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Minimizers and Quasiminimizers

14.1 Quasiminimizers

In addition to currents and varifolds, there are several other ways to model minimal surfaces and related objects, see [139, 161]. Quasiminimizers provide a very natural and general setting for many variational problems. Let $E \subset \mathbb{R}^n$ be closed and unbounded such that for a fixed positive integer m, $0 < \mathcal{H}^m(E \cap B(x,r)) < \infty$ for $x \in E, r > 0$. We say that E is an *m-quasiminimizer* if for some $M < \infty$,

$$\mathcal{H}^m(E \cap W) \leq M\mathcal{H}^m(f(E \cap W))$$

for all Lipschitz mappings $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $W = \{x\colon f(x) \neq x\}$ is bounded. If this holds with $M = 1$, then E minimizes m-dimensional Hausdorff measure. The setting in the papers quoted below is more general. In particular, there is also a local, often very useful, version, but we skip it here. The quasiminimizers were introduced by Almgren in [9] under the name restricted sets. He proved that they are AD-m-regular and m-rectifiable. David and Semmes investigated them in [150]. They re-proved Almgren's results and went further. The following is a special case of their results:

Theorem 14.1 *If $E \subset \mathbb{R}^n$ is a closed m-quasiminimizer, then E is AD-m-regular, uniformly m-rectifiable and it contains big pieces of Lipschitz graphs (recall Section 5.2).*

Both Almgren's and David–Semmes's proofs use Lipschitz projections into k-dimensional cubical skeleta like in the Federer–Fleming proof of the deformation theorem of currents. First this gives AD-regularity. Then, by David and Semmes, via many complicated constructions, the big pieces of the Lipschitz graphs condition are verified.

The codimension 1 case was studied by different methods in [149] and [264].

All these papers contain many interesting results on and connections with various geometric variational problems.

There is much later work along these lines, see David's long paper [139] for a very general setting, for discussion and references. It seems to give the most general rectifiability results. In particular, there he used sliding conditions; the deformations were required to preserve given boundary pieces but were allowed to slide along them.

When minimizing Hausdorff measure the existence of minimizers is often a difficult question, both for the lack of lower semicontinuity and compactness. De Lellis, Ghiraldin and Maggi [162] established a general result to deal with this. For this they used Preiss's Theorem 4.11.

14.2 Mumford–Shah Functional

Let $\Omega \subset \mathbb{R}^n$ be a domain and g a bounded measurable function in Ω. The *Mumford–Shah functional* J is then defined by

$$J(u, K) = \int_{\Omega \setminus K} (u - g)^2 + \mathcal{H}^{n-1}(K) + \int_{\Omega \setminus K} |\nabla u|^2$$

for

$$(u, K) \in \mathcal{A}(\Omega) := \{(u, K) \colon K \subset \Omega \text{ relatively closed and } u \in W^{1,2}_{loc}(\Omega \setminus K)\}.$$

We assume that there are $(u, K) \in \mathcal{A}(\Omega)$ with $J(u, K) < \infty$, which is always true if $\Omega \subset \mathbb{R}^n$ is bounded. For many aspects of the Mumford–Shah functional, including applications to image segmentation and conjectures and results on minimizers, see the books [15] and [138]. Here we restrict the discussion to things related to rectifiability.

A minimizer for J is a pair $(u, K) \in \mathcal{A}(\Omega)$ which gives the smallest value for J. Minimizers always exist, although it is far from obvious since Hausdorff measure is not lower semicontinuous. One way to prove the existence is to first minimize

$$\int_{\Omega} (u - g)^2 + \mathcal{H}^{n-1}(S_u) + \int_{\Omega} |\nabla u|^2$$

for $u \in SBV(\Omega)$, recall Section 12.3. Minimizers for this exist by the compactness properties of SBV. To get from this a minimizer for J, the problem that S_u need not be closed has to be dealt with. Here one cannot use the full $BV(\Omega)$, since it would give 0 for the infimum. Anyway, now S_u is $(n-1)$-rectifiable by Theorem 12.13. This approach is discussed in [15]. In [138], a different approach without SBV is explained.

For a minimizer (u, K), u is in $C^1(\Omega \setminus K)$, which follows from the fact that it solves the PDE $\Delta u = u - g$. For K there are conjectures which are only partially solved. David and Semmes proved the following in [148], see also [138]:

Theorem 14.2 *If (u, K) is a minimizer for J and $B(x, 2r) \subset \Omega$, then $K \cap B(x, r)$ is contained in an AD-$(n-1)$-regular uniformly $(n-1)$-rectifiable set.*

The key to the proof is that the failure of the Poincaré inequality in the complement of an AD-$(n-1)$-regular set E at most scales implies uniform rectifiability of E. This is understandable because the validity of the Poincaré inequality requires that E does not separate the space too much. More precisely: E is uniformly $(n-1)$-rectifiable if there exists a positive number c such that for all $M \geq 1$ the set $F(E, c, M)$ of pairs $(x, r), x \in E, 0 < r < d(E)$, satisfying the following condition, is a Carleson set: for all balls $B(x_i, r_i) \subset B(x, r) \setminus E, i = 1, 2$, with $r_i > cr$ and for all $f \in W^{1,1}(B(x, Mr) \setminus E)$,

$$\left| r_1^{-n} \int_{B(x_1,r_1)} f - r_2^{-n} \int_{B(x_2,r_2)} f \right| \leq M r^{1-n} \int_{B(Mx,r)\setminus E} |\nabla f|. \tag{14.1}$$

David and Semmes proved this by showing that this condition implies the local symmetry of Theorem 5.9. Another proof is described in [138]. The converse is false; an example is a coordinate hyperplane with the balls of radius $1/10$ centred in the integer lattice removed.

For slight simplicity, assume $\Omega = \mathbb{R}^n$. To prove that for a minimizer (u, K) the set $F(K, c, M)$ is a Carleson set, one applies (14.1) with $u = f$ and constructs a competitor to get for some $p < 2$,

$$\omega_p(x, Mr) = r^{p/2-n} \int_{B(x,Mr)\setminus K} |\nabla u|^p > \varepsilon(M) > 0.$$

As $r^{1-n} \int_{B(x,r)\setminus K} |\nabla u|^2$ is bounded, it is not very difficult to prove that the set of (x, r) such that $\omega_p(x, r) > \varepsilon$ satisfies a Carleson condition, from which it follows that $F(K, c, M)$ is a Carleson set.

Theorem 14.2 holds for a much larger class of quasiminimizers.

14.3 Some Free Boundary Problems

In [141], David, Engelstein and Toro studied the following two-phase free boundary problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and q_+ and q_- bounded continuous functions on Ω. Let

$$J(u) = \int_\Omega \left(|\nabla u(x)|^2 + q_+(x)^2 \chi_{\{u>0\}}(x) + q_-(x)^2 \chi_{\{u<0\}}(x) \right) dx.$$

Among other things they proved that if u is an almost minimizer (we omit the definition) for J, then, under slight extra conditions, the sets $\Omega \cap \partial\{x \in \Omega\colon u(x) > 0\}$ and $\Omega \cap \partial\{x \in \Omega\colon u(x) < 0\}$ are locally AD-$(n-1)$-regular and uniformly $(n-1)$-rectifiable. The proof is a complicated mixture of potential theory and geometric measure theory. In particular, proving the AD-regularity is quite demanding and achieved with estimates for the harmonic measure.

We shall return to the corresponding one-phase problem in Section 15.6.

Rigot [390] proved the uniform rectifiability of sets almost minimizing perimeter, recall Section 12.1. Let $g(0,\infty) \to (0,\infty)$ with $g(x) = o(x^{(n-1)/n})$.

Theorem 14.3 *Let $E \subset \mathbb{R}^n$ be Lebesgue measurable. If*

$$P(E) \leq P(F) + g(\mathcal{L}^n((E \setminus F) \cup (F \setminus E))$$

whenever $F \subset \mathbb{R}^n$ is Lebesgue measurable and $F = E$ outside some compact set, then E is equivalent to E' for which $\partial E'$ is AD-$(n-1)$-regular and uniformly $(n-1)$-rectifiable.

She proved this by showing that $\partial E'$ is a Semmes surface, recall Section 8.7.