Minimal Disturbance in Quantum Logic

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In this paper I formalize the notion of minimal disturbance, as this seems to be required by usual interpretations of the theory of quantum mechanics, and construct a quantum logical (lattice) model of the type of situation that seems to be at the root of the problem of the interpretation of Luders' projection rule as a criterion of minimal disturbance for individual state transformations. What is particularly interesting in the situation to be depicted here is that, on the basis of a simple model, which depends only on some very general features of the lattice structure of the theory (and its semantical interpretation), usual interpretive assumptions on minimal disturbance appear to be wanting.

If we restrict our attention to the statistics of measurement results, Luders' rule can easily be interpreted as a formula describing 'minimal change' for statistical states. Succinctly, this is so because Luders' rule is a version of the conditional expectation in Hilbert spaces and conditional expectation is the best estimator of the final state given the result of measurement. More concretely, as Herbut has shown in (1969). Luders' rule follows as a mathematical result if we restrict ourselves to preparatory measurements which are minimally disturbing in the metric of the operator Hilbert space representing the systems being measured.[1]

But this derivation of Luders' rule cannot be extended to justify the interpretation of the rule as a description of individual state transformations, since this presupposes the interpretation of the Hilbert space metric as a suitable relation of 'nearness' or 'approximation' for individual state transformations, an interpretation for which there does not seem to be any other ground than that it is suggested by the statistical structure. For example, Friedman and Putnam (1978) claim that in Quantum Logic one can derive Luders' rule without additional assumptions, something they consider to be an advantage of Quantum Logic over usual interpretations. But to the extent that quantum logic considers the statistical structure as directly reflecting the structure of individual systems (events), Luders' rule is simply assumed in the process. That is because this statistical structure includes the Hilbert space metric, and minimal disturbance in that metric is equivalent to the choice of the sasaki conditional (see Hardegree (1976) and Hellman (1981)). Quantum logic, as other individual state interpretations I have surveyed elsewhere (in 1987), cannot claim any advantage from the possibility of deriving Luders' rule in quantum logic. *This derivation is just a lattice theoretical formulation of a mathematical theorem which is available to all interpretations.*

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But what about more positive characterizations of minimal disturbance? That is, could we not try to characterize a concept of minimal disturbance *independently of Luders rule* and *then* show that measurement transformations satisfying this criterion are described by Luders rule? I show next that the most natural way of attempting to make precise this idea flounders, and as it seems, without hope of repair.

Since we are interested in Luders' rule as a state transformation rule we proceed to formulate a criterion of minimal disturbance in terms of the (individual) states of the systems in question. In a first approximation, minimally disturbing measurements should satisfy the condition that there is no other measurement of the same magnitude and with the same result preserving "more" of the original state of the system. Let us call this rough criterion the *natural criterion of minimal disturbance*. It seems reasonable to take this criterion, at least whenever it is unambiguous, as a necessary condition for a physically meaningful concept of minimal disturbance for individual state transformations.

I will make this idea of minimal disturbance precise within an algebraic (quantum logical) framework representing the so-called "propositional structure" of the theory (the Von Neumann-Birkhoff approach to the foundations of mechanics). In the quantum logical framework a physical system is represented by a complete atomic orthomodular lattice (of propositions) with the covering property.[2] We shall call a lattice with these properties a *quantum lattice*. Classical systems are represented by distributive lattices whereas non-classical quantum systems are represented by non-distributive quantum lattices. The connection with quantum mechanics is established through the isomorphism between the lattice of propositions and the lattice of closed subspaces of the Hilbert space representing the system. The magnitudes of the system are identified with the Boolean sublattices of the quantum lattice. The discussion below does not depend on the subtleties of the mathematical construction process leading to the quantum lattice structure. Details can be found in any treatise on Quantum Logic (for example in Beltrametti and Cassinelli 1981).

According to the usual interpretation of the state vector for individual systems the system 'has' the 'properties' (corresponding to propositions) to which the state vector assigns probability one. *Individual states then, as usually interpreted, are represented by maximal filters in the lattice.* By assumption of completeness, states can be identified with the class of principal ultrafilters i.e. with filters generated by atoms of the quantum lattice representing the system. Since, in usual interpretations, every state is uniquely determined by an atom, and vice versa every atom generates a unique state, I will often talk of atoms as 'states'. This harmless ambiguity will simplify notation. The principal filter generated by **x** is denoted by **[x)**. A measurement state transformation will be represented as a function of two variables: **T(a,r) = b**, where **a** is (the atom generating the) initial state, **r** the proposition representing the result of measurement and **b** is (the atom generating) the final state.

Given a classical system **S** in initial state **[a)**, suppose we measure a magnitude **M** (including **r**) and find **r** as the (proposition representing the) result of measurement. Now, an ideal or 'minimally disturbing' state transformation is one that 'least disturbs' the original state of the system. But there is no intrinsic way (intrinsic to the lattice structure and thus, we assume, intrinsic to the theory) of characterizing this idea beyond the following restriction: if $\mathbf{r} \in \mathbf{[a)}$ then the final state is the same initial state, and if $\mathbf{r} \in \mathbf{[a)}$ then all atoms 'below' **r** (i.e. all atoms **x** such that $\mathbf{x} \leq \mathbf{r}$) generate a state transformation that at least in principle could be brought about by the measurement. As an example look at lattice $\mathbf{B_8}$ in Fig. 1. Let $\mathbf{r} = \mathbf{b} \vee \mathbf{c}$, clearly $\mathbf{T_1(a,r)} = \mathbf{c}$ and $\mathbf{T_2(a,r)} = \mathbf{b}$ cannot be distinguished by any intrinsic feature of the lattice structure.[3] The concept of minimal disturbance in classical mechanics is thus non-theoretical, it reduces to pragmatic considerations on the actual physical process of obtaining data from experimental procedures.

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In the case of (non-classical) quantum systems (represented by non-Boolean quantum lattices) the condition that **r** $\in$ [a) also implies that all atoms 'below' **r** generate a state transformation which is a possible description of measurement. *Here, however, Luders' rule provides an additional criterion that allow us to select one state from among these*

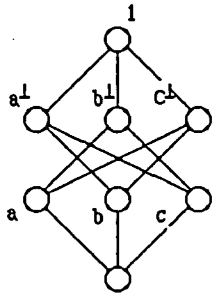

Figure 1. Hasse diagram of an 8-element Boolean lattice

possible final states. Luders' rule, in the algebraic (lattice) framework takes the following form: $L(a,x) = x \wedge (a \vee x^{\perp})$, where **a** is the initial state and **x** is the result of measurement (see Appendix), Luders' rule singles out a unique state transformation since **L(a,x)** is an atom of the lattice (see theorem A3 in the Appendix). So, for example, in the lattice depicted in Fig. 2, if the initial state is a and the result of an autonomous non-maximal measurement of magnitude $(d, d^{\perp})$ is $d^{\perp}$, then according to Luders' rule the final state is $T(a,d^{\perp}) = e$. But now, how can Luders' rule be characterized as a criterion of minimal disturbance for states?.

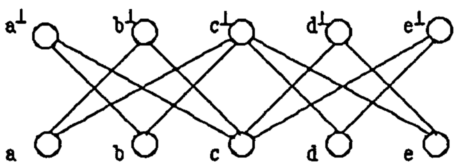

Figure 2. Partial Hasse diagram of a quantum lattice (L_{12})

Consider the following example. Lattice D in Fig. 3 is a sublattice of $L(H_3)$ (This quantum lattice isomorphic to the lattice of propositions of a 3-dimensional Hilbert space). Suppose the initial state is [a) and the result of measurement is $f^{\perp}$. Acceptable state transformations in D are $T_1(a,f^{\perp}) = e$ and $T_2(a,f^{\perp}) = g$. Notice however that $c^{\perp} \in$ [e) whereas $c^{\perp} \notin$ [g). The natural criterion of minimal disturbance (restricted to D) forces us to conclude that $T_1(a,f^{\perp}) = e$ is minimally disturbing. But **e** is not the final state given by Luders rule. As a matter of fact in this case Luders' rule does not select a unique final state. It does not distinguishes between $T_1(a,f^{\perp}) = e$ and $T_2(a,f^{\perp}) = g$. Now, the lattice D of our example (Dilworth's lattice) is not a quantum lattice. In fact this lattice is a well known example of an orthomodular lattice which is not a quantum lattice because (as it is easy to check) it does not satisfy the covering property. Thus, in D, all axioms except the covering property are satisfied, but nonetheless minimal disturbance selects a unique state, Luders' rule does not. On the other hand (see theorem A3 in Appendix), for com-

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plete atomic orthomodular lattices (like D), the validity of the covering property is equivalent to the assumption that Luders' rule selects a unique final state. The example above can be understood, then, as a counterexample to the most natural way of reconstructing (in quantum logic) the main claim of usual interpretations concerning minimal disturbance. Implicit in usual interpretations is the claim that the set of 'ideal' measurement transformations to which Luders' rule applies is a physically distinguishable set of transformations (via minimal disturbance). That is, it is assumed that there is a physical criterion, which can be formulated in terms of minimal disturbance, that allows us to characterize the transformations theoretically singled out by Luders' rule. But if we assume the covering property Luders' rule follows as a mathematical result, and if we do not, minimal disturbance does not coincide with Luders' rule. The existence of the physical criterion for individual state transformations as assumed by usual interpretations is therefore very questionable.

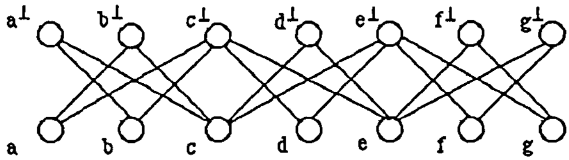

Figure 3. Partial Hasse diagram of Dilworth's lattice

One could argue that the relation between minimal disturbance and Luders rule is not as straightforward as I have depicted above. One could argue, that is, that the relation between minimal disturbance and Luders rule' is more indirect, in a way which is not captured by the above modelling of the situation. But short of defining minimal disturbance in terms of Luders' rule it seems there is not much one could add (at least to the extent that the quantum logical axiomatization faithfully reproduces the structure of the theory). To argue that minimal disturbance is, as in the case of classical mechanics, a non-(lattice) theoretical criterion would seem to sever in principle the desired connection with Luders' projection rule and force us to recognize some sort of hidden variable account of the statistical structure of the theory.

Taking the situation exemplified above in terms of the lattice D as the point of departure there is another argument that can be brought against the usual assumed connection between minimal disturbance and Luders' rule. As we shall see, the natural criterion of minimal disturbance overdetermines the final state (in a sense to be made precise below).

It is possible to find sets of propositions, say **C** and **D**, with $\mathbf{C} \neq \mathbf{D}$ (and **C** and **D** contributing "equally as much" to the original state) such that, for initial state [**a**), preserving **C** is sufficient to determine a final state and preserving **D** is also sufficient to determine another final state. Let $\mathbf{L(H_3)}$ be a (three dimensional) quantum lattice, and let **a**, an atom of **L**, represent an initial state. Suppose we measure non-maximal magnitude $\{\mathbf{f},\mathbf{f}^{\perp}\}$. There are different sublattices $\mathbf{D}_i$ of $\mathbf{L(H_3)}$ isomorphic to Dilworth's lattice D (Fig. 3), which can be 'interpolated' for initial state **a** and non-maximal magnitude $\{\mathbf{f},\mathbf{f}^{\perp}\}$. To see this take a basis $\{\mathbf{a}_i\}$ such that $\mathbf{a} = \mathbf{a_1}$, and such that a rotation around $\mathbf{a_3}$ gives a new basis $\{\mathbf{b}_j\}$ such that $\mathbf{b_1} = \mathbf{a_3}$ and $\mathbf{b_3} \perp \mathbf{f}$. In general there will be different ways in which this can be done. Each of these different 'interpolations' of a lattice D provides us sufficient information on the measurement to select a unique final state according to the criterion of minimal disturbance.

It is clear then that in our example partial information is sufficient to give a unique state. *Further more, bits of partial information give (in general) different unique states,*

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leading thus to contradictory conclusions concerning the final state. This is a challenge to the privileged status assigned to Luders' rule as a criterion of minimal disturbance. One has to wonder at this juncture how one could possibly justify the selection of Luders' rule among all possible rules obtained as above.[4]

One could argue that the criterion of minimal disturbance should be applied in $L(H_3)$ to "all relevant information", But what would this "relevant" information be? The immediate answer, that "relevant information" is information corresponding to propositions generated by compatible magnitudes does not work. The notion of "all relevant information" has a clear meaning only for statistical transformations. Simply to require minimal disturbance is not enough to select a unique state transformation according to Luders' rule. One has to qualify which (sort of) propositions are to be minimally disturbed, that is, what is *the* relevant information, and here the problem of justifying Luders' rule as a description of individual physical processes would start anew.

Appendix: Lattice Formulation of Luders' Rule.

The fundamental algorithm of Quantum Mechanics states that a measurement of the quantity **A** in a system represented by Hilbert space **H** in the state **Q** gives the result a with probability **Pro(a) = tr(QP)**. Where **P, Q** are projections in **H**. In the Hilbert space formulation of Quantum Mechanics Luders rule states that after an (ideal) measurement with result represented by **P**, the state of the system is given by

$$PQP = (trQP)P$$

Let $E_1(PQP)$ denote the support of the projection **PQP**, that is $E_1(PQP)$ is the orthocomplement of the null space of **PQP**: $\{x \in H : PQPx = 0\} = E_0(PQP)$

A1 Theorem: For all projections **P,Q** in **H**
$E_0(PQP) = P^{\perp} \vee (P \wedge Q^{\perp})$
For the proof see Hardegree (1976).

A2 Corollary: $E_0{}^{\perp}(PQP) = E_1(PQP) = P \wedge (P^{\perp} \vee Q)$

In our abstract lattice theoretical (quantum logical) framework then, for **q** an atom of the lattice and **p** an arbitrary element, the transformation $L(q,p) = p \wedge (p^{\perp} \vee q)$ represents Luders' transformation.

A3 Theorem: Let **L** be a complete orthomodular atomic lattice, The following two conditions are equivalent:

(i) The covering property holds

(ii) Luders rule selects a unique final state
(for any atom **p** and lattice element **q**, $q \neq 0$)

See Piron (1976) for a proof of this theorem. In my dissertation and in (1988) I provide a different proof that allows for a physical interpretation of the covering property independently of Luders' rule.

Notes

[1]I have shown that all derivations of Luders' rule in the literature that are known to me are, as statistical derivations, variants of Herbut's derivation in a sense that I made precise elsewhere (in 1987). Herbut's derivation can be seen in turn as a converse to Luders' theorem in Luders (1951).

[2]We say that a covers **b** (notation: **a** $\cdot\!>$ **b**) if **a** > **b** and for no **x** **a** > **x** > **b**. A lattice (with **0**) has the covering property if **a** $\wedge$ **x** = **0** implies **x** $<\!\cdot$ **x** $\vee$ **a** for any atom **a** and element **x**.

[3]This idea can be made precise by appealing to the fact that the equivalence classes generated by both state transformations (via the embedding theorem) are isomorphic.

[4]One could argue similarly to Hellman to show some of the difficulties in justifying Luders' rule among the different rules satisfying the natural criterion of minimal disturbance as presented above. See also Teller.

References

Beltrametti, E. and Cassinelli G. (1981). *The Space Logic of Quantum Mechanics*. Reading, MA: Addison-Wesley.

Friedman M. and Putnam H., (1978). "Quantum Logic, Conditional Probability and Interference," *Dialectica* 32: 305-315.

Hardegree, G. (1976) "The conditional in Quantum Logic." In *Logic and Probability in Quantum Mechanics*. Edited by P. Suppes. Dordrecht: Reidel Press.

Hellman G. (1981). "Quantum Logic and the Projection Postulate." *Philosophy of Science* 48, 3: 469-486.

Herbut F. (1969). "Derivation of the Change of State in Measurement from the Concept of Minimal Measurement." *Annals of Physics* 55: 271-300.

Luders G. (1951). "Uber die Zustansanderung durch den Messprozess." In *Ann. d. Phys*. 8: 322-328.

Piron C. (1976). *Foundations of Quantum Physics*. New York: W.A. Benjamin, Inc.

Martinez S. (1987). "A Search for the Physical Content of Luders Rule," forthcoming.

_______. (1988). "Luders' Rule as a Description of Individual State Transformation," forthcoming.

Teller P. (1983). "The Projection Postulate as a Fortuitous Approximation." *Philosophy of Science* 50: 413-431.