (W, W^\infty) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$,

$$R^i \rho_{W*} F_{(Y, \tilde{D}_1 + \cdots + \tilde{D}_r + E_Z + \tilde{D}')\otimes\mathcal{W}} = 0 \quad \textit{for } i \geq 1,$$

where $\rho_W : Y \times W \to X \times W$ denotes the base change of ρ.

Proof. This proof is adapted from [BRS22, Lemma 2.14]. The question is Nisnevich local around the points in $Z \times W$. Let $z \in Z \times W$ be a point and set $A := \mathcal{O}^h_{X\times W,z}$. For $V \subset Y \times W$, we denote $V_{(z)} := V \times_{X\times W} \mathrm{Spec}\, A$. By assumption we find a regular system of local parameters $t_1, \ldots, t_m$ of A, such that

$$(D_i \times W)_{(z)} = V(t_i) \quad \text{for } 1 \leq i \leq r, \quad (Z \times W)_{(z)} = V(t_1, \ldots, t_c),$$

$$(D' \times W)_{(z)} = V(t_{c+1}^{e_{c+1}} \cdots t_{m_0}^{e_{m_0}}) \quad \text{with } c + 1 \leq m_0 \leq m,$$

$$(X \times W^\infty)_{(z)} = V(t_{m_0+1}^{e_{m_0+1}} \cdots t_{m_1}^{e_{m_1}}) \quad \text{with } m_0 \leq m_1 \leq m.$$

Letting K be the residue field of A, we can choose a ring homomorphism $K \hookrightarrow A$ which is a section of $A \to K$. Then we obtain an isomorphism

$$K\{t_1, \ldots, t_m\} \xrightarrow{\simeq} A.$$

Let $\rho_1 : \widetilde{\mathbf{A}^c} \to \mathbf{A}^c$ be the blowup in 0. By the above,

$$\rho_W : (Y, \tilde{D}_1 + \cdots + \tilde{D}_r + E_Z + \tilde{D}') \otimes \mathcal{W} \to (X, D) \otimes \mathcal{W}$$

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is Nisnevich locally around z isomorphic over k to the morphism

$$(\widetilde{\mathbf{A}^c}, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}' \to (\mathbf{A}^c, L_1 + \cdots + L_r) \otimes \mathcal{W}',$$

$$\left(\mathcal{W}' = \left(\mathbf{A}^{m-c}_K, \left(\prod_{i=c+1}^{m_1} t_i^{e_i}\right)\right)\right)$$

induced by a map $(\widetilde{\mathbf{A}^c}, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \to (\mathbf{A}^c, L_1 + \cdots + L_r)$ as in Proposition 5.3. Hence, the statement follows from the proposition for $n = c < N$. □

Proof of Proposition 5.3. The proof is by induction on $n \geq 2$. The case $n = 2$ follows from Lemma 5.4. Assume that $n > 2$ and that the proposition is proven for $\mathbf{A}^m$ with $m < n$. For $r = 1$, Proposition 5.3 is proved in [BRS22, Theorem 2.12]. Assume that $r \geq 2$. Let $Z := L_1 \cap L_2 \subset \mathbf{A}^n$ and $\tilde{Z} \subset Y$ be the strict transform of Z. Denote by $\rho' : Y' \to Y$ the blowup of Y in $\tilde{Z}$, let $\tilde{L}'_i, E' \subset Y'$ be the strict transforms of $\tilde{L}, E$ respectively, and let $E'' = (\rho')^{-1}(\tilde{Z})$. Note that $\tilde{Z} = \tilde{L}_1 \cap \tilde{L}_2$ intersecting transversally with $\tilde{L}_3 + \cdots + \tilde{L}_r + E$ and $\operatorname{codim}(\tilde{Z}, Y) = 2$. Hence, by Lemma 5.5,

$$R^i \rho'_{W*} F_{(Y', \tilde{L}'_1 + \cdots + \tilde{L}'_r + E' + E'') \otimes \mathcal{W}} = 0 \quad \text{for } i \geq 1.$$

Since Theorem 5.2 has been proved for $i = 0$, we have

$$\rho'_* F_{(Y', \tilde{L}'_1 + \cdots + \tilde{L}'_r + E' + E'') \otimes \mathcal{W}} = F_{(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}}.$$

Hence, we obtain

$$R^i \rho_{W*} F_{(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}} = R^i (\rho\rho')_{W*} F_{(Y', \tilde{L}'_1 + \cdots + \tilde{L}'_r + E' + E'') \otimes \mathcal{W}}. \tag{5.5.1}$$

Denote by $\sigma : \hat{Y} \to \mathbf{A}^n$ the blowup in Z, let $\hat{L}_i \subset \hat{Y}$ be the strict transform of L_i, and let $\Xi = \sigma^{-1}(Z)$. By Lemma 5.5 we get

$$R^i \sigma_{W*} F_{(\hat{Y}, \hat{L}_1 + \cdots + \hat{L}_r + \Xi) \otimes \mathcal{W}} = 0 \quad \text{for } i \geq 1. \tag{5.5.2}$$

Denote by $\sigma' : \hat{Y}' \to \hat{Y}$ the blowup in $\hat{Z} = \sigma^{-1}(0) \subset \Xi$, let $\hat{L}'_i$, $\Xi' \subset \hat{Y}'$ be the strict transforms of $\hat{L}_i$, Ξ respectively, and let $\Xi'' = \sigma'^{-1}(\hat{Z})$. Note that $\hat{Z} \subset \hat{L}_3 \cap \cdots \cap \hat{L}_n \cap \Xi$ and $\operatorname{codim}(\hat{Z}, \hat{Y}) = n - 1$ and $\hat{Z}$ intersects transversally with $\hat{L}_1 + \hat{L}_2$. Thus, by Lemma 5.5 and the case $i = 0$ of Theorem 5.2, we obtain

$$R\sigma'_{W*} F_{(\hat{Y}', \hat{L}'_1 + \cdots + \hat{L}'_r + \Xi' + \Xi'') \otimes \mathcal{W}} = F_{(\hat{Y}, \hat{L}_1 + \cdots + \hat{L}_r + \Xi) \otimes \mathcal{W}}. \tag{5.5.3}$$

Finally, by [BRS22, Lemma 2.15], there is an isomorphism of $\mathbf{A}^n \times W$-schemes

$$(\hat{Y}', \hat{L}'_1, \ldots, \hat{L}_r, \Xi', \Xi'') \cong (Y', \tilde{L}'_1, \ldots, \tilde{L}'_r, E', E''). \tag{5.5.4}$$

Altogether we obtain, for $i \geq 1$,

$$\begin{aligned}
R^i \rho_{W*} F_{(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}} &= R^i (\rho\rho')_{W*} F_{(Y', \tilde{L}'_1 + \cdots + \tilde{L}'_r + E' + E'') \otimes \mathcal{W}}, && \text{by (5.5.1)}, \\
&= R^i (\sigma\sigma')_{W*} F_{(\hat{Y}', \hat{L}'_1 + \cdots + \hat{L}'_r + \Xi' + \Xi'') \otimes \mathcal{W}}, && \text{by (5.5.4)}, \\
&= R^i \sigma_{W*} F_{(\hat{Y}, \hat{L}_1 + \cdots + \hat{L}_r + \Xi) \otimes \mathcal{W}}, && \text{by (5.5.3)}, \\
&= 0, && \text{by (5.5.2)}.
\end{aligned}$$

This completes the proof of the proposition. □

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Remark 5.6. For simplicity, we write

$$H^i_{\log}(-,F) = H^i_{\log}(-,\underline{\omega}^{\mathbf{CI}}F) \quad \text{for } F \in \mathbf{RSC}_{\mathrm{Nis}}.$$

By [RS21a, Corollary 6.8], if $\mathrm{ch}(k) = 0$ and $F = \Omega^i$, we have

$$H^i_{\log}(-,\Omega^i) = H^i(X, \Omega^i(\log|D|) \quad \text{for } (X,D) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}.$$

Hence, $H^i_{\log}(-,F)$ for $F \in \mathbf{RSC}_{\mathrm{Nis}}$ is a generalization of cohomology of sheaves of logarithmic differentials.

6. Relation with logarithmic sheaves with transfers

In this section we use the same notation as [BPØ22].

Let $\mathrm{l}\mathbf{Sm}$ be the category of log smooth and separated fs log schemes of finite type over the base field k and $\mathbf{Sm}\mathrm{l}\mathbf{Sm} \subset \mathrm{l}\mathbf{Sm}$ be the full subcategory consisting of objects whose underlying schemes are smooth over k. Let $\mathrm{l}\mathbf{Cor}$ be the category with the same objects as $\mathrm{l}\mathbf{Sm}$ and whose morphisms are log correspondences defined in [BPØ22, Definition 2.1.1]. Let $\mathrm{l}\mathbf{Cor}_{\mathbf{Sm}\mathrm{l}\mathbf{Sm}} \subset \mathrm{l}\mathbf{Cor}$ be the full subcategory consisting of all objects in $\mathbf{Sm}\mathrm{l}\mathbf{Sm}$.

Let $\mathbf{PSh}^{\mathrm{ltr}}$ be the category of additive presheaves of abelian groups on $\mathrm{l}\mathbf{Cor}$ and $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}} \subset \mathbf{PSh}^{\mathrm{ltr}}$ be the full subcategory consisting of those $\mathcal{F}$ whose restrictions to $\mathrm{l}\mathbf{Sm}$ are dividing Nisnevich sheaves (see [BPØ22, Definition 3.1.4]). It is shown in [BPØ22, Theorem 1.2.1 and Proposition 4.7.5] that $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$ is a Grothendieck abelian category and there is an equivalence of categories

$$\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}} \simeq \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(\mathbf{Sm}\mathrm{l}\mathbf{Sm}), \tag{6.0.1}$$

where the right-hand side denotes the full subcategory of the category $\mathbf{PSh}^{\mathrm{ltr}}(\mathbf{Sm}\mathrm{l}\mathbf{Sm})$ of additive presheaves of abelian groups on $\mathrm{l}\mathbf{Cor}_{\mathbf{Sm}\mathrm{l}\mathbf{Sm}}$ consisting of those $\mathcal{F}$ whose restrictions to $\mathbf{Sm}\mathrm{l}\mathbf{Sm}$ are dividing Nisnevich sheaves.

We now construct a functor

$$\mathcal{L}og : \underline{\mathbf{M}}\mathbf{NST}_{\log} \to \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}. \tag{6.0.2}$$

For $\mathfrak{X} = (X, \mathcal{M}) \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$, we put $\mathfrak{X}^{\mathrm{MP}} = (X, \partial\mathfrak{X})$, where $\partial\mathfrak{X} \subset X$ is the closed subscheme consisting of the points where the log structure $\mathcal{M}$ is not trivial. By [BPØ22, Lemma A.5.10], $\partial\mathfrak{X}$ with reduced structure is a normal crossing divisor on X, so that we can view $\mathfrak{X}^{\mathrm{MP}}$ as an object of $\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$. For $F \in \underline{\mathbf{M}}\mathbf{PST}_{\log}$ and $\mathfrak{X} \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$, we put

$$F^{\log}(\mathfrak{X}) = F(\mathfrak{X}^{\mathrm{MP}}). \tag{6.0.3}$$

Take $\mathfrak{Y} \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$ and $\alpha \in \mathrm{l}\mathbf{Cor}(\mathfrak{Y}, \mathfrak{X})$. By [BPØ22, Definition 2.1.1 and Remark 2.1.2(iii)], we have

$$\alpha \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}((Y, n\cdot\partial\mathfrak{Y}), (X, \partial\mathfrak{X})) \quad \text{for some } n > 0,$$

where $n\cdot\partial\mathfrak{Y} \hookrightarrow Y$ is the nth thickening of $\partial\mathfrak{Y} \hookrightarrow Y$. By the assumption $F \in \underline{\mathbf{M}}\mathbf{PST}_{\log}$, the induced map

$$F^{\log}(\mathfrak{X}) = F(\mathfrak{X}^{\mathrm{MP}}) \xrightarrow{\alpha^*} F(Y, n\cdot\partial\mathfrak{Y})$$

factors through $F^{\log}(\mathfrak{Y}) = F(Y, \partial\mathfrak{Y}) \subset F(Y, n\cdot\partial\mathfrak{Y})$ and we get a map

$$\alpha^{*\log} : F^{\log}(\mathfrak{X}) \to F^{\log}(\mathfrak{Y}).$$

Moreover, for a map $\gamma : F \to G$ in $\underline{\mathbf{M}}\mathbf{PST}_{\log}$, the diagram

$$\begin{array}{ccc} F^{\log}(\mathfrak{X}) & \xrightarrow{\gamma} & G^{\log}(\mathfrak{X}) \\ \downarrow{\scriptstyle \alpha^{*\log}} & & \downarrow{\scriptstyle \alpha^{*\log}} \\ F^{\log}(\mathfrak{Y}) & \xrightarrow{\gamma} & G^{\log}(\mathfrak{Y}) \end{array}$$

is obviously commutative. Hence, the assignment $\mathcal{X} \to F^{\log}(\mathcal{X})$ gives an object $F^{\log}$ of $\mathbf{PSh}^{\mathrm{ltr}}(\mathbf{SmlSm})$ and we get a functor

$$\mathcal{L}og : \underline{\mathbf{M}}\mathbf{PST}_{\log} \to \mathbf{PSh}^{\mathrm{ltr}}(\mathbf{SmlSm}), \quad F \to F^{\log}. \tag{6.0.4}$$

By the definitions of sheaves ([KMSY21a, Definition 1], [BPØ22, Definition 3.1.4] and [KMSY21a, Proposition 1.9.2]), this induces a functor

$$\underline{\mathbf{M}}\mathbf{NST}_{\log} \to \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(\mathbf{SmlSm})$$

which induces the desired functor (6.0.2) using (6.0.1). By the construction, for $F \in \underline{\mathbf{M}}\mathbf{NST}_{\log}$ and $\mathfrak{X} \in \mathbf{SmlSm}$ with $\mathcal{X} = \mathfrak{X}^{\mathrm{MP}} \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$, we have

$$H^i_{\mathrm{Nis}}(X, F_{\mathcal{X}}) = H^i_{s\mathrm{Nis}}(\mathfrak{X}, F^{\log}) \ (F^{\log} = \mathcal{L}og(F)), \tag{6.0.5}$$

where the right-hand side is the cohomology for the strict Nisnevich topology (see [BPØ22, Definition 4.3.1]).

Theorem 6.1. *For $F \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$, $F^{\log} = \mathcal{L}og(F) \in \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$ is strictly $\overline{\square}$-invariant in the sense of [BPØ22, Definition 5.2.2]. For $\mathfrak{X} \in \mathbf{SmlSm}$ with $\mathcal{X} = \mathfrak{X}^{\mathrm{MP}} \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$, we have a natural isomorphism*

$$H^i_{\mathrm{Nis}}(X, F_{\mathcal{X}}) \simeq \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(M(\mathfrak{X}), F^{\log}[i]), \tag{6.1.1}$$

where $\mathbf{logDM}^{\mathrm{eff}}$ is the triangulated category of logarithmic motives defined in [BPØ22, Definition 5.2.1].

Proof. Let $\mathfrak{X}^{\mathrm{Sm}}_{\mathrm{div}}$ be the category of log modifications $\mathfrak{Y} \to \mathfrak{X}$ such that $\mathfrak{Y} \in \mathbf{SmlSm}$ (see [BPØ22, Definition A.11.12]) and $\mathfrak{X}^{\mathrm{Sm}}_{\mathrm{divsc}} \subset \mathfrak{X}^{\mathrm{Sm}}_{\mathrm{div}}$ be the full subcategory given by those maps $\mathfrak{Y} \to \mathfrak{X}$ that are isomorphic to compositions of log modifications along smooth centers (see [BPØ22, Definitions 4.4.4 and A.14.10]). We have isomorphisms

$$\begin{aligned} H^i_{\mathrm{Nis}}(X, F_{\mathcal{X}}) &\overset{(6.0.5)}{\simeq} H^i_{s\mathrm{Nis}}(\mathfrak{X}, F^{\log}) \overset{(*1)}{\simeq} \varinjlim_{\mathfrak{Y} \in \mathfrak{X}^{\mathrm{Sm}}_{\mathrm{divsc}}} H^i_{s\mathrm{Nis}}(\mathfrak{Y}, F^{\log}) \\ &\overset{(*2)}{\simeq} \varinjlim_{\mathfrak{Y} \in \mathfrak{X}^{\mathrm{Sm}}_{\mathrm{div}}} H^i_{s\mathrm{Nis}}(\mathfrak{Y}, F^{\log}) \overset{(*3)}{\simeq} H^i_{d\mathrm{Nis}}(\mathfrak{X}, F^{\log}), \end{aligned}$$

where $(*2)$ follows from [BPØ22, Corollary 4.4.5] and $(*3)$ from [BPØ22, Theorem 5.1.8], and $(*1)$ is a consequence of Theorem 5.2 in view of (6.0.5) and the fact that a log modification of $\mathfrak{X} = (X, \mathcal{M}) \in \mathbf{SmlSm}$ along smooth center is induced Zariski locally by a blowup of X in an intersection of irreducible components of $\partial\mathfrak{X}$ so that it corresponds to a morphism in $\Lambda^{\mathrm{fin}}_{\mathrm{ls}}$ from Definition 5.1.

Hence, the strict $\overline{\square}$-invariance of $F^{\log}$ follows from [Sai20, Theorem 0.6]. Finally, (6.1.1) follows from [BPØ22, Proposition 5.2.3]. □

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We now consider the composite functor

$$\mathcal{L}og' : \mathbf{RSC}_{\mathrm{Nis}} \xrightarrow{\underline{\omega}^{\mathbf{CI}}} \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}} \xrightarrow{\mathcal{L}og} \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}},$$

where $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \subset \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$ is the full subcategory consisting of strictly $\overline{\square}$-invariant objects. By [BM12, Theorem 5.7], $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ is a Grothendieck abelian category.

Lemma 6.2. *$\mathcal{L}og$ and $\mathcal{L}og'$ have the same essential image.*

Proof. This follows directly from the construction and Corollary 2.6(3). □

In what follows, we let

$$\mathcal{L}og : \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \ : \ F \to F^{\mathrm{log}} \tag{6.2.1}$$

denote $\mathcal{L}og'$ defined as above. By (6.0.3), we have

$$F^{\mathrm{log}}(X, \mathrm{triv}) = F(X) \quad \text{for } F \in \mathbf{RSC}_{\mathrm{Nis}},\ X \in \mathbf{Sm}, \tag{6.2.2}$$

where (X, triv) denotes the log scheme with the trivial log structure.

Theorem 6.3. *$\mathcal{L}og$ is exact and fully faithful.*

Proof. First we prove the full faithfulness. Faithfulness follows from (6.2.2). Let $F, G \in \mathbf{RSC}_{\mathrm{Nis}}$ and $\gamma : F^{\mathrm{log}} \to G^{\mathrm{log}}$ be a map in $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$. By (6.2.2) it induces maps $\gamma_X : F(X) \to G(X)$ for all $X \in \mathbf{Sm}$. They are compatible with the action of $\mathbf{Cor}$ since by [BPØ22, Example 2.1.3(3)],

$$\mathbf{Cor}(Y, X) = \mathrm{l}\mathbf{Cor}(Y, \mathrm{triv}), (X, \mathrm{triv})) \quad \text{for } X, Y \in \mathbf{Sm}.$$

Thus, γ_X for $X \in \mathbf{Sm}$ give a map $\gamma_{\mathbf{RSC}_{\mathrm{Nis}}} : F \to G$ in $\mathbf{RSC}_{\mathrm{Nis}}$. To see $\mathcal{L}og(\gamma_{\mathbf{RSC}_{\mathrm{Nis}}}) = \gamma$, it suffices by (6.0.1) to show that $\mathcal{L}og(\gamma_{\mathbf{RSC}_{\mathrm{Nis}}})$ and γ induce the same map $F^{\mathrm{log}}(\mathfrak{X}) \to G^{\mathrm{log}}(\mathfrak{X})$ for $\mathfrak{X} \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$. If $\mathfrak{X}$ has the trivial log structure, this follows immediately from the construction of $\gamma_{\mathbf{RSC}}$. The general case follows from this in view of the commutative diagram

$$\begin{array}{ccc} F^{\mathrm{log}}(\mathfrak{X}) & \xrightarrow{\quad\gamma\quad} & G^{\mathrm{log}}(\mathfrak{X}) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ F^{\mathrm{log}}(X \backslash \partial\mathfrak{X}, \mathrm{triv}) & \xrightarrow{\gamma} & G^{\mathrm{log}}(X \backslash \partial\mathfrak{X}, \mathrm{triv}) \end{array}$$

where j^* are induced by the natural map $(X \backslash \partial\mathfrak{X}, \mathrm{triv}) \to \mathfrak{X}$ of log schemes and are injective by the construction and the semipurity of $\underline{\omega}^{\mathbf{CI}} F$. This completes the proof of the full faithfulness.

Next we show the exactness of $\mathcal{L}og$. It suffices to show the following claim.

Claim 6.3.1. Given an exact sequence $0 \to F \to G \to H \to 0$ in $\mathbf{RSC}_{\mathrm{Nis}}$, the induced sequence

$$0 \to F^{\mathrm{log}}(\mathfrak{X}) \to G^{\mathrm{log}}(\mathfrak{X}) \to H^{\mathrm{log}}(\mathfrak{X}) \to 0$$

is exact for every $\mathfrak{X} \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$ with X henselian local.

Indeed, by the definition of $\mathcal{L}og$, this is reduced to the exactness of

$$0 \to \underline{\omega}^{\mathbf{CI}} F(\mathfrak{X}^{\mathrm{MP}}) \to \underline{\omega}^{\mathbf{CI}} G(\mathfrak{X}^{\mathrm{MP}}) \to \underline{\omega}^{\mathbf{CI}} H(\mathfrak{X}^{\mathrm{MP}}) \to 0,$$

which follows from Corollary 2.6(2). This completes the proof of Theorem 6.3. □

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