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Introduction

This book is about the geometry and topology of symplectic manifolds carrying pairs of complementary Lagrangian foliations. The resulting structure is very rich, intertwining symplectic geometry, the theory of foliations, dynamical systems, and pseudo-Riemannian geometry in interesting ways.

Before describing the contents of the book in detail, we would like to discuss a few motivating vignettes. The first two of these are to be kept in mind as motivational background, whereas the third and fourth ones will be taken up again and again later in the book.

1.1 Motivation

1.1.1 Pairs of Complementary Foliations

Let M be a smooth manifold, and $E \subset TM$ a smooth subbundle of the tangent bundle. We say that E is integrable if through every point $p \in M$ there is a local submanifold L_p with the property that for all $q \in L_p$ the tangent space T_qL_p agrees with E_q. In particular, the dimension of L_p equals the rank of E. Such a submanifold is an integral manifold (of maximal dimension) for E. If E is integrable, then the maximal connected integral manifolds are the leaves of a foliation $\mathcal{F}$ with $T\mathcal{F} = E$. Any foliation is locally trivial in the sense that, in a suitable chart around any point, intersections of the leaves with the domain of the chart look like parallel affine subspaces in $\mathbb{R}^n$, generalising the flowbox picture for one-dimensional foliations.

Now assume that we have a foliation $\mathcal{F}$ on M. The question of whether $\mathcal{F}$ admits a complementary foliation $\mathcal{G}$ is interesting, and often very difficult. The complementarity condition is that $TM = T\mathcal{F} \oplus T\mathcal{G}$, where we do not mean that TM is only abstractly isomorphic to the Whitney sum on the right-hand side, but the more stringent condition that the bundles on the right really are subbundles that form complements of each other inside TM at every point in M.

We can always choose a complement F for the subbundle $E = T\mathcal{F} \subset TM$, but in general an arbitrary complement is not integrable. Only when $\mathcal{F}$ has codimension 1, which means that the complement has rank 1, is it always integrable because all line fields are. Therefore the first interesting case to look at is that of a one-dimensional foliation on a three-manifold. In this case one looks for a two-dimensional foliation that is complementary to a given line field.

Consider the Hopf fibration $\pi\colon S^3 \longrightarrow S^2$, and the one-dimensional foliation $\mathcal{F}$ whose leaves are the fibres of π. In this case there is no complementary foliation. For if $\mathcal{G}$ were complementary to $\mathcal{F}$, then every leaf of $\mathcal{G}$ would be a connected covering space of S^2, and therefore diffeomorphic to S^2. We would then conclude that S^3 is diffeomorphic to $S^2 \times S^1$.

By the same argument, the foliation $\mathcal{F}$ whose leaves are the fibres of the non-trivial S^2-bundle over S^2 does not have a complementary foliation.

In the language of G-structures, a splitting $TM = E \oplus F$ into the direct sum of complementary subbundles of ranks p and q is a G-structure for the group $G = GL_p(\mathbb{R}) \times GL_q(\mathbb{R}) \subset GL_n(\mathbb{R})$, where n is the dimension of M. The question about the existence of such a splitting can often be answered in terms of algebraic topological invariants of M. Such a G-structure is integrable if and only if it is induced from a bifoliation, that is, a local product structure given by a pair of complementary foliations.

The question of the integrability of G-structures has been around since at least the 1950s. For example, it was raised by Calabi as Problem 9 in Hirzebruch's celebrated problem list [Hi-54]. We refer to [Hi-87] and [Kot-13] for accounts of what is now known about those problems.

For the particular type of G-structure at hand, if one does not require full integrability, but requires only the weaker condition that one of the two distributions is integrable, then a lot is known, since one is just asking for the existence of a foliation, of dimension p say, on M, assuming that the tangent bundle of M admits a rank p subbundle. In many cases the integrability of all distributions up to homotopy has been proved by Thurston, for example if $p = n - 1$ (see [Thu-76a]), and also if $p = 2$ (see [Thu-74]). In other cases there are additional obstructions coming from the Bott Vanishing Theorem that forces the vanishing of certain characteristic classes of the normal bundle of a foliation.

Returning to the full integrability of $GL_p(\mathbb{R}) \times GL_q(\mathbb{R})$-structures, it is still a very difficult problem to understand when a splitting of the tangent bundle can be induced by a bifoliation. As far as we know, the technology of h-principles emanating from Thurston's work does not apply in this case. Even in situations where both distributions are separately homotopic to integrable ones, it is very unclear whether they can simultaneously be made integrable in such a way that they remain complementary. It is certainly not possible to fix one foliation and then homotope the normal bundle to obtain a second, complementary, foliation. This problem appears already for $p = 1$ and $q = 2$, since there are circle bundles over surfaces which do not admit any horizontal foliation complementary to the fibres, as in the example of the Hopf fibration above. Although the two-dimensional horizontal subbundle is homotopic to a foliation, that foliation will never be complementary to the fibres. Of course in this case one can just switch the rôles of the two distributions and argue that one makes the two-dimensional distribution integrable without worrying about the complement since every one-dimensional distribution is integrable. This switching does not work even for $p = q = 2$. In this case all distributions are homotopic to integrable ones [Thu-74], but if we take for M^4 the non-trivial S^2-bundle over S^2, then again there is no two-dimensional foliation complementary to the fibres of the fibration. If one just homotopes the horizontal distribution to make it integrable (and no longer horizontal), then one does not know whether an integrable complement exists for the homotoped distribution.

In the case of a four-manifold whose tangent bundle splits as a Whitney sum of two rank 2 bundles, Thurston's Theorem [Thu-74] can be applied to each of the two subbundles to obtain two foliations. However, it is unknown whether one can always keep them complementary while making them both integrable. For example, it is an open problem whether $S^2 \tilde{\times} S^2$ admits a pair of complementary two-dimensional foliations.

For general surface bundles over surfaces the existence or non-existence of a horizontal foliation is an interesting problem that has attracted quite a bit of attention in recent years, but is still open. We refer the interested reader to [KM-05, Bow-11, BCS-13] for discussions of this problem.

The existence of a bifoliation with $p = q$ is a special situation, which appears, for example, in the paper of Harvey and Lawson [HL-12], where it is called a double manifold, or a $\mathbb{D}$-manifold. On a surface this kind of structure can exist only if the Euler characteristic vanishes, but even in dimension 4 there are lots of examples. An example with non-zero Euler characteristic is given by the product of two surfaces of non-zero Euler characteristic. For vanishing Euler characteristic one can take the product of an arbitrary three-manifold with the circle. The existence of two-dimensional foliations on three-manifolds together with the integrability of line fields shows that every such four-manifold is a 'double manifold', but in general the foliations have complicated dynamics, and have little to do with the global product structure.

1.1.2 Hamiltonian Dynamics

The phase spaces M considered in classical mechanics are symplectic manifolds, so there is a symplectic form ω, which is a closed non-degenerate 2-form. Non-degeneracy means that the map

$$\begin{aligned} \mathfrak{X}(M) &\longrightarrow \Omega^1(M) \\ X &\longmapsto i_X\omega \end{aligned}$$

given by contraction is an isomorphism between vector fields and 1-forms. Therefore, for any Hamiltonian function $H\colon M \longrightarrow \mathbb{R}$ there is a unique vector field $X_H \in \mathfrak{X}(M)$ defined by the equation $i_{X_H}\omega = dH$. The Hamiltonian dynamical system corresponding to H is the flow φ_t of the Hamiltonian vector field X_H. Using that ω is closed, we have

$$L_{X_H}\omega = i_{X_H}d\omega + di_{X_H}\omega = 0 + d^2H = 0\,,$$

by the Cartan formula, which implies that $\varphi_t^*\omega = \omega$, so the Hamiltonian flow is a flow by symplectomorphisms.

To understand the dynamics of the system, it is useful to find conserved quantities, or first integrals. Using that ω is skew-symmetric, we calculate

$$L_{X_H}H = i_{X_H}dH = \omega(X_H, X_H) = 0\,,$$

so H is always conserved under the flow, which is therefore along the level sets of H. For any function $f \in C^\infty(M)$ the condition that f be preserved under the flow φ_t is $df(X_H) = 0$, which can be rewritten as $\omega(X_f, X_H) = 0$. This motivates the definition of the Poisson bracket

$$\{f, g\} = \omega(X_f, X_g)$$

for any pair of smooth functions on M. If this vanishes, one says that the two functions are in involution. In this case the formula

$$i_{[X,Y]}\omega = L_X i_Y\omega - i_Y L_X\omega$$

shows that the corresponding vector fields X_f and X_g commute.

Since phase space is even-dimensional, say of dimension $2n$, the nicest possible situation is when there are n conserved quantities that are independent in a suitable sense. So let $H = f_1, \ldots, f_n$ be n conserved quantities for the

Hamiltonian flow φ_t, and assume that they are pairwise in involution, so all their Poisson brackets vanish. We consider the map

$$F\colon M \longrightarrow \mathbb{R}^n$$
$$x \longmapsto (f_1(x), \ldots, f_n(x)) \ .$$

If c is a regular value of F, then the level set $M_c = F^{-1}(c)$ is an n-dimensional smooth submanifold of M. The condition that c is regular for F means that at every point in M_c the one-forms $df_1, \ldots, df_n$ are linearly independent, and therefore the corresponding vector fields $X_{f_1}, \ldots, X_{f_n}$ are also linearly independent. However, these vector fields are all tangent to M_c, and they commute. So M_c has a locally free $\mathbb{R}^n$-action. If M_c is compact and connected, it follows that it is a torus T^n. Moreover, on M_c all the contractions $i_{X_{f_i}}\omega$ vanish since the f_i are constant, and since the X_{f_i} span the tangent spaces to M_c at all points, we conclude that the restriction $\omega|M_c$ vanishes identically. Thus M_c is an example of a Lagrangian submanifold in a symplectic manifold.

If we look at the open set of M consisting of the regular points of F, this subset carries a foliation by Lagrangian submanifolds which are the individual level sets. This is the prototypical example of a Lagrangian foliation. A global perspective on this situation was discussed by Duistermaat [Dui-80], among others.

1.1.3 Anosov Symplectomorphisms

We now consider certain special discrete dynamical systems, which exhibit hyperbolic behaviour everywhere. They will be discussed in more detail in Subsection 5.3.1 of Chapter 5.

A diffeomorphism $f\colon M \to M$ of a compact manifold is Anosov if there is a continuous splitting of the tangent bundle into invariant subbundles of positive rank $TM = E^s \oplus E^u$ such that for all $k > 0$

$$\|Df^k(v)\| \le a \cdot e^{-bk}\|v\| \quad \forall v \in E^s \ ,$$

$$\|Df^k(v)\| \ge a \cdot e^{bk}\|v\| \quad \forall v \in E^u \ ,$$

for some positive constants a and b. Here the norms are taken with respect to some arbitrary Riemannian metric g. While the precise values of the constants a and b depend on the choice of g, the property of being Anosov does not. If the defining inequalities hold for some g, then they hold for every g (with different constants).

The defining property of an Anosov diffeomorphism is sometimes referred to as the existence of an Anosov splitting $TM = E^s \oplus E^u$ into stable (or contracting) and unstable (or dilating) subbundles E^s and E^u respectively. This means that f is hyperbolic everywhere. It is easy to see that when an Anosov splitting exists, it is uniquely determined by f, as the contracting and dilating subspaces have to be maximal with these properties.

The subbundles E^s and E^u are actually tangent to foliations of M with smooth leaves, although the distributions are only assumed continuous. The resulting foliations are called the stable and unstable foliations of f.

Suppose now that M is closed and symplectic, and f is an Anosov diffeomorphism which preserves the symplectic form, $f^*\omega = \omega$, so f is an Anosov symplectomorphism. Then E^s and E^u are Lagrangian with respect to ω, and therefore are tangent to a pair of complementary Lagrangian foliations.

To see this, suppose $v, w \in E^s$. Then

$$\omega(v,w) = (f^*\omega)(v,w) = \omega(Df(v), Df(w)) = \ldots = \omega(Df^k(v), Df^n(w)) .$$

Using the auxiliary metric g, we find that there is a constant c such that

$$|\omega(v,w)| \leq c \cdot ||\omega|| \cdot ||Df^k(v)|| \cdot ||Df^k(w)|| \leq c \cdot ||\omega|| \cdot a^2 \cdot e^{-2bk} \cdot ||v|| \cdot ||w|| .$$

Letting k go to infinity, the right-hand side becomes arbitrarily small. Therefore $\omega(v,w) = 0$, and E^s is ω-isotropic. By the same argument with f^{-1} replacing f we conclude that E^u is also ω-isotropic. As the two distributions are complementary, they must be equidimensional and Lagrangian.

We have seen that an Anosov symplectomorphism of M induces a Lagrangian bifoliation, and so one would naturally like to know how common this situation is. Even without the assumption that f preserves a symplectic form, the mere existence of an Anosov diffeomorphism seems to be a very strong assumption on M, and most manifolds should not admit any such diffeomorphism. The earliest problems and conjectures to this effect go back to Anosov and Smale. For example, Smale [Sma-67, Problem (3.5)] asked whether a closed manifold admitting an Anosov diffeomorphism must be covered by Euclidean space. This would be even stronger than just saying that M must be aspherical, a conclusion which is also still unknown. We refer the interested reader to [GL-16] for a recent discussion of the status of this problem.

In the situation of an Anosov symplectomorphism much more can be said. First of all, since the top-degree power of a symplectic form is a volume form, such diffeomorphisms are volume-preserving, and, in particular, topologically transitive. Second of all, an Anosov symplectomorphism preserves its associated Lagrangian bifoliation, and so is an automorphism of this structure. We will see in Chapter 5 that the automorphism group is in fact a Lie group. More generally, the pseudogroup of structure-preserving local diffeomorphisms of a bi-Lagrangian structure is a Lie pseudogroup, and bi-Lagrangian structures are rigid structures in the terminology of Gromov [Gro-88, DG-91]. This can be seen as the starting point for the work of Benoist and Labourie [BL-93], who proved that an Anosov symplectomorphism of a compact manifold M with smooth stable and unstable foliations is smoothly conjugate to a hyperbolic infranil automorphism. In particular, the manifold M has a nilpotent Lie group for its universal covering, answering Smale's question affirmatively. The proof of [BL-93] relies on the fact that Anosov symplectomorphisms are topologically transitive and act by automorphisms of a rigid structure, so that one can apply Gromov's open orbit theorem [Gro-88].

The result of [BL-93] is part of a long line of investigations which show that much more can be proved for Anosov diffeomorphisms with smooth stable and unstable foliations than for arbitrary Anosov diffeomorphisms, for which the Anosov splitting usually has very little regularity.

1.1.4 Affinely Flat Manifolds

A manifold M is called affinely flat if its (co-)tangent bundle admits a flat torsion-free connection. Equivalently, M has an atlas whose transition maps are affine transformations between open sets in Euclidean space, which, particularly in this case, should really be thought of as affine space.

For any connection on a vector bundle $E \longrightarrow M$, the horizontal subbundle is integrable if and only if the connection is flat. If E admits a flat connection, then M, embedded in E as the image of the zero-section, is a leaf of the horizontal foliation. The horizontal foliation together with the vertical foliation, whose leaves are the fibres of E, make up a bifoliation on the total space of E. If the rank of E equals the dimension of the base manifold M, then we have a bifoliation with equidimensional foliations, or what is called a double manifold in [HL-12].

If $E = T^*M$ happens to be the cotangent bundle of M, then the total space of this bundle has a tautological exact symplectic form, for which the fibres are Lagrangian. The condition for the horizontal foliation defined by a flat connection to be Lagrangian turns out to be precisely the torsion-freeness of the flat connection. This shows that the cotangent bundle of an affinely flat manifold carries a pair of complementary Lagrangian foliations. Moreover, M is a leaf of one of the two foliations, namely the horizontal one. We will give the details of these arguments in Section 4.2 of Chapter 4. These results are due to Weinstein [Wei-71], who proved them as a converse to his observation that the leaves of Lagrangian foliations are affinely flat with respect to the Bott connection. He thus obtained the characterisation of affinely flat manifolds as the manifolds that occur as leaves of Lagrangian foliations.

In spite of this, by now classical, characterisation of affinely flat manifolds in terms of Lagrangian foliations, symplectic geometry and the theory of foliations have so far not been used to address the many open problems about affinely flat manifolds. For example, there is a long-standing conjecture, usually attributed to Chern, suggesting that the Euler characteristics of closed affinely flat manifolds must vanish. This has been proved in many special cases; for example, Klingler [Kli-17] resolved the case of affinely flat manifolds with a parallel volume form. However, the general case of Chern's conjecture is still open, and one might hope that the theory of Lagrangian bifoliations might provide some insight into it.

1.2 What is in This Book?

We have seen that symplectic manifolds with pairs of complementary Lagrangian foliations arise naturally in various parts of geometry. It is this bi-Lagrangian structure we investigate in this book, studying its geometry, and also the topology of manifolds admitting such a structure. For reasons explained in Section 5.4 of Chapter 5, we call a symplectic structure together with a Lagrangian bifoliation a Künneth structure. In this book we set out the basics of Künneth geometry starting from symplectic geometry and the theory of foliations. We think of these considerations as *a priori* a part of differential topology. It turns out that there is an essentially canonical pseudo-Riemannian metric of neutral signature associated to a Künneth structure, but this arises *a posteriori* and is not part of our definition. When discussing this metric we do not assume that the reader has any expertise in pseudo-Riemannian geometry. Instead, along the way we explain how to adapt standard arguments in Riemannian geometry to the pseudo-Riemannian setting. We use only a little complex geometry, and no para-complex geometry at all, except to show that para-Kähler structures are in fact the same as Künneth structures.

In **Chapter 2** we discuss linear symplectic geometry, including the Linear Darboux Theorem, the space of Lagrangian subspaces, and bi-Lagrangian splittings. First we carry out this discussion in a single symplectic vector space and then extend it to symplectic vector bundles. We introduce linear Künneth structures, which are just bi-Lagrangian splittings of symplectic vector bundles. The existence of a Künneth structure on a vector bundle turns out to impose strong restrictions on its characteristic classes.

Chapter 3 constitutes a quick introduction to symplectic manifolds and their Lagrangian submanifolds. We introduce the Moser homotopy method, and use it to prove the Darboux Theorem for symplectic forms and Weinstein's Tubular Neighbourhood Theorem for Lagrangian submanifolds.

In **Chapter 4** we give a brief introduction to foliations and flat bundles. We relate the integrability of subbundles of the tangent bundle to both the flatness and the torsion-freeness of certain affine connections. We also extend this discussion to almost product structures, obtaining criteria for the integrability of such a structure to a bifoliation. We introduce Lagrangian foliations of symplectic manifolds, and we discuss the Bott connection, first for general foliations, and then in more detail for Lagrangian foliations. We also adapt the Moser argument from Chapter 3 to prove the Darboux Theorem for a symplectic form together with a Lagrangian foliation.

In **Chapter 5** we begin the development of Künneth geometry itself. We give the basic definitions, and we note that instead of the usual Darboux Theorem one can prove a local normal form statement that involves a function that plays the rôle of the Kähler potential in Kähler geometry. We also introduce not necessarily integrable almost Künneth structures, which are linear Künneth structures on the tangent bundles of manifolds. We explain why every almost Künneth structure has a natural pseudo-Riemannian metric, making it into a rigid geometric structure with a small automorphism group. Most of this chapter is taken up with constructions of examples. Our emphasis is on global constructions yielding examples of Künneth structures on closed manifolds. Some of the examples we obtain have not appeared in the literature before now.

In **Chapter 6** we prove that every almost Künneth structure gives rise to a preferred affine connection for which the structure is parallel. This connection is torsion free if and only if the structure is integrable to a Künneth structure. In the integrable case only, the Künneth connection is the Levi–Civita connection of the associated pseudo-Riemannian metric. Moreover, its restriction to the two Lagrangian foliations equals the respective Bott connection. At the end of this chapter we prove the equivalence between Künneth and para-Kähler structures.

In **Chapter 7** we investigate the curvature of the Künneth connection. Some of this is done for arbitrary almost Künneth structures, but after the initial discussion of the general case we soon restrict to integrable structures, for which more can be said. We prove that a standard Darboux theorem holds for a Künneth structure if and only if the curvature vanishes. Unfortunately this does not yield a uniformisation result, because the Künneth connection is usually not complete. We work out explicit formulas for the Ricci and scalar curvatures. The formula for the Ricci curvature gives a criterion for when the neutral pseudo-Riemannian metric associated with a Künneth structure is an Einstein metric. At the end of this chapter we investigate parallel Künneth structures on Kähler manifolds.

In **Chapter 8** we introduce hypersymplectic structures. In keeping with our discussion of Künneth structures, we give a purely symplectic formulation that does not involve a pseudo-Riemannian metric or a connection as part of the definition. We do, however, show that our definition is equivalent to the usual metric definition. We show that every hypersymplectic structure gives rise to a family of Künneth structures parametrised by the circle. The leaves of the corresponding Lagrangian foliations are not just affinely flat, which is true for all Lagrangian foliations, but are also symplectic, equipped with symplectic forms that are parallel with respect to the flat affine connection. At the end of this chapter we prove that hypersymplectic structures, equivalently, their subordinate Künneth structures, are Ricci-flat, or neutral Calabi–Yau.

Chapter 9 contains a quick introduction to nil- and infra-nilmanifolds. This is motivated by the fact that Anosov symplectomorphisms can exist only on infra-nilmanifolds [BL-93]. More generally, nilmanifolds offer the possibility of reducing the construction of geometric structures to linear algebra by passing back and forth between left-invariant structures on a Lie group and the corresponding linear structures on its Lie algebra. We use this approach to give explicit examples of Anosov symplectomorphisms and of hypersymplectic – and therefore Künneth – structures on nilmanifolds. We also classify left-invariant Künneth structures on four-dimensional nilpotent Lie groups. At the end of this chapter we indicate how to generalise to solvmanifolds in place of nilmanifolds.

In **Chapter 10** we investigate (almost) Künneth structures on closed four-manifolds. After a brief introduction to the classical invariants of closed smooth four-manifolds, we use these invariants to characterise those closed four-manifolds that admit an almost Künneth structure. In particular, we prove that the existence of an almost Künneth structure does not constrain the fundamental group. We then show that the candidates for having an integrable Künneth structure are the symplectic Calabi–Yau manifolds, whose topology is very restricted. In particular, their fundamental groups are very special. The known examples of symplectic Calabi–Yau manifolds of real dimension 4 are, up to finite coverings, the $K3$ manifold and T^2-bundles over T^2. For the latter we make a systematic study of Lagrangian foliations and of Künneth structures. Many of the results in this chapter are new.

Chapter 10 freely uses results about the Seiberg–Witten invariants of (symplectic) four-manifolds. We do not explain those results, but quote them as needed, giving precise references. Those results are not used elsewhere in this book, and treating them fully would require us to write a completely different book.

1.3 How to Read This Book

Throughout this book we assume familiarity with the basic language of smooth manifolds. No specialised knowledge of differential or symplectic geometry is required.

The first five chapters can be used as a textbook for a rapid introduction to symplectic geometry and the study of Lagrangian foliations aimed at undergraduates. For this audience one could leave out the parts of Chapter 2 that discuss characteristic classes. A course covering these five chapters would fit neatly into a term with only eight or nine weeks of lectures.

In a longer course for beginning graduate students, with twelve to fifteen weeks of lectures, one can cover most of the book. For this audience one can probably cover Chapters 1 to 3 quite quickly, then treat Chapters 4 to 8 in considerable detail, and finally switch to a survey mode for Chapters 9 and 10. Indeed the original manuscript for this book was formed by the lecture notes of such a course that we taught at the University of Munich in the spring semester of 2016.

1.4 What is *Not* in This Book

The only curvature conditions we discuss for Künneth structures are flatness, leading to the best possible local normal form, and the Einstein condition. The latter is satisfied, for example, for the Künneth structures arising from hypersymplectic structures. There are many other curvature conditions one could consider, and that have been considered in the literature, but that we do not discuss here, and those lead to many special results in (local) pseudo-Riemannian geometry.

We do not discuss homogeneous Künneth structures, except on nilmanifolds. It was proved by Hou, Deng and Kaneyuki [HDK-97] that a manifold with a Künneth structure homogeneous under a compact Lie group must be a torus. For structures homogeneous under a non-compact semisimple Lie group, Hou, Deng, Kaneyuki and Nishiyama [HDKN-99] proved that the manifold must be an adjoint orbit of a hyperbolic element. This means that in both these cases one does not find any interesting compact examples. We refer to the survey by Alekseevsky, Medori and Tomassini [AMT-09] for further information on the homogeneous situation.

We also do not discuss calibrations and special Lagrangian submanifolds in Künneth geometry, but refer the reader to the paper by Harvey and Lawson [HL-12] and the references therein. As explained in those references, calibrations in Künneth geometry appear naturally in the classical Monge–Kantorovich mass transport problem.

There have been many other instances in which Künneth structures have arisen in connection with various differential equations. As noted by Hitchin [Hit-90] when he first introduced hypersymplectic structures, these structures arise naturally on moduli spaces of harmonic maps from Riemann surfaces to compact Lie groups, and on moduli spaces of solutions for the KdV equation and for the non-linear Schrödinger equation. The common feature of these equations that leads to the connection with hypersymplectic geometry is that they are dimensional reductions of the self-dual Yang–Mills equations in signature $(2, 2)$. Very recently, a variant of Nahm's equations was added to this list, again making contact with hypersymplectic geometry; see Bielawski, Romão and Röser, [BiRR-17]. Another class of differential equations, this time arising from hydrodynamics, was connected to hypersymplectic geometry by Banos, Roubtsov and Roulstone [BaRR-16].

Quite recently Künneth structures have found applications in Teichmüller theory; see Loustau and Sanders [LS-17]. This is perhaps not surprising given the appearance of Künneth vector bundles in the guise of symplectic Anosov structures in the work of Burger, Iozzi, Labourie and Wienhard [BILW-05].

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