




RESEARCH ARTICLE

# Relative Dehn functions, hyperbolically embedded subgroups and combination theorems

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## Abstract

Consider the following classes of pairs consisting of a group and a finite collection of subgroups:

- $\mathcal{C} = \{(G, \mathcal{H}) \mid \mathcal{H} \text{ is hyperbolically embedded in } G\}$
- $\mathcal{D} = \{(G, \mathcal{H}) \mid \text{the relative Dehn function of } (G, \mathcal{H}) \text{ is well-defined}\}.$

Let  $G$  be a group that splits as a finite graph of groups such that each vertex group  $G_v$  is assigned a finite collection of subgroups  $\mathcal{H}_v$ , and each edge group  $G_e$  is conjugate to a subgroup of some  $H \in \mathcal{H}_v$ , if  $e$  is adjacent to  $v$ . Then there is a finite collection of subgroups  $\mathcal{H}$  of  $G$  such that

1. If each  $(G_v, \mathcal{H}_v)$  is in  $\mathcal{C}$ , then  $(G, \mathcal{H})$  is in  $\mathcal{C}$ .
2. If each  $(G_v, \mathcal{H}_v)$  is in  $\mathcal{D}$ , then  $(G, \mathcal{H})$  is in  $\mathcal{D}$ .
3. For any vertex  $v$  and for any  $g \in G_v$ , the element  $g$  is conjugate to an element in some  $Q \in \mathcal{H}_v$ , if and only if  $g$  is conjugate to an element in some  $H \in \mathcal{H}$ .

That edge groups are not assumed to be finitely generated and that they do not necessarily belong to a peripheral collection of subgroups of an adjacent vertex are the main differences between this work and previous results in the literature. The method of proof provides lower and upper bounds of the relative Dehn functions in terms of the relative Dehn functions of the vertex groups. These bounds generalize and improve analogous results in the literature.

## 1. Introduction

Consider the following classes of pairs consisting of a group and a finite collection of subgroups:

- $\mathcal{C} = \{(G, \mathcal{H}) \mid \mathcal{H} \text{ is hyperbolically embedded in } G\}$
- $\mathcal{D} = \{(G, \mathcal{H}) \mid \text{the relative Dehn function of } (G, \mathcal{H}) \text{ is well-defined}\}.$

**Theorem 1.1.** *Let  $G$  be a group that splits as a finite graph of groups such that each vertex group  $G_v$  is assigned a finite collection of subgroups  $\mathcal{H}_v$ , and each edge group  $G_e$  is conjugate to a subgroup of some  $H \in \mathcal{H}_v$ , if  $e$  is adjacent to  $v$ . Then there is a finite collection of subgroups  $\mathcal{H}$  of  $G$  such that*

1. *If each  $(G_v, \mathcal{H}_v)$  is in  $\mathcal{C}$ , then  $(G, \mathcal{H})$  is in  $\mathcal{C}$ .*
2. *If each  $(G_v, \mathcal{H}_v)$  is in  $\mathcal{D}$ , then  $(G, \mathcal{H})$  is in  $\mathcal{D}$ .*
3. *For any vertex  $v$  and for any  $g \in G_v$ , the element  $g$  is conjugate in  $G_v$  to an element of some  $Q \in \mathcal{H}_v$ , if and only if  $g$  is conjugate in  $G$  to an element of some  $H \in \mathcal{H}$ .*

The theorem is trivial without the third item in the conclusion; indeed, the pair  $(G, \{G\})$  belongs to both  $\mathcal{C}$  and  $\mathcal{D}$ . In comparison with previous results in the literature, our main contribution is that our combination results do not assume that edge groups are finitely generated or contained in  $\mathcal{H}_v$ .

The notion of a hyperbolically embedded collection of subgroups was introduced by Dahmani, Guirardel, and Osin [9]. A pair  $(G, \mathcal{H})$  in  $\mathcal{C}$  is called a *hyperbolically embedded pair*, and we write  $\mathcal{H} \hookrightarrow_h G$ . Our combination results for hyperbolically embedded pairs  $(G, \mathcal{H})$  generalize analogous results for relatively hyperbolic pairs in [1, 7, 8, 16, 17] and for hyperbolically embedded pairs [9, 14].

The notions of finite relative presentation and relative Dehn function  $\Delta_{G,\mathcal{H}}$  of a group  $G$  with respect to a collection of subgroups  $\mathcal{H}$  were introduced by Osin [18] generalizing the notions of finite presentation and Dehn function of a group. A pair  $(G, \mathcal{H})$  is called *finitely presented* if  $G$  is finitely presented relative to  $\mathcal{H}$ , and  $\Delta_{G,\mathcal{H}}$  is called the *Dehn function of the pair*  $(G, \mathcal{H})$ . While a finitely presented group has a well-defined Dehn function; in contrast, the Dehn function of a finitely presented pair  $(G, \mathcal{H})$  is not always well defined, for a characterization see [12, Thm.E(2)]. Our result generalizes combination results for pairs  $(G, \mathcal{H})$  with well-defined Dehn function by Osin [17, Thms. 1.2 and 1.3].

We prove Theorem 1.1 for the case of graphs of groups with a single edge, since then the general case follows directly by induction on the number of edges of the graph. This particular case splits into three subcases corresponding to the three results stated below. The proofs of these subcases use characterizations of pairs  $(G, \mathcal{H})$  being hyperbolically embedded [15, Thm. 5.9] and having a well defined Dehn function [12, Thm. 4.7] in terms of existence of  $G$ -graphs with certain properties that relate to Bowditch’s fineness [5]. These characterizations are discussed in Section 2. The proof of Theorem 1.1 for the case of a graph of groups with a single edge entails the construction of graphs satisfying the conditions of those characterizations for the fundamental group of the graph of groups. We use the existing graphs for the vertex groups as building blocks.

Our method of proof provides lower and upper bounds for the relative Dehn function of the fundamental group of the graph of groups in the terms of the relative Dehn functions of the vertex groups; see Section 6. Specifically, Theorem 1.6 below generalizes results of Brick [6] on bounds for the Dehn functions of free products (see the improvement by Guba and Sapir [11]) and improve the bounds found by Osin for relative Dehn functions in [18, Thms 1.2 and 1.3].

Our main result reduces to the following statements.

**Theorem 1.2** (Amalgamated Product). *For  $i \in \{1, 2\}$ , let  $(G_i, \mathcal{H}_i \cup \{K_i\})$  be a pair and  $\partial_i : C \rightarrow K_i$  a group monomorphism. Let  $G_1 *_C G_2$  denote the amalgamated product determined by  $G_1 \xleftarrow{\partial_1} C \xrightarrow{\partial_2} G_2$ , and let  $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ . Then:*

1. *If  $\mathcal{H}_i \cup \{K_i\} \hookrightarrow_h G_i$  for each  $i$ , then  $\mathcal{H} \cup \{K_1, K_2\} \hookrightarrow_h G_1 *_C G_2$ .*
2. *If  $(G_i, \mathcal{H}_i \cup \{K_i\}) \in \mathcal{D}$  for each  $i$ , then  $(G_1 *_C G_2, \mathcal{H} \cup \{K_1, K_2\}) \in \mathcal{D}$ .*
3. *For any  $g \in G_i$ , the element  $g$  is conjugate in  $G_i$  to an element of some  $Q \in \mathcal{H}_i \cup \{K_i\}$  if and only if  $g$  is conjugate in  $G$  to an element of some  $H \in \mathcal{H} \cup \{K_1, K_2\}$ .*

In the following statements, for a subgroup  $K$  of a group  $G$  and an element  $g \in G$ , the conjugate subgroup  $gKg^{-1}$  is denoted by  $K^g$ .

**Theorem 1.3** (HNN-extension I). *Let  $(G, \mathcal{H} \cup \{K, L\})$  be a pair with  $K \neq L$ ,  $C$  a subgroup of  $K$ , and  $\varphi : C \rightarrow L$  a group monomorphism. Let  $G *_\varphi$  denote the HNN-extension  $\langle G, t \mid tct^{-1} = \varphi(c) \text{ for all } c \in C \rangle$ . Then:*

1. *If  $\mathcal{H} \cup \{K, L\} \hookrightarrow_h G$  then  $\mathcal{H} \cup \{K^t, L\} \hookrightarrow_h G *_\varphi$ .*
2. *If  $(G, \mathcal{H} \cup \{K, L\}) \in \mathcal{D}$ , then  $(G *_\varphi, \mathcal{H} \cup \{K^t, L\}) \in \mathcal{D}$ .*
3. *For any  $g \in G$ , the element  $g$  is conjugate in  $G$  to an element of some  $Q \in \mathcal{H} \cup \{K, L\}$  if and only if  $g$  is conjugate in  $G *_\varphi$  to an element of some  $H \in \mathcal{H} \cup \{K^t, L\}$ .*

Note that the third items of Theorems 1.2 and 1.3 follow directly from standard arguments in combinatorial group theory. This article focuses on proving the other statements.

**Corollary 1.4** (HNN-extension II). *Let  $(G, \mathcal{H} \cup \{K\})$  be a pair,  $C$  a subgroup of  $K$ ,  $s \in G$ , and  $\varphi : C \rightarrow K^s$  a group monomorphism. Let  $G*_{\varphi}$  denote the HNN-extension  $\langle G, t \mid tct^{-1} = \varphi(c) \text{ for all } c \in C \rangle$ . Then:*

1. *If  $\mathcal{H} \cup \{K\} \hookrightarrow_h G$  then  $\mathcal{H} \cup \{\langle K, s^{-1}t \rangle\} \hookrightarrow_h G*_{\varphi}$ .*
2. *If  $(G, \mathcal{H} \cup \{K\}) \in \mathcal{D}$ , then  $(G*_{\varphi}, \mathcal{H} \cup \{\langle K, s^{-1}t \rangle\}) \in \mathcal{D}$ .*
3. *For any  $g \in G$ , the element  $g$  is conjugate in  $G$  to an element of some  $Q \in \mathcal{H} \cup \{K\}$  if and only if  $g$  is conjugate in  $G*_{\varphi}$  to an element of some  $H \in \mathcal{H} \cup \{\langle K, s^{-1}t \rangle\}$ .*

*Proof.* First, we prove the statement in the case that  $s$  is the identity element of  $G$ . Let  $L$  be the HNN-extension  $L = K*_{\varphi}$ . Observe that there is a natural isomorphism between  $G*_{\varphi}$  and the amalgamated product  $G *_K L$ . In this case, the conclusion of the corollary is obtained directly by invoking Theorem 1.2, since the pair  $(L, \{L\})$  is in both classes  $\mathcal{C}$  and  $\mathcal{D}$ .

Now we argue in the case that  $s \in G$  is arbitrary. Let  $\psi : C \rightarrow K$  the composition  $I_s \circ \varphi$  where  $I_s$  is the inner automorphism  $I_s(x) = s^{-1}xs$ . Since

$$G*_{\varphi} = \langle G, t \mid c^{s^{-1}t} = \varphi(c)^{s^{-1}} \text{ for all } c \in C \rangle,$$

there is a natural isomorphism  $G*_{\varphi} \rightarrow G*_{\psi}$  which restricts to the identity on the base group  $G$ , and the stable letter of  $G*_{\psi}$  corresponds to  $s^{-1}t$  in  $G*_{\varphi}$ . Since  $\psi$  maps  $C \leq K$  into  $K$ , we have reduced the case of arbitrary  $s \in G$  to the case that  $s$  is the identity in  $G$  and the statement of the corollary follows.  $\square$

Let us describe the argument proving our main result using the three previous statements. The argument relies on the following observation.

**Remark 1.5.** *If a pair  $(G, \mathcal{H} \cup \{L\})$  belongs to  $\mathcal{C}$  (respectively  $\mathcal{D}$ ) and  $g \in G$  then  $(G, \mathcal{H} \cup \{L^g\})$  belongs to  $\mathcal{C}$  (respectively  $\mathcal{D}$ ). This statement can be seen directly from the original definitions of hyperbolically embedded collection of subgroups [9] and relative Dehn function [18]. It can be also deduced directly from Theorems 2.2 and 2.9, respectively, in the main body of the article.*

*Proof of Theorem 1.1.* The case of a tree of groups satisfying the hypothesis of the theorem follows from Theorem 1.2 and Remark 1.5. Then the general case reduces to the case of a graph of groups with a single vertex, where the vertex group corresponds to the fundamental group of a maximal tree of groups. In the case of a graph of groups with a single vertex, each edge corresponds to applying either Theorem 1.3 or Corollary 1.4 together with Remark 1.5.  $\square$

The following theorem generalizes results of Brick [6, Proposition 3.2] on bounds on Dehn functions of free products and improve bounds for relative Dehn functions found by Osin [18, Theorems 1.2 and 1.3].

**Theorem 1.6.**

1. *Under the assumptions of Theorem 1.2(2), if  $\Delta$  is a relative Dehn function of  $(G_1 *_C G_2, \mathcal{H} \cup \{\langle K_1, K_2 \rangle\})$  and  $\Delta_i$  is a relative Dehn function of  $(G_i, \mathcal{H}_i \cup \{K_i\})$  then*

$$\max\{\Delta_1, \Delta_2\} \leq \Delta \leq \max\{\overline{\Delta_1}, \overline{\Delta_2}\},$$

where  $\overline{\Delta_i}$  denotes the super-additive closure of  $\Delta_i$ .

2. *Under the assumptions of Theorem 1.3(2), if  $\Delta$  is a relative Dehn function of  $(G*_{\varphi}, \mathcal{H} \cup \{\langle K', L \rangle\})$  and  $\Delta_0$  is a relative Dehn function of  $(G, \mathcal{H} \cup \{K, L\})$  then*

$$\Delta_0 \leq \Delta \leq \overline{\Delta_0},$$

where  $\overline{\Delta_0}$  is the super-additive closure of  $\Delta_0$

3. Under the assumptions of Corollary 1.4(2), if  $\Delta$  is a relative Dehn function of  $(G*_\varphi, \mathcal{H} \cup \{K, s^{-1}t\})$  and  $\Delta_0$  is a relative Dehn function of  $(G, \mathcal{H} \cup \{K\})$  then

$$\Delta_0 \leq \Delta \leq \overline{\Delta_0},$$

where  $\overline{\Delta_0}$  is the super-additive closure of  $\Delta_0$

We conclude the introduction with a more detailed comparison of our results with previous results in the literature.

1. Dahmani, Guirardel, and Osin proved Theorem 1.2(1) in the case that  $\partial_1 : C \rightarrow K_1$  is an isomorphism and  $K_1$  is finitely generated [9, Thm 6.20]; and Theorem 1.3(1) in the case that  $C = K$  and  $K$  is finitely generated [9, Thm 6.19].
2. Osin proved Theorem 1.2(2) in the case that  $\partial_1 : C \rightarrow K_1$  is an isomorphism and  $K_1$  is finitely generated, see [17, Thm 1.3]; and Theorem 1.3(2) in the case that  $C = K$  and  $K$  is finitely generated, see [17, Thm 1.2].
3. Under the assumptions of Theorem 1.1, if each  $(G_v, \mathcal{H}_v) \in \mathcal{C}$  for every vertex  $v$ , and there is at least one  $v$  such that  $\mathcal{H}_v$  is nontrivial in  $G_v$ , the existence of a nontrivial collection  $\mathcal{H}$  such that  $(G, \mathcal{H}) \in \mathcal{C}$  follows from results of Minasyan and Osin [14, Cor. 2.2 and 2.3] and the characterization of acylindrical hyperbolicity in terms of existence of proper infinite hyperbolically embedded subgroups by Osin [19]; by a nontrivial collection we mean that it contains a proper infinite subgroup. This alternative approach does not guarantee that the collection  $\mathcal{H}$  satisfies the third condition of Theorem 1.1.
4. Theorems 1.2(1) and 1.3(1), in the case that  $G_i$  is hyperbolic relative to  $\mathcal{H}_i$  for  $i = 1, 2$ , follow from results of Wise and the first author [7, Thm. A].

### 1.1. Organization

The rest of the article consists of five sections. In Section 2, we review characterizations of pairs  $(G, \mathcal{H})$  being hyperbolically embedded and having well-defined Dehn functions in terms of actions on graphs. In Section 3, we reduce the proof of Theorems 1.2 and 1.3 to prove two technical results, Theorems 3.1 and 3.2. Their proofs are the content of Sections 4 and 5, respectively. The last section contains the proof of Theorem 1.6.

## 2. Characterizations using fineness

In this section, we describe a characterization of pairs  $(G, \mathcal{H})$  being hyperbolically embedded, Theorem 2.2; and a characterization of the pairs having a well-defined Dehn function, Theorem 2.9. These characterizations are in terms of existence of  $G$ -graphs with certain properties that relate to Bowditch's fineness [5], a notion that is defined below. The characterizations are re-statements of previous results in the literature [15, Thm. 5.9] and [12, Thm. 4.7]. This section also includes a couple of lemmas that will be of use in later sections.

All graphs  $\Gamma = (V, E)$  considered in this section are simplicial, so we consider the set of edges  $E$  to be a collection of subsets of cardinality two of the vertex set  $V$ .

Let  $\Gamma$  be a simplicial graph, let  $v$  be a vertex of  $\Gamma$ , and let  $T_v\Gamma$  denote the set of the vertices adjacent to  $v$ . For  $x, y \in T_v\Gamma$ , the angle metric  $\angle_v(x, y)$  is the combinatorial length of the shortest path in the graph  $\Gamma - \{v\}$  between  $x$  and  $y$ , with  $\angle_v(x, y) = \infty$  if there is no such path. The graph  $\Gamma$  is fine at  $v$  if  $(T_v\Gamma, \angle_v)$  is a locally finite metric space. A graph is *fine* if it is fine at every vertex.

It is an observation that a graph  $\Gamma$  is fine if and only if for every pair of vertices  $x, y$  and every positive integer  $n$ , there are finitely many embedded paths between  $x$  and  $y$  of length at most  $n$ ; for a proof see [5].

### 2.1. Hyperbolically embedded pairs

In [19, Definition 2.9], Osin defines the notion of a collection of subgroups  $\mathcal{H}$  being hyperbolically embedded into a group  $G$ . This relation is denoted as  $\mathcal{H} \hookrightarrow_h G$  and, in this case, we say that the pair  $(G, \mathcal{H})$  is a hyperbolically embedded pair. In this article, we use the following characterization of hyperbolically embedded collection proved in [15] as our working definition.

**Definition 2.1** (Proper pair). *A pair  $(G, \mathcal{H})$  is proper if  $\mathcal{H}$  is a finite collection of subgroups such that no two distinct infinite subgroups are conjugate in  $G$ .*

**Theorem 2.2** (Criterion for hyperbolically embedded pairs). [15, Theorem 5.9] *A proper pair  $(G, \mathcal{H})$  is a hyperbolically embedded pair if and only if there is a connected  $G$ -graph  $\Gamma$  such that*

1. *There are finitely many  $G$ -orbits of vertices.*
2. *Edge  $G$ -stabilizers are finite.*
3. *Vertex  $G$ -stabilizers are either finite or conjugates of subgroups in  $\mathcal{H}$ .*
4. *Every  $H \in \mathcal{H}$  is the  $G$ -stabilizer of a vertex of  $\Gamma$ .*
5.  *$\Gamma$  is hyperbolic.*
6.  *$\Gamma$  is fine at  $V_\infty(\Gamma) = \{v \in V(\Gamma) \mid v \text{ has infinite stabilizer}\}$ .*

**Definition 2.3.** *We refer to a graph  $\Gamma$  satisfying the conditions of Theorem 2.2 as a  $(G, \mathcal{H})$ -graph*

Let us observe that in [15], Theorem 2.2 is proved for the case that  $\mathcal{H}$  consists of a single infinite subgroup, and the authors observe that the argument in the case that  $\mathcal{H}$  is a finite collection of infinite subgroups (such that no pair of distinct infinite subgroups in  $\mathcal{H}$  are conjugate in  $G$ ) follows by the same argument. Then the general case in which  $\mathcal{H}$  is a finite collection of subgroups follows from the following statement: if  $\mathcal{H}$  is a collection of subgroups and  $K$  a finite subgroup of a group  $G$ , then:

1.  $\mathcal{H} \hookrightarrow_h G$  if and only if  $\mathcal{H} \cup \{K\} \hookrightarrow_h G$ .
2. There is  $(G, \mathcal{H})$ -graph if and only if there is a  $(G, \mathcal{H} \cup \{K\})$ -graph.

The first statement is a direct consequence of the definition of hyperbolically embedded collection by Osin [19]. The if part of the second statement is trivial, and the only if part follows directly from [2, Thm. 3.4].

### 2.2. Relative presentations

In [18, Chapter 2], Osin introduces the notions of relative presentation of a group with respect to a collection of subgroups, and relative Dehn functions. We briefly recall these notions below.

Let  $G$  be a group and let  $\mathcal{H}$  be a collection of subgroups. A subset  $S$  of  $G$  is a *relative generating set* of  $G$  with respect to  $\mathcal{H}$  if the natural homomorphism

$$F(S, \mathcal{H}) = F(S) *_{H \in \mathcal{H}} H \longrightarrow G \tag{1}$$

is surjective, where  $F(S)$  denotes the free group with free generating set  $S$ . A relative generating set of  $G$  with respect to  $\mathcal{H}$  is called a *generating set of the pair  $(G, \mathcal{H})$* . A pair that admits a finite generating set is called a *finitely generated pair*. Let  $R \subseteq F(S, \mathcal{H})$  be a subset that normally generates the kernel of the above homomorphism. In this case, we have a short exact sequence of groups

$$1 \rightarrow \langle\langle R \rangle\rangle \rightarrow F(S, \mathcal{H}) \rightarrow G \rightarrow 1,$$

and the triple

$$\langle S, \mathcal{H} \mid R \rangle \tag{2}$$

is called a *relative presentation of  $G$  with respect to  $\mathcal{H}$* , or just a *presentation of the pair  $(G, \mathcal{H})$* . Abusing notation, we write  $G = \langle S, \mathcal{H} \mid R \rangle$ . If both  $S$  and  $R$  are finite we say that the pair  $(G, \mathcal{H})$  is *finitely presented*.

**Lemma 2.4.** *Let  $G$  be a group and let  $\mathcal{H}_0 \sqcup \mathcal{H}$  be a collection of subgroups. Let  $P$  denote the subgroup of  $G$  generated by  $S_0$  and the subgroups in  $\mathcal{H}_0$ . If*

$$G = \langle S_0 \cup S, \mathcal{H}_0 \cup \mathcal{H} \mid R_0 \cup R \rangle \quad \text{and} \quad P = \langle S_0, \mathcal{H}_0 \mid R_0 \rangle$$

then

$$G = \langle S, \mathcal{H} \cup \{P\} \mid R' \rangle,$$

where  $R'$  is the image of  $R$  under the natural epimorphism  $\varphi : F(S_0 \cup S, \mathcal{H}_0 \cup \mathcal{H}) \rightarrow F(S, \mathcal{H} \cup \{P\})$ .

*Proof.* Let  $A = F(S, \mathcal{H})$ ,  $B = F(S_0, \mathcal{H}_0)$ ,  $K$  the normal subgroup of  $B$  generated by  $R_0$  and  $N$  the normal subgroup of  $A * B = F(S_0 \cup S, \mathcal{H}_0 \cup \mathcal{H})$  generated by  $R$ . Our hypotheses imply that the natural epimorphisms  $A * B \rightarrow G$  and  $B \rightarrow P$  induce short exact sequences:

$$1 \rightarrow \langle\langle N, K \rangle\rangle \rightarrow A * B \rightarrow G \rightarrow 1, \quad \text{and} \quad 1 \rightarrow K \rightarrow B \rightarrow P \rightarrow 1.$$

Let us identify  $P = B/K$ . The natural epimorphism of the statement of the lemma

$$\varphi : A * B \rightarrow A * (B/K)$$

induces an isomorphism:

$$\hat{\varphi} : \frac{A * B}{\langle\langle N, K \rangle\rangle} \rightarrow \frac{A * (B/K)}{\varphi(N)} = \frac{A * P}{\varphi(N)}.$$

By the definition of  $N$ , we have that  $\varphi(N)$  is the normal subgroup of  $A * P$  generated by  $R' = \varphi(R)$ . Therefore, the natural epimorphism  $A * P \rightarrow G$  induces a short exact sequence:

$$1 \rightarrow \langle\langle R' \rangle\rangle \rightarrow A * P \rightarrow G \rightarrow 1$$

which concludes the proof. □

The following pair of lemmas allow us to conclude that certain amalgamated products and HNN-extensions preserve relative finite presentability.

**Lemma 2.5** (Amalgamated products). *For  $i \in \{1, 2\}$ , let  $(G_i, \mathcal{H}_i \cup \{K_i\})$  be a pair,  $\partial_i : C \rightarrow K_i$  a group monomorphism. Let  $G_1 *_C G_2$  denote the amalgamated product determined by  $G_1 \xleftarrow{\partial_1} C \xrightarrow{\partial_2} G_2$ , and  $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ . If*

$$G_i = \langle S_i, \mathcal{H}_i \cup \{K_i\} \mid R_i \rangle$$

then

$$G_1 *_C G_2 = \langle S_1 \cup S_2, \mathcal{H} \cup \{\{K_1, K_2\}\} \mid R_1 \cup R_2 \rangle.$$

*Proof.* Observe that  $\langle S_1 \cup S_2, \mathcal{H} \cup \{K_1, K_2\} \mid R_1 \cup R_2, \partial_1(c) = \partial_2(c) \text{ for all } c \in C \rangle$  is a relative presentation of  $G_1 *_C G_2$ . Since the subgroup  $\langle K_1, K_2 \rangle \leq G_1 *_C G_2$  is isomorphic to the amalgamated product  $K_1 *_C K_2$ , we have that  $\langle K_1, K_2 \mid \partial_1(c) = \partial_2(c) \text{ for all } c \in C \rangle$  is a relative presentation of  $\langle K_1, K_2 \rangle$ . The proof concludes by invoking Lemma 2.4. □

**Lemma 2.6** (HNN-extension). *Let  $(G, \mathcal{H} \cup \{K, L\})$  be a pair with  $K \neq L$ ,  $C$  a subgroup of  $K$ ,  $\varphi : C \rightarrow L$  a group monomorphism, and let  $G *_\varphi$  denote the HNN-extension  $\langle G, t \mid tct^{-1} = \varphi(c) \text{ for all } c \in C \rangle$ . If*

$$G = \langle S, \mathcal{H} \cup \{K, L\} \mid R \rangle$$

then

$$G*_{\varphi} = \langle S, t, \mathcal{H} \cup \{K', L\} \mid R' \rangle,$$

where  $R'$  is the set of relations obtained by taking each element of  $R$  and replacing all occurrences of elements  $k \in K$  by words  $t^{-1}k't$ . In particular,  $R$  and  $R'$  have the same cardinality.

*Proof.* Let  $J$  denote the subgroup  $K'$ , and let  $\psi : K \rightarrow J$  be the isomorphism  $\psi(k) = tkt^{-1}$ . Observe that  $\langle S, t, \mathcal{H} \cup \{K, L\} \mid R, tct^{-1} = \varphi(c) \text{ for all } c \in C \rangle$  is a presentation for the pair  $(G*_{\varphi}, \mathcal{H} \cup \{K, L\})$ . Therefore,

$$G*_{\varphi} = \langle S, t, \mathcal{H} \cup \{J, L\} \mid R', \psi(c) = \varphi(c) \text{ for all } c \in C \rangle.$$

A consequence of Britton's lemma is that the subgroup  $\langle J, L \rangle \leq G*_{\varphi}$  is isomorphic to the amalgamated product  $J *__{\varphi(C)} L$ . Hence,

$$\langle J, L \rangle = \langle \{J, L\} \mid \psi(c) = \varphi(c) \text{ for all } c \in C \rangle.$$

The proof concludes by invoking Lemma 2.4. □

### 2.3. Relative Dehn functions

Suppose that  $\langle S, \mathcal{H} \mid R \rangle$  is a finite relative presentation of the pair  $(G, \mathcal{H})$ . For a word  $W$  over the alphabet  $\mathcal{S} = S \sqcup \bigsqcup_{H \in \mathcal{H}} (H - \{1\})$  representing the trivial element in  $G$ , there is an expression:

$$W = \prod_{i=1}^k f_i^{-1} R f_i \tag{3}$$

where  $R_i \in R$  and  $f_i \in F(\mathcal{S})$ . We say a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a *relative isoperimetric function* of the relative presentation  $\langle S, \mathcal{H} \mid R \rangle$  if, for any  $n \in \mathbb{N}$ , and any word  $W$  over the alphabet  $\mathcal{S}$  of length  $\leq n$  representing the trivial element in  $G$ , one can write  $W$  as in (3) with  $k \leq f(n)$ . The smallest relative isoperimetric function of a finite relative presentation  $\langle S, \mathcal{H} \mid R \rangle$  is called the *relative Dehn function of  $G$  with respect to  $\mathcal{H}$* , or the *Dehn function of the pair  $(G, \mathcal{H})$* . This function is denoted by  $\Delta_{G, \mathcal{H}}$ . Theorem 2.7 below justifies the notation  $\Delta_{G, \mathcal{H}}$  for the Dehn function of a finitely presented pair  $(G, \mathcal{H})$ .

For functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $f \leq g$  if there exist constants  $C, K, L \in \mathbb{N}$  such that  $f(n) \leq Cg(Kn) + Ln$  for every  $n$ . We say  $f$  and  $g$  are *asymptotically equivalent*, denoted as  $f \asymp g$ , if  $f \leq g$  and  $g \leq f$ .

**Theorem 2.7.** [18, Theorem 2.34] *Let  $G$  be a finitely presented group relative to the collection of subgroups  $\mathcal{H}$ . Let  $\Delta_1$  and  $\Delta_2$  be the relative Dehn functions associated with two finite relative presentations. If  $\Delta_1$  takes only finite values, then  $\Delta_2$  takes only finite values, and  $\Delta_1 \asymp \Delta_2$ .*

The Dehn function of a pair  $(G, \mathcal{H})$  is *well defined* if it takes only finite values. This can be characterized in terms of fine graphs as follows.

**Definition 2.8** (Cayley–Abels graph for pairs). *A Cayley–Abels graph of the pair  $(G, \mathcal{H})$  is a connected cocompact simplicial  $G$ -graph  $\Gamma$  such that:*

1. *edge  $G$ -stabilizers are finite,*
2. *vertex  $G$ -stabilizers are either finite or conjugates of subgroups in  $\mathcal{H}$ ,*
3. *every  $H \in \mathcal{H}$  is the  $G$ -stabilizer of a vertex of  $\Gamma$ , and*
4. *any pair of vertices of  $\Gamma$  with the same  $G$ -stabilizer  $H \in \mathcal{H}$  are in the same  $G$ -orbit if  $H$  is infinite.*



**Theorem 2.9.** *Let  $(G, \mathcal{H})$  be a proper pair. The following statements are equivalent.*

1. *The Dehn function  $\Delta_{G,\mathcal{H}}$  is well defined.*
2.  *$(G, \mathcal{H})$  is finitely presented and there is a fine Cayley–Abels graph of  $(G, \mathcal{H})$ .*
3.  *$(G, \mathcal{H})$  is finitely presented and every Cayley–Abels graph of  $(G, \mathcal{H})$  is fine.*

Theorem 2.9 is essentially [12, Theorem E] together with a result on Cayley–Abels graphs from [2, Theorem H]. This is described below.

Concrete examples of Cayley–Abels graphs can be exhibited using the following construction introduced by Farb [10]; see also [13].

**Definition 2.10** (Coned-off Cayley graph). *Let  $(G, \mathcal{H})$  be a pair, and let  $S$  be a finite relative generating set of  $G$  with respect to  $\mathcal{H}$ . Denote by  $G/\mathcal{H}$  the set of all cosets  $gH$  with  $g \in G$  and  $H \in \mathcal{H}$ . The coned-off Cayley graph  $\hat{\Gamma}(G, \mathcal{H}, S)$  is the graph with vertex set  $G \cup G/\mathcal{H}$  and edges of the following type*

- $\{g, gs\}$  for  $s \in S$  and  $g \in G$ ,
- $\{x, gH\}$  for  $g \in G, H \in \mathcal{H}$  and  $x \in gH$ .

That a pair  $(G, \mathcal{H})$  has a well-defined function is characterized in terms of fineness of coned-off Cayley graphs.

**Theorem 2.11.** [12, Theorem E] *Let  $(G, \mathcal{H})$  be a finitely presented pair with a finite generating set  $S$ . The Dehn function  $\Delta_{G,\mathcal{H}}$  is well defined if and only if the coned-off Cayley graph  $\hat{\Gamma}(G, \mathcal{H}, S)$  is fine.*

Every coned-off Cayley graph  $\hat{\Gamma}(G, \mathcal{H}, S)$  with  $S$  a finite relative generating set is a Cayley–Abels graph. The following result implies that coned-off Cayley graphs are, up to quasi-isometry, independent of the choice of finite generating set, and we denote them by  $\hat{\Gamma}(G, \mathcal{H})$ . Observe now that Theorem 2.9 also follows from the following result.

**Theorem 2.12.** [2, Theorem H] *If  $\Gamma$  and  $\Delta$  are Cayley–Abels graphs of the proper pair  $(G, \mathcal{H})$ , then:*

1.  $\Gamma$  and  $\Delta$  are quasi-isometric, and
2.  $\Gamma$  is fine if and only if  $\Delta$  is fine.

### 3. Combination theorems for graphs

In this section, we state two technical results, Theorems 3.1 and 3.2, which will be proven in the subsequent sections. The section includes how to deduce the main results of the article, Theorems 1.2 and 1.3, from these technical results.

**Theorem 3.1.** *For  $i \in \{1, 2\}$ , let  $(G_i, \mathcal{H}_i \cup \{K_i\})$  be a pair and  $\partial_i : C \rightarrow K_i$  a group monomorphism. Let  $G = G_1 *_C G_2$  denote the amalgamated product determined by  $G_1 \xleftarrow{\partial_1} C \xrightarrow{\partial_2} G_2$ , and  $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ . Let  $\Gamma_i$  be a  $G_i$ -graph that has a vertex  $x_i$  with  $G_i$ -stabilizer  $K_i$ . Then there is a  $G$ -graph  $\Gamma$  with the following properties:*

1.  $\Gamma$  has a vertex  $z$  such that the  $G$ -stabilizer  $G_z = \langle K_1, K_2 \rangle$ , and there is a  $G_i$ -equivariant inclusion  $\Gamma_i \hookrightarrow \Gamma$  that maps  $x_i$  to  $z$ .
2. If  $\Gamma_i$  is connected for  $i = 1, 2$ , then  $\Gamma$  is connected.
3. If every  $H \in \mathcal{H}_i \cup \{K_i\}$  is the  $G_i$ -stabilizer of a vertex of  $\Gamma_i$  for  $i = 1, 2$ , then every  $H \in \mathcal{H} \cup \{K_1, K_2\}$  is the  $G$ -stabilizer of a vertex of  $\Gamma$ .
4. If vertex  $G_i$ -stabilizers in  $\Gamma_i$  are finite or conjugates of subgroups in  $\mathcal{H}_i \cup \{K_i\}$  for  $i = 1, 2$ , then vertex  $G$ -stabilizers in  $\Gamma$  are finite or conjugates of subgroups in  $\mathcal{H} \cup \{K_1, K_2\}$ .
5. If  $\Gamma_i$  has finite edge  $G_i$ -stabilizers for  $i = 1, 2$ , then  $\Gamma$  has finite edge  $G$ -stabilizers.



6. If  $\Gamma_i$  has finitely many  $G_i$ -orbits of vertices (edges) for  $i = 1, 2$ , then  $\Gamma$  has finitely many  $G$ -orbits of vertices (resp. edges).
7. If  $\Gamma_i$  is fine for  $i = 1, 2$ , then  $\Gamma$  is fine.
8. If  $\Gamma_i$  is fine at  $V_\infty(\Gamma_i)$  for  $i = 1, 2$ , then  $\Gamma$  is fine at  $V_\infty(\Gamma)$ .
9. If  $\Gamma_i$  is hyperbolic for  $i = 1, 2$ , then  $\Gamma$  is hyperbolic.
10. If  $\Gamma_i$  is simplicial for  $i = 1, 2$ , then  $\Gamma$  is simplicial.

Let us explain how Theorem 1.2 follows from the above result.

*Proof of Theorem 1.2.* For the first statement, suppose  $\mathcal{H}_i \cup \{K_i\}$  is hyperbolically embedded in  $G_i$ . Then  $\mathcal{H}_i \cup \{K_i\}$  is an almost malnormal collection of subgroups of  $G_i$  by [9, Prop. 4.33]. In particular,  $(G_i, \mathcal{H}_i \cup \{K_i\})$  is a proper pair. By Theorem 2.2, there is a  $(G_i, \mathcal{H}_i \cup \{K_i\})$ -graph  $\Gamma_i$ . Let  $x_i$  be a vertex of  $\Gamma_i$  with  $G_i$ -stabilizer  $K_i$ . Applying Theorem 3.1 to  $\Gamma_1, \Gamma_2, x_1$  and  $x_2$ , we obtain a  $(G_1 *_C G_2, \mathcal{H} \cup \{\langle K_1, K_2 \rangle\})$ -graph. Note that  $(G_1 *_C G_2, \mathcal{H} \cup \{\langle K_1, K_2 \rangle\})$  is a proper pair by a standard argument using normal forms. Then invoke Theorem 2.2 to obtain that  $\mathcal{H} \cup \{\langle K_1, K_2 \rangle\}$  is hyperbolically embedded in  $G_1 *_C G_2$ .

The second statement is proved analogously. Suppose the relative Dehn function of  $(G_i, \mathcal{H}_i \cup \{K_i\})$  is well defined. By [17, Prop. 2.36], the pair  $(G_i, \mathcal{H}_i \cup \{K_i\})$  is proper. It follows that  $(G_1 *_C G_2, \mathcal{H} \cup \{\langle K_1, K_2 \rangle\})$  is also a proper pair by a standard argument using normal forms. By Theorem 2.9,  $(G_i, \mathcal{H}_i \cup \{K_i\})$  is finitely presented and admits a fine Cayley–Abels graph  $\Gamma_i$ . In particular, there is a vertex  $x_i \in \Gamma_i$  with  $G_i$ -stabilizer equal to  $K_i$ . Apply Theorem 3.1 to  $\Gamma_1, \Gamma_2$  and the vertices  $x_1, x_2$  to obtain a fine Cayley–Abels graph  $\Gamma$  for the pair  $(G_1 *_C G_2, \mathcal{H} \cup \{\langle K_1, K_2 \rangle\})$ . Since  $(G_1 *_C G_2, \mathcal{H} \cup \{\langle K_1, K_2 \rangle\})$  is finitely presented by Lemma 2.5, then Theorem 2.9 implies that the relative Dehn function of  $(G_1 *_C G_2, \mathcal{H} \cup \{\langle K_1, K_2 \rangle\})$  is well defined.  $\square$

**Theorem 3.2.** *Let  $(G, \mathcal{H} \cup \{K, L\})$  be a pair with  $K \neq L, C \leq K$ , and  $\varphi : C \rightarrow L$  a group monomorphism. Let  $G*_\varphi$  denote the HNN-extension  $\langle G, t \mid tct^{-1} = \varphi(c) \text{ for all } c \in C \rangle$ . Let  $\Delta$  be a  $G$ -graph that has vertices  $x$  and  $y$  such that their  $G$ -stabilizers are  $K$  and  $L$ , respectively, and their  $G$ -orbits are disjoint. Then there is a  $G*_\varphi$ -graph  $\Gamma$  with the following properties:*

1.  $\Gamma$  has a vertex  $z$  such that  $G_z = \langle K^t, L \rangle$ , and there is a  $G$ -equivariant inclusion  $\Delta \hookrightarrow \Gamma$  such that  $x \mapsto t^{-1} \cdot z$  and  $y \mapsto z$ .
2. If  $\Delta$  is connected, then  $\Gamma$  is connected.
3. If every  $H \in \mathcal{H} \cup \{K, L\}$  is the  $G$ -stabilizer of a vertex of  $\Delta$ , then every  $H \in \mathcal{H} \cup \{\langle K^t, L \rangle\}$  is the  $G*_\varphi$ -stabilizer of a vertex of  $\Gamma$ .
4. If vertex  $G$ -stabilizers in  $\Delta$  are finite or conjugates of subgroups in  $\mathcal{H} \cup \{K, L\}$ , then vertex  $G*_\varphi$ -stabilizers in  $\Gamma$  are finite or conjugates of subgroups in  $\mathcal{H} \cup \{\langle K^t, L \rangle\}$ .
5. If  $\Delta$  has finite edge  $G$ -stabilizers, then  $\Gamma$  has finite edge  $G*_\varphi$ -stabilizers.
6. If  $\Delta$  has finitely many  $G$ -orbits of vertices (edges), then  $\Gamma$  has finitely many  $G*_\varphi$ -orbits of vertices (resp. edges).
7. If  $\Delta$  is fine, then  $\Gamma$  is fine.
8. If  $\Delta$  is fine at  $V_\infty(\Delta)$ , then  $\Gamma$  is fine at  $V_\infty(\Gamma)$ .
9. If  $\Delta$  is hyperbolic, then  $\Gamma$  is hyperbolic.

*Proof of Theorem 1.3.* This proof is completely analogous to the proof of Theorem 1.2: invoke Theorem 3.2 and Lemma 2.6 instead of Theorem 3.1 and Lemma 2.5, respectively.  $\square$

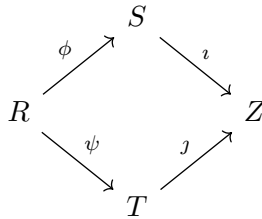
#### 4. Amalgamated products and graphs

This section describes an argument proving Theorem 3.1. While the statement of this result seems intuitive, we are not aware of a full account of those techniques in a common framework, so this section provides a detailed construction.

4.1. Pushouts in the category of  $G$ -sets

Let  $\phi : R \rightarrow S$  and  $\psi : R \rightarrow T$  be  $G$ -maps. The pushout of  $\phi$  and  $\psi$  is defined as follows. Let  $Z$  be the  $G$ -set obtained as the quotient of the disjoint union of  $G$ -sets  $S \sqcup T$  by the equivalence relation generated by all pairs  $s \sim t$  with  $s \in S$  and  $t \in T$  satisfying that there is  $r \in R$  such that  $\phi(r) = s$  and  $\psi(r) = t$ . There are canonical  $G$ -maps  $\iota : S \rightarrow Z$  and  $j : T \rightarrow Z$  such that  $\iota \circ \phi = j \circ \psi$ . This construction satisfies the universal property of pushouts in the category of  $G$ -sets.

**Proposition 4.1.** *Let  $\phi : R \rightarrow S$  and  $\psi : R \rightarrow T$  be  $G$ -maps. Consider the pushout*



of  $\phi$  and  $\psi$ . Suppose there is  $r \in R$  such that  $R = G.r$ . If  $s = \phi(r)$ ,  $t = \psi(r)$  and  $z = \iota(s)$ , then the  $G$ -stabilizer  $G_z$  equals the subgroup  $\langle G_s, G_t \rangle$ .

*Proof.* Since  $\iota$  and  $j$  are  $G$ -maps,  $\langle G_s, G_t \rangle \leq G_z$ . Conversely, let  $g \in G_z$ . If  $g \in G_s$  then  $g \in \langle G_s, G_t \rangle$ . Suppose  $g \notin G_s$ .

Let  $r_0$  denote the element  $r \in R$  in the statement, in particular,  $s = \phi(r_0)$ ,  $t = \psi(r_0)$  and  $R = G.r_0$ . Since  $j(t) = \iota(g.s)$ , the definition of  $Z$  as a collection of equivalence classes in  $S \sqcup T$  implies that there is a sequence  $r'_0, r_1, r'_1, \dots, r_k, r'_k$  of elements of  $R$  such that

$$t = \psi(r'_0), \phi(r'_0) = \phi(r_1), \psi(r_1) = \psi(r'_1), \dots, \psi(r_k) = \psi(r'_k), \phi(r'_k) = g.s.$$

Let  $s_i = \phi(r'_{i-1}) = \phi(r_i)$  and  $t_i = \psi(r_i) = \psi(r'_i)$ . Since  $R = G.r_0$ , there are elements  $a_0, a_1, \dots, a_k$  and  $b_0, b_1, \dots, b_{k-1}$  of  $G$  such that

$$a_i.r_i = r'_i \quad \text{and} \quad b_j.r'_j = r_{j+1}$$

for  $0 \leq i \leq k$  and  $0 \leq j < k$ . Then

$$g.s = \phi(r'_k) = \phi(a_k b_{k-1} a_{k-1} \dots b_0 a_0 . r_0) = a_k b_{k-1} a_{k-1} \dots b_0 a_0 . s$$

and hence  $a_k b_{k-1} a_{k-1} \dots b_0 a_0 \in G_s$ . Since  $G_s \leq \langle G_s, G_t \rangle$ , to prove that  $g \in \langle G_s, G_t \rangle$  is enough to show that  $a_i, b_j \in \langle G_s, G_t \rangle$ . We will argue by induction.

First note that since  $\phi$  and  $\psi$  are  $G$ -maps

$$a_i.s_i = s_{i+1} \quad \text{and} \quad b_j.t_j = t_{j+1},$$

and hence

$$G_{s_{i+1}} = a_i G_{s_i} a_i^{-1} \quad \text{and} \quad G_{t_{j+1}} = b_j G_{t_j} b_j^{-1}.$$

Moreover,  $t_i = \psi(r_i) = \psi(r'_i) = \psi(a_i.r_i) = a_i.t_i$  implies

$$a_i \in G_{t_i},$$

and analogously  $s_{j+1} = \phi(r_{j+1}) = \phi(b_j.r'_j) = b_j.s_{j+1}$  implies

$$b_j \in G_{s_{j+1}}.$$

Since  $t_0 = t$  and  $s_0 = s$ , we have that

$$a_0 \in G_{t_0} \leq \langle G_s, G_t \rangle, \quad \text{and} \quad b_0 \in G_{s_1} = a_0 G_{s_0} a_0^{-1} \leq \langle G_s, G_t \rangle.$$

Suppose  $i < k$ ,  $a_i, b_i \in \langle G_s, G_t \rangle$ ,  $G_{s_i} \leq \langle G_s, G_t \rangle$  and  $G_{t_i} \leq \langle G_s, G_t \rangle$ . Then

$$a_{i+1} \in G_{t_{i+1}} = b_i G_{t_i} b_i^{-1} \leq \langle G_s, G_t \rangle,$$

and hence

$$G_{s_{i+1}} = a_i G_{s_i} a_i^{-1} \leq \langle G_s, G_t \rangle.$$

In the case that  $i + 1 < k$ ,

$$b_{i+1} \in G_{s_{i+2}} = a_{i+1} G_{s_{i+1}} a_{i+1}^{-1} \leq \langle G_s, G_t \rangle.$$

Therefore, by induction,  $a_i, b_j \in \langle G_s, G_t \rangle$  for  $0 \leq i \leq k$  and  $0 \leq j < k$ . □

### 4.2. Extending actions on sets

In the case that  $K$  is a subgroup of  $G$  and  $S$  is a  $K$ -set, one can extend the  $K$ -action on  $S$  to a  $G$ -set  $G \times_K S$  that we now describe. Up to isomorphism of  $K$ -sets, we can assume that  $S$  is a disjoint union of  $K$ -sets:

$$S = \bigsqcup_{i \in I} K/K_i$$

where  $K/K_i$  is the  $K$ -set consisting of left cosets of a subgroup  $K_i$  of  $K$ . Then the  $G$ -set  $G \times_K S$  is defined as a disjoint union of  $G$ -sets:

$$G \times_K S := \bigsqcup_{i \in I} G/K_i.$$

Observe that the canonical  $K$ -map

$$\iota : S \rightarrow G \times_K S, \quad K_i \mapsto K_i$$

is injective. This construction satisfies a number of useful properties that we summarize in the following proposition.

For  $n$  a natural number and a set  $X$ , let  $[X]^n$  denote the collection of subsets of  $X$  of cardinality  $n$ . If  $X$  is a  $G$ -set, then  $[X]^n$  is a  $G$ -set with action defined as  $g \cdot \{x_1, \dots, x_n\} = \{g \cdot x_1, \dots, g \cdot x_n\}$ .

**Proposition 4.2.** *Let  $K \leq G$  and  $S$  a  $K$ -set.*

1. *The canonical  $K$ -map  $\iota : S \rightarrow G \times_K S$  induces a bijection of orbit spaces  $S/K \rightarrow (G \times_K S)/G$ .*
2. *For each  $s \in S$ , the  $K$ -stabilizer  $K_s$  equals the  $G$ -stabilizer  $G_{\iota(s)}$ .*
3. *If  $T$  is a  $G$ -set and  $f : S \rightarrow T$  is  $K$ -equivariant, then there is a unique  $G$ -map  $\tilde{f} : G \times_K S \rightarrow T$  such that  $\tilde{f} \circ \iota = f$ .*
4. *If  $\iota(S) \cap g \cdot \iota(S) \neq \emptyset$  for  $g \in G$ , then  $g \in K$  and  $\iota(S) = g \cdot \iota(S)$ .*
5. *In part three, if  $f$  induces an injective map  $S/K \rightarrow T/G$  and  $K_s = G_{f(s)}$  for every  $s \in S$ , then  $\tilde{f}$  is injective.*
6. *Let  $j : [S]^n \rightarrow G \times_K [S]^n$  be the canonical map. Then for every  $n \in \mathbb{N}$ , there is a  $G$ -equivariant injection  $\hat{\iota} : G \times_K [S]^n \rightarrow [G \times_K S]^n$  such that  $\hat{\iota} \circ j = \bar{\iota}$  where  $\bar{\iota} : [S]^n \rightarrow [G \times_K S]^n$  is the natural  $K$ -map induced by  $\iota : S \rightarrow G \times_K S$ .*

*Proof.* The first four statements are observations. For the fifth statement, suppose  $\tilde{f}(\iota(s_1)) = \tilde{f}(g \cdot \iota(s_2))$ . Then  $f(s_1) = g \cdot f(s_2)$ . Since the map  $S/K \rightarrow T/G$  induced by  $f$  is injective, we have that  $s_1$  and  $s_2$  are in the same  $K$ -orbit in  $S$ , say  $s_2 = k \cdot s_1$  for  $k \in K$ . It follows that  $f(s_1) = gk \cdot f(s_1)$ , and since  $K_{s_1} = G_{f(s_1)}$ , we have that  $gk \in K_{s_1}$ . Therefore  $\iota(s_1) = \iota(gk \cdot s_1) = g \cdot \iota(ks_1) = g \cdot \iota(s_2)$ .

The sixth statement is proved as follows. The  $K$ -map  $\iota : S \rightarrow G \times_K S$  naturally induces a  $K$ -map  $\bar{\iota} : [S]^n \rightarrow [G \times_K S]^n$ . By the third statement, there is a unique  $G$ -map  $\hat{\iota} : G \times_K [S]^n \rightarrow [G \times_K S]^n$  such that  $\hat{\iota} \circ j = \bar{\iota}$  where  $j : [S]^n \rightarrow G \times_K [S]^n$ . As a consequence of the fourth statement,  $\iota : [S]^n \rightarrow [G \times_K S]^n$  induces an injective map  $[S]^n/K \rightarrow [G \times_K S]^n/G$  and  $K_A = G_{\iota(A)}$  for every  $A \in [S]^n$ ; therefore,  $\hat{\iota}$  is injective. □

As the reader might have noticed, this construction is an instance of general categorical phenomena; that formulation will have no use in this article so we will not discuss it.

### 4.3. Graphs as 1-dimensional complexes

While the objectives of this section only require us to consider simplicial graphs, the category of simplicial graphs does not have pushouts [20]. For this reason, it is convenient to work within the framework of one-dimensional complexes or equivalently graphs in the sense that we describe below. We will only consider a particular class of pushouts of graphs that behaves well over simplicial graphs. A *graph* is a triple  $(V, E, r)$ , where  $V$  and  $E$  are sets, and  $r : E \rightarrow [V]^2$  is a function where  $[V]^n$  is the collection of nonempty subsets of  $V$  of cardinality at most  $n$ . Elements of the set  $V$  and  $E$  are called *vertices* and *edges*, respectively; the function  $r$  is called the *attaching map*. For a graph  $\Gamma$ , we denote  $V(\Gamma)$  and  $E(\Gamma)$  its vertex and edge set, respectively. If  $v \in V(\Gamma)$ ,  $e \in E(\Gamma)$  and  $v \in r(e)$ , then  $v$  is *incident* to  $e$ , and  $v$  is called an *endpoint* of  $e$ . Vertices incident to the same edge are called *adjacent*.

The graph  $(V, E, r)$  is *simplicial* if every edge has two distinct endpoints and  $r$  is injective. Equivalently,  $(V, E, r)$  is simplicial if  $r : E \rightarrow [V]^2$  is injective and its image does not intersect  $[V]^1$ .

A graph  $\Delta$  is a *subgraph* of a graph  $\Gamma$  if  $V(\Delta) \subset V(\Gamma)$ ,  $E(\Delta) \subset E(\Gamma)$  and  $r_\Delta$  equals the restriction of  $r_\Gamma$  to  $E(\Delta)$ . Abusing notation, we consider any vertex of a graph  $\Gamma$  as an edgeless subgraph with a single vertex, and any edge  $e$  of  $\Gamma$  as the subgraph with vertex set the set of vertices incident to  $e$  in  $\Gamma$  and edge set consisting of only  $e$ .

For a vertex  $u$  of a simplicial graph  $\Gamma = (V, E, r)$ , let  $\text{star}_\Gamma(u)$  denote the subgraph with vertex set  $V(\text{star}(u)) = \{u\} \cup \{v \in V \mid v \text{ is adjacent to } u\}$  and edge set  $E(\text{star}(u)) = \{e \in E \mid \text{the endpoints of } e \text{ belong to } V(\text{star}(u))\}$  and the attaching map is the corresponding restriction of  $r$ .

Our notion of morphism allows the collapse of edges to single vertices. Specifically, a *morphism* of  $\phi : (V, E, r) \rightarrow (V', E', r')$  of graphs is a pair of maps  $\phi_0 : V \rightarrow V'$  and  $\phi_1 : E \rightarrow V' \cup E'$  such that there is a commutative diagram

$$\begin{array}{ccc}
 \phi_1^{-1}(E') & \xrightarrow{\phi_1} & E' \\
 \downarrow r & & \downarrow r' \\
 [V]^2 & \xrightarrow{\phi_0} & [V']^2
 \end{array}
 \qquad
 \begin{array}{ccc}
 \phi_1^{-1}(V') & \xrightarrow{\phi_1} & V' \\
 \downarrow r & & \cong \downarrow \\
 [V]^2 & \xrightarrow{\phi_0} & [V']^1
 \end{array}$$

where the horizontal bottom arrow  $\phi_0$  is the natural  $G$ -map induced by  $\phi_0 : V \rightarrow V'$ , and  $V' \rightarrow [V']^1$  is the natural bijection given by  $v \mapsto \{v\}$ . Observe that in general for a morphism  $\phi = (\phi_0, \phi_1) : \Gamma \rightarrow \Delta$  of graphs, the map  $\phi_0$  does not determine  $\phi_1$ ; however if  $\Delta$  is simplicial then  $\phi_0$  determines  $\phi_1$ . A morphism  $(\phi_0, \phi_1)$  is a *monomorphism* (also called an *embedding*) if both maps are injective.

Given a graph morphism  $\phi = (\phi_0, \phi_1) : \Gamma \rightarrow \Delta$  and a subgraph  $\Theta$  of  $\Delta$ , the *preimage*  $\phi^{-1}(\Theta)$  is the subgraph of  $\Gamma$  with vertex set  $\phi_0^{-1}(V(\Theta))$  and edge set  $\phi_1^{-1}(V(\Theta) \cup E(\Theta))$ .

Let  $G$  be a group. A  $G$ -graph is a graph  $(V, E, r)$  where  $V$  and  $E$  are  $G$ -sets, and  $r$  is a  $G$ -map with respect to the natural  $G$ -action on  $[V]^2$  induced by the  $G$ -set  $V$ . A morphism  $(\phi_0, \phi_1)$  of  $G$ -graphs is a morphism of graphs such that each  $\phi_i$  is a  $G$ -map. A  $G$ -equivariant embedding is a monomorphisms of  $G$ -graphs. A  $G$ -action on a graph  $\Gamma$  has *no inversions* if for every  $e \in E$  and  $g \in G$  such that  $g.e = e$ ,  $g.v = v$  for every  $v \in r(e)$ . For a  $G$ -action without inversions on a graph  $\Gamma$  and  $K \leq G$ , let  $\Gamma^K$  denote subgraph of  $\Gamma$  defined by  $V(\Gamma^K) = \{v \in V(\Gamma) \mid k.v = v \text{ for all } k \in K\}$  and  $E(\Gamma^K) = \{e \in E(\Gamma) \mid k.e = e \text{ for all } e \in K\}$ .

### 4.4. Extending group actions on graphs

Let  $K$  be a subgroup of  $G$ , and let  $\Lambda = (V, E, r)$  be a  $K$ -graph. Define

$$G \times_K \Lambda = (G \times_K V, G \times_K E, \tilde{r})$$

where  $\tilde{r}$  is unique  $G$ -map induced by the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & G \times_K E \\ \downarrow r & & \downarrow \tilde{r} \\ [V]^2 & \xrightarrow{i} & [G \times_K V]^2 \end{array}$$

where  $i : V \hookrightarrow G \times_K V$  and  $j : E \hookrightarrow G \times_K E$  are the canonical  $K$ -maps, see Lemma 4.2(3). Note that there is a *canonical  $K$ -equivariant embedding*

$$\Lambda \hookrightarrow G \times_K \Lambda$$

induced by  $i$  and  $j$ . We consider  $\Lambda$  a  $K$ -subgraph of  $G \times_K \Lambda$ .

**Remark 4.3.** Proposition 4.2, parts 2 and 4 imply:

1. If  $\Lambda$  is a simplicial  $K$ -graph without inversions, then  $G \times_K \Lambda$  is a simplicial  $G$ -graph without inversions.
2. For any connected subgraph  $\Delta$  of  $G \times_K \Lambda$ , there is  $g \in G$  such that  $g.\Delta$  is a subcomplex of  $\Lambda$ , in a commutative diagram,

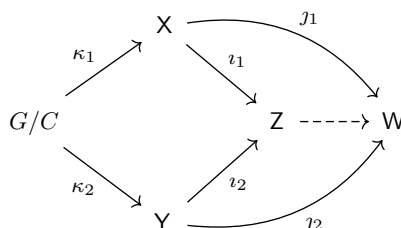
$$\begin{array}{ccc} G \times_K \Lambda & \xrightarrow{g} & G \times_K \Lambda \\ \uparrow & & \uparrow \\ \Delta & \xrightarrow{g} & \Lambda \end{array}$$

In particular, if  $\Lambda$  is connected, then every connected component of  $G \times_K \Lambda$  is isomorphic to  $\Lambda$ .

### 4.5. Pushouts of graphs

Let  $X$  and  $Y$  be  $G$ -graphs, let  $C \leq G$  be a subgroup and suppose  $X^C$  and  $Y^C$  are nonempty. Let  $x \in X^C$  and  $y \in Y^C$  be vertices. The  $C$ -pushout  $Z$  of  $X$  and  $Y$  with respect to the pair  $(x, y)$  is the  $G$ -graph  $Z$  obtained by taking the disjoint union of  $X$  and  $Y$  and then identifying the vertex  $g.x$  with the vertex  $g.y$  for every  $g \in G$ .

Equivalently, the  $C$ -pushout  $Z$  of  $X$  and  $Y$  with respect to the pair  $(x, y)$  is the  $G$ -graph  $Z$  whose vertex set  $V(Z)$  is the pushout of the  $G$ -maps  $\kappa_1 : G/C \rightarrow V(X)$  and  $\kappa_2 : G/C \rightarrow V(Y)$  given by  $C \mapsto x$  and  $C \mapsto y$ ; and edge set the disjoint union of the  $G$ -sets  $E(X)$  and  $E(Y)$ , and attaching map  $E(Z) \rightarrow V(Z)^2$  defined as the union of the attaching maps for  $X$  and  $Y$  postcomposed with the maps  $V(X) \rightarrow V(Z)$  and  $V(Y) \rightarrow V(Z)$  defining the pushout.



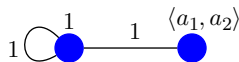
The standard universal property of pushouts holds for this construction: if  $J_1 : X \rightarrow W$  and  $J_2 : Y \rightarrow W$  are morphisms of  $G$ -graphs such that  $J_1 \circ \kappa_1 = J_2 \circ \kappa_2$ , then there is a unique morphism of  $G$ -graphs  $Z \rightarrow W$  such that above diagram commutes.

**Remark 4.4.** Let  $Z$  be the  $C$ -pushout of  $X$  and  $Y$  with respect to a pair  $(x, y)$ .

1. For any vertex  $x$  in  $X$ ,  $G_x = G_{i_1(x)}$  or  $x$  is in the image of  $\kappa_1$ .
2. For any edge  $e$  of  $X$ ,  $G_e = G_{i_1(e)}$ .
3. If  $X/G$  and  $Y/G$  both have finitely many vertices (resp. edges), then  $Z/G$  has finitely many vertices (resp. edges).

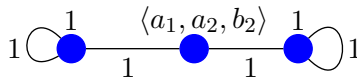
**Example 4.1.** Let  $G = A *_C B$  where  $A$  and  $B$  are free abelian groups of rank two, and  $C$  is maximal cyclic subgroup of  $A$  and  $B$ . Let  $X$  be the  $A$ -graph consisting of a single vertex with the trivial  $A$ -action and define  $Y$  analogously for  $B$ . Then the graph  $G \times_A X$  is the edgeless  $G$ -graph with vertex set the collection of left cosets of  $G/A$ ; and analogously  $G \times_B Y$  is the edgeless graph with vertex set  $G/B$ . Let  $Z$  be the  $C$ -pushout of  $X$  and  $Y$ . By parts (4) and (7) of Proposition 4.5,  $Z$  is a connected edgeless  $G$ -graph and hence a single vertex.

**Example 4.2.** Let  $A = \langle a_1, a_2, a_3 \mid [a_1, a_2] \rangle$  and  $B = \langle b_1, b_2, b_3 \mid [b_1, b_2] \rangle$ , and let  $X = \hat{\Gamma}(A, \langle a_1, a_2 \rangle, a_3)$  and  $Y = \hat{\Gamma}(B, \langle b_1, b_2 \rangle, b_3)$  be the coned-off Cayley graphs. Note that  $X$  is the Bass–Serre tree of the splitting of  $A$  as the graph of groups



with two vertices and two edges with trivial edge group.

Let  $G = A *_C B$  be the amalgamated product where  $C$  corresponds to the cyclic subgroup  $\langle a_1 \rangle \leq A$  and  $\langle b_1 \rangle \leq B$ . Consider the  $C$ -pushout  $Z$  of  $G \times_A X$  and  $G \times_B Y$ . By the fourth, fifth and sixth statements of Proposition 4.5 below,  $Z$  is a tree, it contains three distinct  $G$ -orbits of vertices, two of these  $G$ -orbits have all representatives with trivial stabilizer, and there is a vertex  $z$  with stabilizer  $\langle a_1, a_2, b_2 \rangle = \langle a_1, a_2 \rangle *_{\langle a_1 \rangle = \langle b_1 \rangle} \langle b_1, b_2 \rangle$ , and there are four distinct orbits of edges all with representatives having trivial stabilizer. Hence,  $Z$  is the Bass–Serre tree of a splitting of  $G$  given by the graph of groups



with three vertices and four edges. In particular,  $Z$  is the coned-off Cayley graph of  $\hat{\Gamma}(G, G_y, \{a_3, b_3\})$ .

**Proposition 4.5.** Let  $G$  be the amalgamated free product group  $A *_C B$ , let  $X$  be a  $A$ -graph, and let  $Y$  be a  $B$ -graph. Let  $x \in X^C$  and  $y \in Y^C$  be vertices. Let  $Z$  be the  $C$ -pushout of  $G \times_A X$  and  $G \times_B Y$  with respect to  $(x, y)$ . Let  $z = i_1(x) = i_2(y)$ . The following properties hold:

1. The homomorphism  $A_x *_C B_y \rightarrow G$  induced by the inclusions  $A_x \leq G$  and  $B_y \leq G$  is injective and has image  $G_z$ . In particular,  $G_z = \langle A_x, B_y \rangle$  is isomorphic to  $A_x *_C B_y$ .
2. The morphism  $X \hookrightarrow G \times_A X \xrightarrow{i_1} Z$  is an  $A$ -equivariant embedding. Analogously,  $Y \hookrightarrow G \times_B Y \xrightarrow{i_2} Z$  is a  $B$ -equivariant embedding.  
From here on, we consider  $X$  and  $Y$  as subgraphs of  $Z$  via these canonical embeddings.
3. For every vertex  $v$  (resp. edge  $e$ ) of  $Z$ , there is  $g \in G$  such that  $g.v$  is a vertex (resp. is an edge  $g.e$ ) of the subgraph  $X \cup Y$ .
4. For every vertex  $v$  of  $X$  which is not in the  $A$ -orbit of  $x$ ,  $A_v = G_v$  where  $G_v$  is the  $G$ -stabilizer of  $v$  in  $Z$ . Analogously for every vertex  $v$  of  $Y$  not in the  $B$ -orbit of  $y$ ,  $B_v = G_v$ .
5. For every edge  $e$  of  $X$  (resp.  $Y$ ),  $A_e = G_e$  (resp.  $B_e = G_e$ ) where  $G_e$  is the  $G$ -stabilizer of  $e$  in  $Z$ .
6. If the complexes  $X/A$  and  $Y/B$  both have finitely many vertices (resp. edges), then  $Z/G$  has finitely many vertices (resp. edges).
7. If  $X$  and  $Y$  are connected, then  $Z$  is connected.

8. There is a  $G$ -tree  $T$  and a morphism  $\xi : Z \rightarrow T$  of  $G$ -graphs with the following properties: The  $G$ -orbit of  $\xi(z)$  and its complement in the set of vertices of  $T$  make  $T$  a bipartite graph; the preimage  $\xi^{-1}(\xi(z))$  is a single vertex; and if a vertex  $v$  of  $T$  is not in the  $G$ -orbit of  $\xi(z)$ , then the preimage of the star of  $v$  is a subgraph of  $Z$  isomorphic to  $X$  or  $Y$ .

*Proof.* The first item is a direct consequence of Proposition 4.1. For the second item, first note that the composition  $X \hookrightarrow G \times_A X \xrightarrow{v_1} Z$  is a morphism of  $A$ -graphs. Observe that to prove the embedding part is enough to consider only vertices of  $X$  that are in the  $A$ -orbit of  $x$ . Suppose that  $a.x$  and  $x$  with  $a \in A$  both map to  $z \in Z$ . Then,  $a \in G_z = A_x *_C B_y$  and therefore  $a \in A_x$  and hence  $a.x = x$ . Item three follows directly from the definition of  $Z$ , and items four to six are consequences of Proposition 4.2.

To prove the seventh statement suppose that  $X$  and  $Y$  are connected graphs. The subgraph  $X \cup Y$  of  $Z$  is connected since both  $X$  and  $Y$  contain the vertex  $z$ . On the other hand, any vertex of  $Z$  belongs to a translate of  $X \cup Y$  by an element of  $G$ . Therefore to prove that  $Z$  is connected, it is enough to show that for any  $g \in G$  there is a path in  $Z$  from  $z$  to  $g.z$ . For any  $g \in G$  and  $a \in A$ , there is a path from  $g.z$  to  $ga.z$  in  $Z$ : indeed, there is a path from  $z$  to  $a.z$  in the connected  $A$ -subgraph  $X$  of  $Z$ , and hence there is a path from  $g.z$  to  $ga.z$  in  $Z$ . Analogously, for any  $g \in G$  and  $b \in B$ , there is a path from  $g.z$  to  $gb.z$ . Since any element of  $G$  is of the form  $a_1 b_1 \dots a_n b_n$  with  $a_i \in A$  and  $b_i \in B$ , there is a path from  $z$  to  $g.z$  for any  $g \in G$ .

Now we prove the eighth statement. Observe that  $G$  splits as  $G = A *_A (A_x *_C B_y) *_B B$  where the subgroups  $A_x, B_y$  and  $A_x *_C B_y$  are naturally identified with the  $G$ -stabilizers of  $x \in G \times_A X, y \in G \times_B Y$ , and  $z \in Z$ . Let  $T$  denote the Bass–Serre tree of this splitting. The vertex and edge sets of  $T$  can be described as:

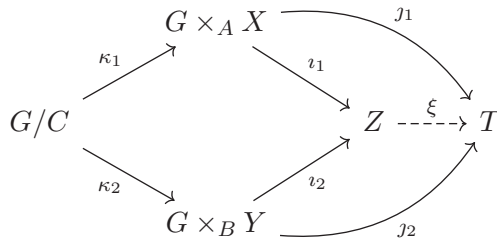
$$V(T) = G/A \sqcup G/(A_x *_C B_y) \sqcup G/B$$

and

$$E(T) = \{ \{gA, g(A_x *_C B_y)\} \mid g \in G \} \sqcup \{ \{g(A_x *_C B_y), gB\} \mid g \in G \}$$

respectively. Note that  $T$  is a bipartite  $G$ -graph, the equivariant bipartition of the vertices given by  $G/A \sqcup G/B$  and  $G/(A_x *_C B_y)$ .

Consider the  $A$ -map from  $X$  to  $T$  that maps every vertex of  $X$  not in the  $A$ -orbit of  $x$  to the vertex  $A$ , and  $x \mapsto A_x *_C B_y$ . Since  $T$  is simplicial, this induces a unique morphism of  $G$ -graphs  $J_1 : G \times_A X \rightarrow T$ . Analogously, there is  $B$ -map  $Y \rightarrow T$  that maps every vertex not in the  $B$ -orbit of  $y$  to the vertex  $B$  and  $y \mapsto A_x *_C B_y$ ; this induces a unique  $G$ -map  $J_2 : G \times_B Y \rightarrow T$ .



Consider the  $G$ -maps  $\kappa_1 : G/C \rightarrow G \times_A X$  and  $\kappa_2 : G/C \rightarrow G \times_B Y$  given by  $C \mapsto x$  and  $C \mapsto y$ , respectively. Since  $J_1 \circ \kappa_1 = J_2 \circ \kappa_2$ , the universal property implies that there is a surjective  $G$ -map  $\xi : Z \rightarrow T$ .

Note that  $\xi^{-1}(\xi(z))$  is contained in the orbit  $G.z$ . Suppose  $g.z \in \xi^{-1}(\xi(z))$ . Then  $g(A_x *_C B_y) = A_x *_C B_y$  and hence  $g \in A_x *_C B_y$ . Since  $A_x *_C B_y$  is the  $G$ -stabilizer of  $z$ , we have that  $g.z = z$ . This shows that  $\xi^{-1}(\xi(z)) = \{z\}$ .

Let us conclude by proving that if  $v \in V(T)$  is not in the  $G$ -orbit of  $\xi(z) = A_x *_C B_y$ , then  $\xi^{-1}(\text{star}_T(v))$  is a graph isomorphic to either  $X$  or  $Y$ . Note that such a vertex  $v$  is an element of  $G/A \cup G/B$ . By equivariance, it is enough to consider the two symmetric cases, namely  $v = A$  or  $v = B$ . Let us prove



that  $\xi^{-1}(\text{star}_T A)$  is isomorphic to  $X$ . Observe that any edge of  $\text{star}_T(A)$  is of the form  $\{A, a(A_x *_C B_y)\}$  with  $a \in A$ . Since  $(\xi \circ \iota_1)^{-1}(\text{star}_T(A)) = J_1^{-1}(\text{star}_T(A))$  and  $\xi^{-1}(\text{star}_T(A)) \subset \iota_1(G \times_A X)$ , we have that  $\xi^{-1}(\text{star}_T(A)) = \iota_1(J_1^{-1}(\text{star}_T(A)))$ . Recall that the canonical  $A$ -map  $X \rightarrow G \times_A X$  is injective, and this defines a natural identification of  $X$  with a subgraph of  $G \times_A X$  which equals  $J^{-1}(\text{star}_T(A))$  by definition of  $J$ . Then we have that  $\iota_1(J_1^{-1}(\text{star}_T(A))) = \iota_1(X)$  is isomorphic to  $X$  by the second item of the proposition.  $\square$

4.6. Proof of Theorem 3.1

**Lemma 4.6.** *Let  $\xi : \Gamma \rightarrow T$  be a morphism of graphs where  $T$  is a bipartite tree, say  $V(T) = K \cup L$ . Suppose  $\xi^{-1}(v)$  is a single vertex for every  $v \in L$ , and  $\xi^{-1}(\text{star}(v))$  is a connected subgraph for every  $v \in K$ . Let  $\Omega = \{\xi^{-1}(\text{star}(v)) \mid v \in K\}$ . Then:*

1.  $\Gamma$  is a simplicial graph if and only if every  $\Delta \in \Omega$  is simplicial.
2.  $\Gamma$  is a  $\delta$ -hyperbolic graph if and only if every  $\Delta \in \Omega$  is a  $\delta$ -hyperbolic graph.
3. For any vertex  $u$  of  $\Gamma$ , the following statements are equivalent:
  - $\Gamma$  is fine at  $u$ .
  - For every  $\Delta \in \Omega$ , if  $u$  is a vertex of  $\Delta$ , then  $\Delta$  is fine at  $u$ .

*Proof.* Note for any vertex  $u$  of  $\Gamma$ , if  $\xi(u) \in L$  then  $u$  is a cut vertex of  $\Gamma$ . The bipartite assumption on  $T$  implies that if  $\Theta$  is the closure of a connected component of  $\Gamma \setminus \xi^{-1}(L)$ , then  $\Theta$  equals some  $\Delta \in \Omega$ .

The first and second statements follow from the previous observation, the second one with important generalizations [3]. For the third statement, if  $\Gamma$  is fine at  $u$ , then any subgraph containing  $u$  is fine at  $u$ . Conversely, let  $u$  be a vertex of  $\Gamma$  such that any  $\Delta$  containing  $u$  is fine at  $u$ . There are two cases to consider.

Suppose that  $\xi(u) \in K$ . Then there is a unique  $\Delta \in \Omega$  that contains  $u$ . The bipartite assumption implies that  $T_u \Delta = T_u \Gamma$ . Since every vertex of  $\Gamma$  that maps to  $L$  disconnects  $\Gamma$ , the metric spaces  $(T_u \Delta, \angle_u)$  and  $(T_u \Gamma, \angle_u)$  coincide. Since  $\Delta$  is fine at  $u$ , then  $\Gamma$  is fine at  $u$ .

Suppose that  $\xi(u) \in L$ . Observe if  $x, y \in T_u \Gamma$  and  $x$  and  $y$  belong to different subgraphs in  $\Omega$ , then  $\angle_u(x, y) = \infty$ . Therefore, for any  $x \in T_u \Gamma$ , every ball of finite radius in  $T_u \Gamma$  centered at  $x$  is a ball of finite radius in  $T_u \Delta$  centered at  $x$  for some  $\Delta$ . Since by assumption,  $\Delta$  is fine at  $v$ , every ball of finite radius in  $T_v \Gamma$  centered at  $x$  is finite.  $\square$

*Proof of Theorem 3.1.* Let  $\Gamma$  be the  $C$ -pushout of the  $G$ -graphs  $G \times_{G_1} \Gamma_1$  and  $G \times_{G_2} \Gamma_2$  with respect to  $(x_1, x_2)$ , and let  $z$  be the image of  $x_1$  in  $\Gamma$ . The first six properties of  $\Gamma$  are direct corollaries of Proposition 4.5. The last four properties follow directly by invoking Proposition 4.5(8) and Lemma 4.6.  $\square$

5. HNN-extensions

This section describes a proof of Theorem 3.2. The argument is analogous to the one proving Theorem 3.1. In this case, we need to construct a  $G *_\varphi$ -graph from a given  $G$ -graph that we call the  $\varphi$ -coalescence.

**Definition 5.1** (Coalescence in sets). *Let  $H$  be a subgroup of a group  $A$ , let  $\varphi : H \rightarrow A$  be a monomorphism and let  $G$  be the HNN-extension:*

$$G = A *_\varphi = \langle G, t \mid tct^{-1} = \varphi(c) \text{ for all } c \in C \rangle.$$

*Let  $X$  be an  $A$ -set,  $x \in X^H$  and  $y \in X^{\varphi(H)}$ . The  $\varphi$ -coalescence of  $X$  with respect to  $(x, y)$  is the  $G$ -set  $Z$  arising as quotient of  $G \times_A X$  by the equivalence relation generated by the set of basic relations:*

$$\mathcal{B} = \{(gt.x, g.y) \mid g \in G\}.$$

Note that the quotient map:

$$\rho : G \times_A X \rightarrow Z$$

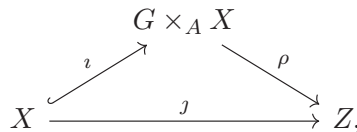
is  $G$ -equivariant.

**Example 5.1.** Let  $\varphi : A \rightarrow A$  be a group automorphism and consider the HNN-extension  $G = A *_{\varphi}$ . Let  $X$  be the  $A$ -set consisting of a single point. Then  $G \times_A X$  is the  $G$ -space  $G/A$ , and then the  $\varphi$ -coalescence of  $X$  is again a single point.

**Example 5.2.** Consider a free product  $A = H_1 * H_2$ . Let  $\varphi : H_1 \rightarrow H_2$  be an isomorphism, and  $G = A *_{\varphi}$ . Let  $X$  be the  $A$ -set  $A/H_1 \cup A/H_2$  of all left cosets of  $H_1$  and  $H_2$  in  $A$ . Then  $G \times_A X$  is the  $G$ -set of left cosets  $G/H_1 \cup G/H_2$ . The  $\varphi$ -coalescence  $Z$  of  $X$  with respect to the pair  $(H_1, H_2)$  is the quotient  $G \times_A X$  by identifying  $gtH_1$  and  $gH_2$  for every  $g \in G$ . Hence,  $Z$  is naturally isomorphic as a  $G$ -set to  $G/H_1$ . Observe that the  $A$ -map  $X \rightarrow Z$  given by  $H_1 \mapsto H_1$  and  $\varphi(H_1) \mapsto tH_1$  is an injective  $A$ -map.

**Lemma 5.2.** Let  $H$  be a subgroup of a group  $A$ , let  $\varphi : H \rightarrow A$  be a monomorphism and let  $G = A *_{\varphi}$ . Let  $X$  be an  $A$ -set, let  $x, y \in X$  be in different  $A$ -orbits such that  $A_x = H, A_y = \varphi(H)$ . If  $Z$  is the  $\varphi$ -coalescence of  $X$  with respect to  $(x, y)$ , and  $z = \rho(y)$ , then:

1. the  $G$ -stabilizer  $G_z$  equals  $\varphi(H)$ , and
2. the  $A$ -map  $j : X \rightarrow Z$  defined by the commutative diagram



is injective.

*Proof.* The inclusion  $\varphi(H) \subseteq G_z$  is a consequence of  $\rho$  being  $G$ -equivariant,  $\varphi(H) = A_y$  and  $\rho(y) = z$ . Conversely, let  $g \in G_z$ . Then  $g.y \sim y$  in  $G \times_A X$ , and it follows that there is an integer  $n \geq 1$  and a sequence  $w_1, w_2, \dots, w_n$  of elements of  $G \times_A X$  such that  $g.y = w_1, w_n = y$  and  $w_i$  and  $w_{i+1}$  are the components of a basic relation (see the definition of coalescence). Since  $x$  and  $y$  are in different  $A$ -orbits in  $X$ , they represent different  $G$ -orbits in  $G \times_A X$ , see the first item of Proposition 4.2. Hence, we have that  $w_i = g_i.x$  if  $i$  is even, and  $w_i = g_i.y$  if  $i$  is odd, for some elements  $g_i$  of  $G$  where  $g_1 = g$  and  $g_n \in \varphi(H)$ . Note that the integer  $n$  is odd, and the chain of basic relations between the  $w_i$ 's in  $G \times_A X$  can be expressed as:

$$g_1.y \sim g_2.x \sim g_3.y \sim g_4.x \sim \dots \sim g_{n-1}.x \sim g_n.y.$$

By definition of basic relation,  $tg_2^{-1}g_1 \in \varphi(H), tg_2^{-1}g_3 \in \varphi(H), tg_4^{-1}g_3 \in \varphi(H), tg_4^{-1}g_5 \in \varphi(H)$ , and so on until  $tg_{n-1}^{-1}g_n \in \varphi(H)$ . Since  $n$  is odd, we have that

$$g^{-1}g_n = (tg_2^{-1}g)^{-1}(tg_2^{-1}g_3)(tg_4^{-1}g_3)^{-1}(tg_4^{-1}g_5) \dots (tg_{n-1}^{-1}g_n) \in \varphi(H),$$

which implies  $g = g_1 \in \varphi(H)$ . This shows that  $G_z = \varphi(H)$

Now we prove the second statement. By Lemma 4.2, the natural  $A$ -map  $X \rightarrow G \times_A X$  is injective. Observe that  $Z$  is obtained as a quotient of  $G \times_A X$  by the  $G$ -equivariant equivalence relation generated by the basic relation  $t.x \sim y$ . Hence to prove injectivity of  $X \rightarrow G \times_A X \rightarrow Z$ , it is enough to show that the restriction to  $A.x \cup A.y$  is injective. Assume there are  $a_1, a_2 \in A$  such that  $a_1.x$  and  $a_2.x$  map to the same element in  $Z$ . Then letting  $a = a_2^{-1}a_1$ , both  $a.x$  and  $x$  map to the same element in  $Z$ . Hence,  $a.x \sim x$  which implies that  $at^{-1}.y \sim t^{-1}.y$ . Therefore  $(ta^{-1}t^{-1}).y \sim y$  and thus by the first statement,  $ta^{-1}t^{-1} \in \varphi(H)$ , and hence  $a^{-1} \in t^{-1}\varphi(H)t = H$ . This results in  $a \in H$ . Therefore,  $a_2^{-1}a_1 \in H$  and  $a_1.x = a_2.x$ . We have shown

that the restriction  $A.x \rightarrow Z$  is injective. With a similar argument one can show that  $A.y \rightarrow Z$  is also injective.  $\square$

**Definition 5.3** (Coalescence in graphs). *Let  $H$  be a subgroup of a group  $A$ , let  $\varphi : H \rightarrow A$  be a monomorphism, and let  $G = A*_\varphi$ . Let  $X$  be an  $A$ -graph, let  $x, y \in V(X)$  such that  $x \in X^H$  and  $y \in X^{\varphi(H)}$ . The  $\varphi$ -Coalescence  $Z$  of  $X$  with respect to  $(x, y)$  is the  $G$ -graph with vertex set the  $\varphi$ -Coalescence of the  $A$ -set  $V(X)$  with respect to  $(x, y)$ , edge set  $G \times_A E(X)$ , and attaching map  $E(Z) \rightarrow [V(Z)]^2$  defined as the composition*

$$\begin{array}{ccc}
 E(X) & \longrightarrow & [V(X)]^2 \\
 \downarrow & & \downarrow \\
 E(Z) = G \times_A E(X) & \longrightarrow & G \times_A [V(X)]^2 \\
 & \searrow & \downarrow \\
 & & [V(Z)]^2
 \end{array}$$

where the horizontal middle arrows are induced by the attaching map  $E(X) \rightarrow [V(X)]^2$  (see Lemma 4.2(3)) and the the bottom vertical map is induced by the quotient map  $G \times_A V(X) \rightarrow V(Z)$ . Let  $\rho : G \times_A X \rightarrow Z$  denote the induced  $G$ -morphism.

**Remark 5.4** (Equivalent definition of coalescence). *Equivalently, the  $\varphi$ -coalescence  $Z$  of the  $A$ -graph  $X$  with respect to  $(x, y)$  is the quotient  $Z$  of the  $G$ -graph  $G \times_A X$  by the equivalence relation generated by  $gt.x \sim g.y$  for all  $g \in G$ .*

**Proposition 5.5.** *Let  $H$  be a subgroup of a group  $A$ , let  $\varphi : H \rightarrow A$  be a monomorphism and let  $G = A*_\varphi$ . Let  $X$  be an  $A$ -graph, let  $x, y \in X$  in different  $A$ -orbits such that  $A_x = H, A_y = \varphi(H)$ . If  $Z$  is the  $\varphi$ -coalescence of  $X$  with respect to  $(x, y)$ , and  $z = \rho(y)$ , then the following properties hold:*

1.  $G_z = \varphi(H)$ .
2. The map  $X \hookrightarrow G \times_A X \rightarrow Z$  is an  $A$ -equivariant embedding.  
From here on, we consider  $X$  as a subgraph of  $Z$  via this canonical embedding.
3. For every vertex  $v$  (resp. edge  $e$ ) of  $Z$ , there is  $g \in G$  such that  $g.v$  is a vertex (resp. is an edge  $g.e$ ) of  $X$ .
4. For every vertex  $v$  of  $X$  which is not in the  $A$ -orbit of  $x, A_v = G_v$  where  $G_v$  is the  $G$ -stabilizer of  $v$  in  $Z$ .
5. For every edge  $e$  of  $X A_e = G_e$  where  $G_e$  is the  $G$ -stabilizer of  $e$  in  $Z$ .
6. If the complex  $X/A$  has finitely many vertices (resp. edges), then  $Z/G$  has finitely many vertices (resp. edges).
7. If  $X$  is connected, then  $Z$  is connected.
8. There is a  $G$ -tree  $T$  and a morphism  $\xi : Z \rightarrow T$  of  $G$ -graphs with the following properties: The  $G$ -orbit of  $\xi(z)$  and its complement in the set of vertices of  $T$  make  $T$  a bipartite  $G$ -graph; the preimage  $\xi^{-1}(\xi(z))$  is a single vertex; and if a vertex  $v$  of  $T$  is not in the  $G$ -orbit of  $\xi(z)$ , then the preimage of the star of  $v$  is a subgraph of  $Z$  isomorphic to  $X$ .

The following argument is analogous to the proof of Proposition 4.5.

*Proof.* The first and second statements are direct consequences of Lemma 5.2 when considering  $V(X)$  and  $E(X)$  as  $A$ -sets. Items three to six follow directly from the definition of  $Z$  and Proposition 4.2.

Suppose  $X$  is connected. By Proposition 4.2, the graph  $G \times_A X$  is a disjoint union of copies of the connected subgraph  $X$ , and hence any element in  $Z$  belongs to a translate of  $X$  by an element of  $G$ .

Therefore, to prove that  $Z$  is connected, it is sufficient to show that for any  $g \in G$  there is a path in  $Z$  from  $z$  to  $g.z$ .

First observe that if there is a path from  $z$  to  $g.z$ , then there is a path from  $z$  to  $gt.z$ . Indeed, there is a path from  $x$  to  $y$  in the connected subgraph  $X$  of  $G \times_A X$ , and hence there is a path from  $z = \rho(t.x)$  to  $t.z = \rho(t.y)$  in  $Z$ . Therefore, there is a path from  $g.z$  to  $gt.z$  in  $Z$ , and in particular a path from  $z$  to  $gt.z$ .

Now observe that if there is a path from  $z$  to  $g.z$ , then there is a path from  $z$  to  $ga.z$  for any  $a \in A$ . Indeed, there is a path from  $z$  to  $a.z$  in the connected  $A$ -subgraph  $X$  of  $Z$ . Hence, there is a path from  $g.z$  to  $ga.z$  in  $Z$ , and in particular a path from  $z$  to  $ga.z$ .

To conclude, any  $g \in G$  is a product of the form  $g = a_1 t^{\theta_1} a_2 t^{\theta_2} \dots a_n t^{\theta_n} a_{n+1}$  with  $a_i \in A$  and  $\theta_i = \pm 1$ . Therefore, an induction argument using the two previous statements shows that there is a path from  $z$  to  $g.z$  in  $Z$  for every  $g \in G$ .

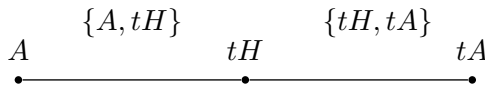
Now we prove the eighth statement. Consider the barycentric subdivision  $T$  of the Bass–Serre tree of the splitting  $G*_\varphi$ . Specifically,  $T$  is the tree with vertex set

$$V(T) = G/A \sqcup G/H$$

edge set

$$E(T) = \{\{gA, gtH\} \mid g \in G\} \sqcup \{\{gA, gH\} \mid g \in G\}.$$

Note that all the edges of  $T$  are  $G$ -translates of the following two edges attached at the vertex  $tH$ ,



Suppose that the  $A$ -set  $V(X) = (\bigsqcup_{i \in I} A/A_i) \sqcup A/H \sqcup A/\varphi(H)$ . Then

$$V(G \times_A X) = \left( \bigsqcup_{i \in I} G/A_i \right) \sqcup G/H \sqcup G/\varphi(H).$$

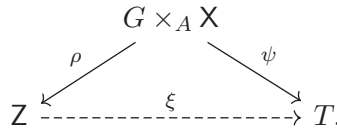
Since  $T$  is a simplicial graph, there is an induced  $G$ -equivariant morphism of graphs

$$\psi : G \times_A X \rightarrow T$$

defined on vertices by:

$$A_i \mapsto A, \quad H \mapsto H, \quad \varphi(H) \mapsto tH.$$

Note that any edge in  $G \times_A X$  of the form  $\{gA_i, gaA_j\}$  for  $g \in G$  and  $a \in A$  is collapsed to the vertex  $gA$  in  $T$ ; and edges of the form  $\{gA_i, gH\}$  and  $\{gA_i, gtH\}$  are mapped to the edges  $\{gA, gH\}$  and  $\{gA, g\varphi(H)\}$  of  $T$ , respectively. This map induces a  $G$ -equivariant morphism of graphs  $\xi : Z \rightarrow T$  such that the following diagram commutes,



Indeed this diagram commutes since  $\psi$  is  $G$ -equivariant and  $\psi(tH) = \psi(\varphi(H))$ .

Observe that  $\xi(z) = \psi(tH)$  and  $\psi^{-1}(tH) = \{tH, \varphi(H)\}$ , then  $\rho(tH) = \rho(\varphi(H))$  implies that  $\xi^{-1}(\xi(z)) = \{z\}$ .

By definition of  $T$ , if a vertex  $v$  is not in the  $G$ -orbit of  $\xi(z) = tH$ , then  $v \in G/A$ . Hence, the partition  $V(T) = G/A \sqcup G/H$  makes  $T$  a bipartite  $G$ -graph. Any edge of  $\text{star}_T(A)$  is of the form  $\{A, atH\}$  for some  $a \in A$ . Hence,  $\xi^{-1}(\text{star}_T(A)) = \rho(\psi^{-1}(\text{star}_T(A))) = \rho(X)$  and then the definition of  $Z$  implies that  $\rho(X)$  is isomorphic  $X$ . □

*Proof of Theorem 3.2.* The argument is the same as the one used to prove Theorem 3.1, the only difference is the use of Proposition 5.5 instead of Proposition 4.5. □

### 6. Dehn functions and coarse isoperimetric functions

In this section, we recall the definition of coarse isoperimetric function of a graph and recall how one can recover the relative Dehn function of a pair via Cayley–Abels graphs; see Theorem 6.4. In the second part of the section, we discuss a technical result that provides bounds for coarse isoperimetric functions of graphs based on maps into trees; see Proposition 6.6. These results are used to obtain bounds on the relative Dehn function of fundamental groups of graph of groups based on the relative Dehn functions of the vertex groups; see Corollary 1.6.

#### 6.1. Coarse isoperimetric functions

A singular combinatorial map  $X \rightarrow Y$  between one-dimensional CW-complexes is a continuous map such that the restriction to each open one-dimensional cell of  $X$  is either a homeomorphism onto an open cell of  $Y$  or its image is contained in the 0-skeleton of  $Y$ . A singular combinatorial loop  $c : I \rightarrow X$  is a singular combinatorial map such that its domain is a CW-complex homeomorphic to a closed interval. The length  $\text{Len}(c)$  of  $c$  is defined as the number of open 1-cells of  $I$  that map homeomorphically to open cells of  $X$ .

Let  $\Gamma$  be a connected graph, regard it as a CW-complex, and consider the path-metric on  $\Gamma$  obtained by regarding each edge as a segment of unit length. Let  $k > 0$ . A  $k$ -filling of a singular combinatorial loop  $c : I \rightarrow \Gamma$  is a pair  $(P, \Phi)$  consisting of a triangulation  $P$  of the 2-disc  $D^2$  and a singular combinatorial map  $\Phi : P^{(1)} \rightarrow \Gamma$  such that  $\Phi|_{\partial D}$  equals the closed path  $c$  (after identifying the endpoints of the domain of  $c$ ) and the image under  $\Phi$  of the boundary of each 2-cell of  $P$  is a set of diameter at most  $k$  in  $\Gamma$ . Define  $|(P, \Phi)|$  to be the number of faces of  $P$  and

$$\text{area}_k^\Gamma(c) := \min\{|(P, \Phi)| : (P, \Phi) \text{ an } k\text{-filling of } c\}.$$

The  $k$ -coarse isoperimetric function  $f_k^\Gamma : \mathbb{N} \rightarrow \mathbb{N}$  of  $\Gamma$  is then defined to be

$$f_k^\Gamma(\ell) := \sup\{\text{area}_k^\Gamma(c) : \text{Len}(c) \leq \ell\}.$$

We say that  $f_k^\Gamma$  is well defined if it takes only finite values. The graph  $\Gamma$  is  $k$ -fillable if  $f_k^\Gamma$  is well defined, and  $\Gamma$  is fillable if it is  $k$ -fillable for some integer  $k$ . Note that if  $f_k^\Gamma$  is well defined then  $f_\ell^\Gamma$  is well defined for all  $\ell \geq k$ .

For two functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$ , define  $f \preceq g$  if there exist constants  $C, K, L \in \mathbb{N}$  such that

$$f(n) \leq Cg(Kn) + Ln.$$

We say that  $f$  is asymptotically equivalent to  $g$  if  $f \asymp g$  if  $f \preceq g$  and  $g \preceq f$ .

**Proposition 6.1.** [4, Proposition III.H.2.2] *If  $\Gamma$  and  $\Gamma'$  are quasi-isometric connected graphs such that  $\Gamma$  is fillable, then  $\Gamma'$  is fillable and  $f_k^{\Gamma'} \asymp f_k^\Gamma$  for all sufficiently large integers  $k$ .*

We conclude the subsection recalling two results in order to deduce Corollary 6.4 which shows that the relative Dehn function of a finitely presented pair is equivalent to coarse isoperimetric functions of Cayley–Abels graphs.

The following theorem is a re-statement of a result of Osin; see [12, Prop. 4.8].

**Theorem 6.2.** [18, Thm. 2.53] *Let  $G$  be a group and let  $\mathcal{H}$  be a collection of subgroups. If  $\Delta_{G,\mathcal{H}}$  is well defined, then  $\hat{\Gamma}(G, \mathcal{H})$  is fillable and  $\Delta_{G,\mathcal{H}} \asymp_k^{\hat{\Gamma}(G,\mathcal{H})}$  for all sufficiently large integers  $k$ .*

**Theorem 6.3.** [12, Theorem E] *Let  $(G, \mathcal{H})$  be a finitely generated pair. If  $\hat{\Gamma}(G, \mathcal{H})$  is fine and fillable, then  $(G, \mathcal{H})$  is finitely presented and  $\Delta_{G,\mathcal{H}}$  is well defined.*

As previously observed, the coned-off Cayley graphs of a finitely generated pair  $(G, \mathcal{H})$  are Cayley–Abels graphs of the pair. Theorem 2.12 states that all Cayley–Abels graphs of a finitely generated pair are quasi-isometric, and if one of them is fine then all of them are fine. Moreover, fillable and the class of coarse isoperimetric functions are quasi-isometry invariants of graphs by Proposition 6.2. Putting these results together with the two results above, and Theorem 2.11, one obtains the following corollary.

**Corollary 6.4.** *Let  $\Gamma$  be a Cayley–Abels graph of finitely generated proper pair  $(G, \mathcal{H})$ .*

1. *If  $\Delta_{G,\mathcal{H}}$  is well defined, then  $\Gamma$  is fine and fillable, and  $\Delta_{G,\mathcal{H}} \asymp f_k^\Gamma$  for all sufficiently large integers  $k$ .*
2. *If  $\Gamma$  is fine and fillable, then  $(G, \mathcal{H})$  is finitely presented and  $\Delta_{G,\mathcal{H}}$  is well defined.*

### 6.2. Relative Dehn functions and splittings

Let  $g : \mathbb{N} \rightarrow \mathbb{N}$  be a function. Then  $g$  is *superadditive* if  $g(m) + g(n) \leq g(m + n)$ . If  $g(0) = 0$  then the *super-additive closure*  $\bar{g} : \mathbb{N} \rightarrow \mathbb{N}$  of  $g$  is the function:

$$\bar{g}(n) = \max \left\{ \sum_{i=1}^k g(n_i) \mid k \in \mathbb{N}, n_i \in \mathbb{N}, \sum_{i=1}^k n_i = n \right\},$$

and it is an observation that  $\bar{g}$  is the least super-additive function such that  $g(n) \leq \bar{g}(n)$  for all  $n$ . Note that the requirement  $g(0) = 0$  is necessary in order for  $\bar{g}$  to be well defined. An outstanding open question raised by Mark Sapir is whether the Dehn function of any finite presentation is asymptotically equivalent to a superadditive function [11].

**Proposition 6.5.** *Let  $r : \Gamma \rightarrow \Delta$  be a retraction of graphs. If  $\Gamma$  is  $k$ -fillable, then  $\Delta$  is  $k$ -fillable and  $f_k^\Delta(n) \leq f_k^\Gamma(n)$ .*

*Proof.* Let  $c : I \rightarrow \Delta$  is a singular combinatorial loop. If  $(P, \Phi)$  is a  $k$ -filling of  $c$  in  $\Gamma$  then it is an observation that  $(P, r \circ \Phi)$  is a  $k$ -filling of  $c$  in  $\Delta$ . Therefore,  $\text{area}_k^\Delta(c) \leq \text{area}_k^\Gamma(c)$  and the result follows. □

The following proposition is the main technical result of the section.

**Proposition 6.6.** *Let  $\xi : \Gamma \rightarrow T$  be a morphism of graphs where  $T$  is a bipartite tree, say  $V(T) = K \cup L$ . Suppose  $\xi^{-1}(v)$  is a single vertex for every  $v \in L$ , and  $\xi^{-1}(\text{star}(v))$  is a connected subgraph for every  $v \in K$ . Let  $\Omega = \{\xi^{-1}(\text{star}(v)) \mid v \in K\}$ .*

*If there is  $k > 0$  such that each  $\Delta \in \Omega$  is a  $k$ -fillable graph and*

$$g(n) := \sup\{f_k^\Delta(n) \mid \Delta \in \Omega\} < \infty \text{ for every } n,$$

*then  $\Gamma$  is  $k$ -fillable and*

$$f_k^\Gamma(n) \leq \bar{g}(n)$$

*where  $\bar{g}$  denotes the super-additive closure of the function  $g : \mathbb{N} \rightarrow \mathbb{N}$ .*

*Proof.* It is an observation that if  $c_1$  and  $c_2$  are singular combinatorial loops in  $\Gamma$  with the same initial point, and both admit  $k$ -fillings, then the concatenated loop  $c_1 \cdot c_2$  admits a  $k$ -filling and

$$\text{Len}(c_1 \cdot c_2) = \text{Len}(c_1) + \text{Len}(c_2) \quad \text{and} \quad \text{area}_k(c_1 \cdot c_2) \leq \text{area}_k(c_1) + \text{area}_k(c_2).$$

To prove that  $f_k^\Gamma(n) \leq \bar{g}(n)$ , we prove that if  $c : I \rightarrow \Gamma$  is a singular combinatorial loop in  $\Gamma$  then  $\text{area}_k^\Gamma(c) \leq \bar{g}(\text{Len}(c))$ .

Let  $c : I \rightarrow \Gamma$  be a singular combinatorial loop in  $\Gamma$ . Consider the loop  $\xi \circ c$  in the tree  $T$ . The image of  $\xi \circ c$  is a finite subtree  $T_c$  of  $T$ . Let  $\#T_c \cap K$  denote the number of vertices of  $T_c$  that belong to  $K$ . To

the loop  $c$  assign the complexity  $|c| = (\#T_c \cap K, \text{Len}(c)) \in \mathbb{N} \times \mathbb{N}$ . Consider the lexicographical order on  $\mathbb{N} \times \mathbb{N}$ , and recall that this is well-ordered set. We prove by induction on  $(\#T_c \cap K, \text{Len}(c))$  that  $\text{area}_k^\Gamma(c) \leq \bar{g}(\text{Len}(c))$ .

Base case  $|c| = (0, m)$ . Suppose that  $T_c$  does not contain vertices in  $K$ . In this case, the bipartite assumption on  $T$  implies that  $T_c$  consists of a single vertex  $v$  in  $L$ . Since  $\xi^{-1}(v)$  is a single vertex of  $\Gamma$ , it follows that  $c$  is constant path and hence  $\text{Len}(c) = 0$  and  $\text{area}_k^\Gamma(c) = 0 \leq \bar{g}(0)$ .

Base case  $|c| = (1, m)$ . Suppose that the vertex set of  $T_c$  contains a single vertex in  $K$ , say  $v$ . Then the bipartite assumption on  $T$  implies that  $T_c$  is a subgraph of  $\text{star}(v)$  and hence the image of  $c$  is contained in the subgraph  $\Delta = \xi^{-1}(\text{star}(v))$ . By assumption,  $\Delta$  is  $k$ -fillable, and hence there is  $k$ -filling of  $c$  in  $\Delta$  which is trivially also a  $k$ -filling in  $\Gamma$ . Hence,

$$\text{area}_k^\Gamma(c) \leq \text{area}_k^\Delta(c) \leq f_k^\Delta(\text{Len}(c)) \leq g(\text{Len}(c)) \leq \bar{g}(\text{Len}(c)).$$

General case  $|c| = (n, m)$  with  $n \geq 2$ . For the inductive step, suppose that  $T_c \cap K$  has at least two vertices in  $K$ . Without loss of generality, we can identify the domain  $I$  of  $c$  with the closed interval  $[0, 1]$  (with some CW-structure). Since  $T_c$  is connected, the bipartite assumption on  $T$ , implies that  $T_c$  contains a vertex  $v \in L$  such that  $v$  is not a leaf of  $T_c$ , in particular,  $T_c - \{v\}$  has at least two connected components. Then  $(\xi \circ c)^{-1}(T_c - \{v\})$  is a disconnected open subset of  $[0, 1]$ . Let  $J_1$  be the closure of a connected component of  $(\xi \circ c)^{-1}(T_c - \{v\})$ . By changing the initial point of the loop  $c : I \rightarrow \Gamma$ , we can assume that  $J_1 = [0, \alpha]$  for some  $\alpha < 1$ . Let  $J_2 = [\alpha, 1]$ , and let  $c_i$  be the restriction of  $c$  to the interval  $J_i$ . Then  $c_i$  is singular combinatorial loop, and  $c$  is the concatenation  $c_1 \cdot c_2$ . Since  $T_c - \{v\}$  is disconnected, it follows that  $0 < \text{Len}(c_i) < \text{Len}(c)$ . Since  $\#T_{c_i} \cap K \leq \#T_c \cap K$ , it follows that  $|c_i| < |c|$ . Hence by induction

$$\begin{aligned} \text{area}_k^\Gamma(c) &\leq \text{area}_k^\Gamma(c_1) + \text{area}_k^\Gamma(c_2) \\ &\leq \bar{g}(\text{Len}(c_1)) + \bar{g}(\text{Len}(c_2)) \\ &\leq \bar{g}(\text{Len}(c_1) + \text{Len}(c_2)) = \bar{g}(\text{Len}(c)) \end{aligned}$$

where the first inequality follows from the observation in the first paragraph of this proof, the second inequality uses the induction hypothesis, and the third one uses that  $\bar{g}$  is superadditive. □

### 6.3. Proof of Theorem 1.6

*Proof.* The proofs of the first two statements are analogous, and the argument goes back to the method of proof of the corresponding theorems in the introduction. The third statement is a consequence of the second one; see the proof of Corollary 1.4(2).

We prove the first statement and leave the proof of the second statement to the reader. We remark that the argument essentially reproves Theorem 1.2(2).

Let  $\Gamma_i$  be a Cayley–Abels graph of  $(G_i, \mathcal{H}_i \cup \{K_i\})$  for  $i = 1, 2$ . Then  $\Gamma_i$  has a vertex  $x_i$  with  $G_i$ -stabilizer  $K_i$ . Let  $\Gamma$  be the  $C$ -pushout of  $G \times_{G_1} \Gamma_1$  and  $G \times_{G_2} \Gamma_2$  with respect to  $(x_1, x_2)$ . Theorem 3.1 implies that  $\Gamma$  is a Cayley–Abels graph of  $(G_1 *_C G_2, \mathcal{H} \cup \{\langle K_1, K_2 \rangle\})$ . By Proposition 4.5(8), there is a morphism of graphs  $\xi : \Gamma \rightarrow T$  that satisfies the hypothesis of Proposition 6.6, namely,  $T$  is a bipartite tree with  $V(T) = K \cup L$  such that  $\xi^{-1}(v)$  is a single vertex for each  $v \in L$ , and  $\xi^{-1}(\text{star}(v))$  is isomorphic to  $\Gamma_i$  for some  $i = 1, 2$  for every  $v \in K$ . Corollary 6.4 implies that  $\Gamma_1$  and  $\Gamma_2$  are both  $k$ -fillable for some  $k$ . Then Proposition 6.6 implies that  $\Gamma$  is  $k$ -fillable and

$$f_k^\Gamma \leq \overline{\max\{f_k^{\Gamma_1}, f_k^{\Gamma_2}\}} \asymp \max\left\{\overline{f_k^{\Gamma_1}}, \overline{f_k^{\Gamma_2}}\right\}.$$

Then Corollary 6.4 implies that

$$\Delta \leq \max\{\overline{\Delta_1}, \overline{\Delta_2}\}.$$

On the other hand, the properties of the morphism  $\Gamma \rightarrow T$  imply that there is a retraction  $\Gamma \rightarrow \Gamma_i$  and hence Proposition 6.5 implies that  $f_k^{\Gamma_i} \leq f_k^\Gamma$  and therefore  $\Delta_i \leq \Delta$ . □



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## References

- [1] E. Alibegović, A combination theorem for relatively hyperbolic groups, *Bull. Lond. Math. Soc.* **37**(3) (2005), 459–466.
- [2] S. Arora and E. Martínez-Pedroza, Topological groups with a compact open subgroup, relative hyperbolicity and coherence, *J. Algebra* **631** (2023), 830–876.
- [3] M. Bestvina and M. Feighn, A combination theorem for negatively curved groups, *J. Differential Geom.* **35**(1) (1992), 85–101.
- [4] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **319** (Springer, Berlin, 1999).
- [5] B. H. Bowditch, Relatively hyperbolic groups, *Int. J. Algebra Comput.* **22**(3) (2012), 66.
- [6] S. G. Brick, On Dehn functions and products of groups, *Trans. Am. Math. Soc.* **335**(1) (1993), 369–384.
- [7] H. Bigdely and D. T. Wise, Quasiconvexity and relatively hyperbolic groups that split, *Mich. Math. J.* **62**(2) (2013), 387–406.
- [8] F. Dahmani, Combination of convergence groups, *Geom. Topol.* **7** (2003), 933–963.
- [9] F. Dahmani, V. Guirardel and D. V. Osin, Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces, *Mem. Am. Math. Soc.* **245**(1156) (2017), v+152.
- [10] B. Farb, Relatively hyperbolic groups, *Geom. Funct. Anal.* **8**(5) (1998), 810–840.
- [11] V. S. Guba and M. V. Sapir, On Dehn functions of free products of groups, *Proc. Am. Math. Soc.* **127**(7) (1999), 1885–1891.
- [12] S. Hughes, E. Martínez-Pedroza and L. J. S. Saldana, Quasi-isometry invariance of relative filling functions (with an appendix by Ashot Minasyan), *Groups Geom. Dyn.* In Press.
- [13] G. C. Hruska, Relative hyperbolicity and relative quasiconvexity for countable groups, *Algebr. Geom. Topol.* **10**(3) (2010), 1807–1856.
- [14] A. Minasyan and D. Osin, Acylindrical hyperbolicity of groups acting on trees, *Math. Ann.* **362**(3-4) (2015), 1055–1105.
- [15] E. Martínez-Pedroza and F. Rashid, A note on hyperbolically embedded subgroups, *Commun. Algebra* **50**(4) (2022), 1459–1468.
- [16] M. Mj and L. Reeves, A combination theorem for strong relative hyperbolicity, *Geom. Topol.* **12**(3) (2008), 1777–1798.
- [17] D. V. Osin, Relative Dehn functions of amalgamated products and HNN-extensions, in *Topological and asymptotic aspects of group theory*, Contemporary Mathematics, vol. **394** (American Mathematical Society, Providence, RI, 2006), 209–220.
- [18] D. V. Osin, Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems, *Mem. Am. Math. Soc.* **179**(843) (2006), vi+100.
- [19] D. Osin, Acylindrically hyperbolic groups, *Trans. Am. Math. Soc.* **368**(2) (2016), 851–888.
- [20] J. R. Stallings, Topology of finite graphs, *Invent. Math.* **71**(3) (1983), 551–565.