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A SUBSTRUCTURAL GENTZEN CALCULUS FOR ORTHOMODULAR QUANTUM LOGIC

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Abstract. We introduce a sequent system which is Gentzen algebraisable with orthomodular lattices as equivalent algebraic semantics, and therefore can be viewed as a calculus for orthomodular quantum logic. Its sequents are pairs of non-associative structures, formed via a structural connective whose algebraic interpretation is the *Sasaki product* on the left-hand side and its De Morgan dual on the right-hand side. It is a *substructural* calculus, because some of the standard structural sequent rules are restricted—by lifting all such restrictions, one recovers a calculus for classical logic.

§1. Introduction. A few decades back, the realm of subclassical logics was fragmented into several research areas that were pursued, to a large extent, independently of one another: intuitionistic logic, intermediate logics, fuzzy logics, relevant logics, linear logic, and whatnot. In the 1990s, the emergence of the powerful unifying framework of *substructural logics* [17, 29] provided a common ground where these logics could be mutually compared and studied by recourse to the same concepts and methods.

At about the same time, a similar impulse towards unification was driving the work of algebraists, who had long noticed conspicuous similarities between the structure theories of classes of algebras introduced with very different motivations, like lattice-ordered groups, Boolean algebras, Heyting algebras, or MV-algebras. The notion of a *residuated lattice*, anticipated as early as in the 1930s by Ward and Dilworth, but formulated and investigated in full generality only at the end of last century by Constantine Tsinakis, Hiroakira Ono and other authors [6, 17], supplied the appropriate generalisation subsuming all these examples. And it was not long before these two processes were shown to be related via the Blok–Pigozzi theory of algebraisation: all the main substructural logics were indeed proved to be algebraisable with varieties of residuated lattices as equivalent algebraic semantics. As a matter of fact, there is good reason to claim that this unification of the above-mentioned research domains has been one of the most successful achievements of Abstract Algebraic Logic (AAL) so far.

Some important logics and classes of algebras, however, are hardly pliant to this approach. Orthomodular lattices and orthomodular quantum logic are cases in point. Orthomodular lattices have, generally speaking, no pair of operations that behaves

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like a residuated pair. Correspondingly, candidate implications in orthomodular quantum logic lack the minimal requirements that would entitle them to be regarded as substructural conditionals [15, Chap. 8]. As a consequence, results, tools and techniques that have proved successful in the area of residuated structures and substructural logics have been by and large unavailable to quantum logicians.

However, the problem can be circumvented. Although orthomodular lattices make no instance of residuated lattices, or even of residuated ℓ-groupoids, Coecke and Smets [12], building on previous work by Hardegree [22], showed that a pointed *left*-residuated ℓ-groupoid can be extracted from an ortholattice $\mathbf{L}$ iff $\mathbf{L}$ is orthomodular. Later, Chajda and Länger [9] extended this result to a full-blown term equivalence between orthomodular lattices and a variety of pointed left-residuated ℓ-groupoids (called *orthomodular groupoids*) having rather strong properties. This result prompted some of the present authors to undertake an investigation of the structure theory of pointed left-residuated ℓ-groupoids, with an eye to building a general theory where orthomodular lattices (in their incarnation as orthomodular groupoids) could find a home alongside residuated lattices and other algebras of substructural logics. Some preliminary results are contained in [30].

The present paper is a continuation of this programme. We intend to press the point that orthomodular quantum logic is indeed a substructural logic and provide a *sequent calculus* obtained from the calculus for classical logic by suitably restricting certain structural rules (as well as modifying some operational rules). At this point, a word of caution is in order. Two different logics have been marketed as "orthomodular quantum logic" in the literature (see below). The one we have in mind is the 1-assertional logic of the variety $\mathbb{OML}$ of orthomodular lattices, which is more closely associated with it because it is regularly algebraisable with $\mathbb{OML}$ as its equivalent variety semantics. We will show that our calculus is *Gentzen algebraisable* with orthomodular groupoids as equivalent algebraic semantics, and in particular that its external consequence relation (in the sense of Avron [1]) coincides with the 1-assertional logic of the variety $\mathbb{OG}$ of orthomodular groupoids. Hence, in light of the term equivalence mentioned above, it can be viewed as a calculus for the 1-assertional logic of $\mathbb{OML}$.

The main differences with the existing calculi for orthomodular quantum logic [13, 28] or for substructural logics are as follows.

(1) First of all, with respect to sequent systems for orthomodular logic, our sequents are pairs of structures of different data-types. In the calculi of Nishimura and Cutland-Gibbins, sequents are pairs consisting of sets of formulas (or of a set of formulas and a formula), and the algebraic interpretation of comma is lattice meet on the left-hand side and lattice join on the right-hand side. On the contrary, we operate with certain non-associative structures, and the algebraic interpretation of comma (here noted as a circle) is the *Sasaki product* on the left-hand side and its De Morgan dual on the right-hand side. This allows us to make the most of the fact that the Sasaki product and the Sasaki hook form a left-residuated pair.

(2) Secondly, certain structural rules of the calculus, like contraction or weakening to the left of the circle, can be applied unrestrictedly. Other structural rules instead, like exchange, associativity or weakening to the right of the circle, can only be applied when the algebraic interpretations of the involved structures *commute* in the sense of the algebraic theory of orthomodular lattices

(see below). A sequent calculus for classical logic, naturally enough, can be obtained by lifting all these restrictions.

(3) Finally, with respect to substructural calculi, some nontrivial modifications in the usual operational rules for the connectives are necessary, because the standard formulations are tailored to classes of algebras whose underlying groupoid operation and order relation are *compatible* and thus form a *po*-groupoid. This is not the case in $\mathbb{OG}$, where the Sasaki product obeys monotonicity from one coordinate, but not generally the other, w.r.t. the lattice ordering.

Rigorously speaking, our calculus is not a *display calculus* in the same way as those examined, e.g., in [4, chap. 6]. Still, it retains at least a flavour of Belnap's Display Logic. In particular, the rule (δ) below, which allows one to move structures back and forth to either side of the sequent arrow, is crucial in yielding cut-free proofs of particular sequents.

Whether this calculus enjoys cut elimination is left as an open problem. Were it cut-free, it would still fail the subformula property, but the other rules that violate the property are harmless for analytic proof search. Even if a positive answer to this problem would not automatically guarantee a decision procedure for the calculus—and thereby an answer to the long-standing open problem of the decidability of the equational theory of orthomodular lattices—it would certainly yield important insights into the issue.

The paper is structured as follows. Although we presuppose that the reader has a working knowledge of universal algebra and AAL, and a certain familiarity with residuated structures and substructural logics, in Section 2 we go over some background notions concerning Gentzen systems, non-associative substructural logics, orthomodular lattices, and pointed left-residuated ℓ-groupoids, in order to make this paper reasonably self-contained. In Section 3 we present our calculus `OGC` and its basic properties. In Section 4 we show that it is Gentzen algebraisable with $\mathbb{OG}$ as equivalent algebraic semantics. In Section 5 we briefly hint at the modifications needed to obtain from `OGC` a calculus for pointed left-residuated ℓ-groupoids. Finally, in Section 6 we conclude with some final remarks and open problems.

§2. Preliminaries.

2.1. Gentzen systems and Gentzen algebraisability. The notion of a Gentzen calculus we consider here is more general than the one developed in [16, Definition 1.18 and Sec. 5.6], while it is somewhat closer to the one in [31]. The same holds for the relation of Gentzen algebraisability, which we view as an instance of the relation of equivalence between (abstract) consequence relations investigated by Blok and Jónsson [5] and Galatos and Tsinakis [19]. We will refrain from giving all the formal details, referring the reader to these papers for any undefined concept. This will involve a certain amount of handwaving, in the interests of brevity.

Generally speaking, a (sentential) *Gentzen calculus* `G` (or *sequent calculus*) is a formal deductive system composed of a collection of inference rules, applying which it is possible to derive sequents from other sequents. A *sequent* is a pair (X, Y), often evocatively written with a *separator* $\Rightarrow$, e.g., $X \Rightarrow Y$, where the *structures* X, Y are of some specified *data-type* (e.g., sets, multisets, etc.) of formulas over some fixed sentential language $\mathcal{L}$. Inference rules are usually distinguished into three kinds:

- *Axiom*: a rule with no premises, whose consequent is called an *initial sequent*.
- *Structural*: a rule which acts solely on the form of the structures involved in the sequents (i.e., ignoring the shape of the formulas).
- *Logical*: a rule that acts on the shape of formulas (usually by introducing a symbol from $\mathcal{L}$ in the conclusion).

An inference rule is typically presented schematically via a *meta-inference rule*, where it is understood that each substitution-instance of that scheme is an inference rule in the system $\mathtt{G}$.

Given a set of sequents $S \cup \{s\}$, a *proof of s from S in* $\mathtt{G}$ is a finite rooted labelled tree $\mathcal{D}$ whose nodes are labelled by sequents, such that the root of $\mathcal{D}$ is labelled by s, the sequents labelling each internal node and its children-nodes form an instance of some (meta-)inference rule from $\mathtt{G}$, and each leaf is labelled by an initial sequent or a member of S. For a Gentzen system $\mathtt{G}$, there is an associated action-invariant consequence relation $\vdash_{\mathtt{G}}$ in the sense of the general theory by Blok and Jónsson [5] and Galatos and Tsinakis [19], often called the *derivability relation for* $\mathtt{G}$, such that $S \vdash_{\mathtt{G}} s$ if and only if there is a proof of s from S in $\mathtt{G}$. A sequent is said to be *provable in* $\mathtt{G}$ if $\vdash_{\mathtt{G}} s$, i.e., there is a proof of s from the empty set in $\mathtt{G}$.

Let $\mathbb{K}$ be a class of algebras in the same language $\mathcal{L}$. The *equational consequence relation for* $\mathbb{K}$ is the relation $\models_{\mathbb{K}}$ such that for every set of $\mathcal{L}$-equations $E \cup \{\varphi \approx \psi\}$, $E \models_{\mathbb{K}} \varphi \approx \psi$ if and only if for every algebra $\mathbf{A} \in \mathbb{K}$ and every *valuation* h on $\mathbf{A}$, i.e., any homomorphism from the algebra of $\mathcal{L}$-formulas to $\mathbf{A}$, if $h(\gamma) = h(\delta)$ for every $\gamma \approx \delta \in E$ then $h(\varphi) = h(\psi)$.

In essence, a Gentzen system $\mathtt{G}$ is said to be *algebraisable* if there exists a class of algebras $\mathbb{K}$ for which the derivability relation of $\mathtt{G}$ and the equational consequence relation for $\mathbb{K}$ are Blok–Jónsson equivalent, in which case $\mathbb{K}$ is called an *equivalent algebraic semantics* for $\mathtt{G}$ (an equivalent *variety* semantics, if $\mathbb{K}$ is a variety). More precisely, and following the conventions of Raftery [31] (as opposed to [16]), $\mathtt{G}$ is algebraisable, with an equivalent algebraic semantics $\mathbb{K}$, if there exists a pair of *transformers* (τ, ρ), both commuting with substitutions, such that for any set of sequents $S \cup \{s\}$ and for all $\mathcal{L}$-formulas φ, ψ:

$$S \vdash_{\mathtt{G}} s \iff \tau[S] \models_{\mathbb{K}} \tau(s) \qquad \text{and} \qquad \varphi \approx \psi \mathrel{=}\joinrel\mathrel{|}\joinrel\mathrel{\models}_{\mathbb{K}} \tau[\rho(\varphi \approx \psi)].$$

2.2. Non-associative substructural logics. As explained in the introduction, in this paper we want to recast orthomodular quantum logic as a non-associative substructural logic. The study of these logics is not unprecedented. Less deviant forms of non-associative substructural logics—with two residuals and a monotonic order—have been considered in the literature. An exhaustive list would be long, but let us mention at least [11, 18] and papers in the tradition of Lambek calculus [8, 27]. Further information on substructural logics and residuated structures can be found in [17, 26].

The sequent calculus $\mathtt{GL}$, as developed by Galatos and Ono [18], can be seen as the basis for substructural logics lacking associativity of the *fusion* connective and of its structural counterpart, *comma*. The system $\mathtt{GL}$ is essentially obtained by relaxing from the full Lambek calculus $\mathtt{FL}$ the data-type of the structures within the sequents from lists to binary trees.

We recall that a *pointed residuated unital* ℓ-groupoid is an algebra $(G, \wedge, \vee, \cdot, \backslash, /, 0, 1)$ that has a lattice reduct $(G, \wedge, \vee)$, a unital groupoid reduct $(G, \cdot, 1)$, and further satisfies the *residuation law*: for all $a, b, c \in G$,

$$b \leq a \backslash c \iff a \cdot b \leq c \iff a \leq c/b,$$

where $\leq$ is the induced lattice-order, with no assumptions placed upon the constant 0. The residuation quasiequations can be replaced by appropriate equations, hence pointed residuated unital ℓ-groupoids form a variety $\mathbb{PRG}$. Just as axiomatic extensions of $\mathtt{FL}$ correspond to subvarieties of pointed residuated lattices, axiomatic extensions of $\mathtt{GL}$ correspond to subvarieties of residuated lattice-ordered groupoids; more precisely, the variety of pointed residuated unital ℓ-groupoids is an equivalent algebraic semantics for $\mathtt{GL}$.

2.3. Orthomodular lattices and orthomodular quantum logic. Orthomodular lattices are the algebras at the centre of the standard Birkhoff–von Neumann approach to quantum logic (for a readable survey of their algebraic theory, see [7]). In this framework, quantum events (or properties) are mathematically represented by projection operators on a complex separable Hilbert space. If $\mathcal{H}$ is a Hilbert space and $\Pi(\mathcal{H})$ is the set of all projection operators on $\mathcal{H}$, the structure

$$(\Pi(\mathcal{H}), \wedge, \vee, ', 0, 1),$$

where 0 (respectively, 1) is the projection onto the one-element (respectively, total) subspace, $(P_X)'$ is the projection onto the subspace $X^{\perp}$ orthogonal to X, $P_X \wedge P_Y = P_{X \cap Y}$, and $P_X \vee P_Y = P_{(X \cup Y)^{\perp\perp}}$, is a canonical example of an *orthomodular lattice*, a structure defined below.

An *ortholattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, ', 0, 1)$, where the reduct $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, with 0 (1) the least (greatest) element, and for which the unary operation $'$ is an ortho-complementation, namely an antitone involution such that the identity $x \wedge x' \approx 0$ holds in $\mathbf{L}$. An ortholattice $\mathbf{L}$ is an *orthomodular lattice* if it satisfies the orthomodular law:

$$x \leq y \implies y \approx (y \wedge x') \vee x,$$

where $\leq$ is the induced lattice-order, or equivalently the identity $x \vee ((x \vee y) \wedge x') \approx x \vee y$. The class of orthomodular lattices thus forms a variety, denoted by $\mathbb{OML}$. Of particular interest for this work are the following term operations which may be defined in any ortholattice:

$$\begin{array}{rcll} x \cdot y & := & (x \vee y') \wedge y & \text{(Sasaki product)}, \\ y/x & := & (y \wedge x) \vee x' & \text{(Sasaki hook)}, \\ x + y & := & (x \wedge y') \vee y & \text{(Sasaki sum)}. \end{array}$$

We note that, in general, the Sasaki product and sum are neither associative nor commutative in ortho(modular) lattices.

A certain relation on orthomodular lattices, the *commuting relation*, is of crucial importance for this paper. Given an orthomodular lattice $\mathbf{L}$ and $a, b \in L$, a is said to *commute* with b in case $(a \wedge b) \vee (a' \wedge b) = b$. We flag some known facts about the commuting relation that will be put to good use hereafter (items (3) and (4) are known as *Kröger's Lemma* [3, Theorem 6.1] and the *Gudder–Schelp Theorem* [2, 21], respectively).

LEMMA 1. *Let* $\mathbf{L}$ *be an orthomodular lattice, and let* $a, b, c \in L$.

(1) *The commuting relation is reflexive and symmetric.*
(2) *If* a *commutes with* b *and with* c, *then it commutes with* $b', b \vee c$ *and* $b \wedge c$.
(3) *If* b *commutes with* c, *then* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *and* $b \cdot c = c \cdot b$.
(4) *If* b *commutes with* c *and* a *commutes with* $b \wedge c$, *then* $a \vee b'$ *commutes with* c.
(5) a *commutes with* b *if and only if* $a \cdot b \leq a$.

The next celebrated result is one of the most useful tools for practitioners of the field:

THEOREM 2 (Foulis–Holland [7, Proposition 2.8]). *If* $\mathbf{L}$ *is an orthomodular lattice and* $a, b, c \in L$ *are such that* a *commutes both with* b *and with* c, *then the set* $\{a, b, c\}$ *generates a distributive sublattice of* $\mathbf{L}$.

Two different logics can be, and have indeed been, associated with $\mathbb{OML}$: its 1-assertional logic $\mathrm{L}^1_{\mathbb{OML}}$ (see e.g., [14, 24]) and its logic of order $\mathrm{L}^{\leq}_{\mathbb{OML}}$ (see e.g., [15, 20, 23]).

In full, $\mathrm{L}^1_{\mathbb{OML}}$ is the logic in the language of $\mathbb{OML}$ whose consequence relation is defined as follows for any set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathrm{L}^1_{\mathbb{OML}}} \varphi \text{ iff } \{\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathbb{OML}} \varphi \approx 1.$$

This logic is regularly algebraisable with $\mathbb{OML}$ as equivalent algebraic semantics. The logic $\mathrm{L}^{\leq}_{\mathbb{OML}}$ has the same language but its consequence relation is defined as follows for any *finite* set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathrm{L}^{\leq}_{\mathbb{OML}}} \varphi \text{ iff for any } \mathbf{A} \in \mathbb{OML}, \text{ any } b \in A \text{ and any valuation } h \text{ to } \mathbf{A},$$
$$\text{if } b \leq h(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } b \leq h(\varphi).$$

This logic is slightly less well-behaved from the point of view of AAL, although it fits in a number of frameworks that have been intensively studied (logics of order, selfextensional logics with conjunction, semilattice-based logics with an algebraisable assertional companion: see [16, Sec. 7.2]). From general facts in these theories we can deduce that it is protoalgebraic but not even equivalential, let alone algebraisable, and yet that the class of algebra reducts of its reduced matrix models is again $\mathbb{OML}$.

The existing calculi for "orthomodular quantum logic"(Nishimura [28] or Cutland and Gibbins [13]) can be considered, in a weak sense, as proof systems for either of these logics. Indeed, a sequent $\Gamma \Rightarrow \varphi$, where Γ is a finite set of formulas, is a theorem of Nishimura's calculus just in case $\Gamma \vdash_{\mathrm{L}^{\leq}_{\mathbb{OML}}} \varphi$; on the other hand $\Rightarrow \varphi$ is derivable from the set $\{\Rightarrow \gamma : \gamma \in \Gamma\}$ just in case $\Gamma \vdash_{\mathrm{L}^1_{\mathbb{OML}}} \varphi$ (and similar reflections hold for the calculus by Cutland and Gibbins). In a stronger sense, however, they might just as well be proof systems for neither, since none of such calculi has been proved to be Gentzen algebraisable with $\mathbb{OML}$ as equivalent algebraic semantics.

2.4. Pointed left-residuated ℓ-groupoids. In light of the left residuation property observed in orthomodular groupoids, a theory of left-residuated lattice-ordered groupoids has been developed [30] with the aim of placing the theory of substructural logics and quantum logics within a common framework. A *pointed left-residuated ℓ-groupoid* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, /, 0, 1)$ that has a lattice reduct $(L, \wedge, \vee)$, a groupoid reduct $(L, \cdot, 1)$ with a left-unit 1, and furthermore satsifies the *left-residuation law*[1]: for all $a, b, c \in L$,

$$a \cdot b \leq c \iff a \leq c/b,$$

where $\leq$ is the induced lattice-order.

In [30] it is shown that the class of all pointed left-residuated ℓ-groupoids forms a variety, denoted by $\mathbb{PLRG}$. As with pointed residuated ℓ-groupoids, 0 is an arbitrary constant for which no assumptions are placed. However, we will make use of the term-defined operation $\varphi' := 0/\varphi$, where φ is any formula in the language of pointed left-residuated ℓ-groupoids.[2] We caution the reader that, in general, this operation is neither involutive nor antitone in $\mathbb{PLRG}$.

An *orthomodular groupoid* is a pointed left-residuated ℓ-groupoid that satisfies the following identities:

$$\begin{aligned} x'' &\approx x && \text{(involutive)},\\ (x \vee y)' &\leq x' && \text{(antitone)},\\ x \cdot (x \vee y) &\approx x && \text{(strongly idempotent)},\\ x \cdot y &\approx (x \vee y') \wedge y && \text{(Sasakian)},\\ x/y &\approx (x \wedge y) \vee y' && \text{(/-Sasakian)}. \end{aligned}$$

As is such, the class of orthomodular groupoids forms a variety, denoted by $\mathbb{OG}$. In [9], it is shown that orthomodular groupoids are term equivalent to orthomodular lattices. The proposition below shows that many of the above identities can be equivalently replaced by the identity (*strongly involutive*): $y/x \approx (y' \cdot x)'$.

Proposition 3 [30]. *A pointed left-residuated ℓ-groupoid is an orthomodular groupoid if and only if it is strongly idempotent and strongly involutive.*

Remark 4. *The following quasi-identities do not generally hold in $\mathbb{OG}$:*

$$\begin{aligned} x \leq w \;\&\; y \leq z &\implies x \cdot y \leq w \cdot z,\\ x \leq w \;\&\; y \leq z &\implies (y/w) \cdot x \leq z. \end{aligned}$$

The above remark draws a stark contrast in $\mathbb{PLRG}$ with $\mathbb{PRG}$, whose groupoid operation $\cdot$ is isotone in both its coordinates, and therefore the two quasi-identities above are satisfied.

§3. `OGC`: a calculus for orthomodular groupoids.

3.1. The morphology of the calculus. Essentially following [18], we first specify in detail the morphological structure of the calculus we are going to present.

Let $\mathcal{L}$ be the language $(2, 2, 2, 2, 0, 0)$ with connectives $\wedge, \vee, \cdot, /, 0, 1$, and let $\mathbf{Fm}_{\mathcal{L}}$ be the formula $\mathcal{L}$-algebra with countably many generators in $Var_{\mathcal{L}}$. Let us denote by $Fm_{\mathcal{L}}^{\sim}$ the set $\{\varphi^{\sim} : \varphi \in Fm_{\mathcal{L}}\}$ and consider $Fm_{\mathcal{L}}^{\delta} = Fm_{\mathcal{L}} \cup Fm_{\mathcal{L}}^{\sim}$. We refer by $\mathbf{G}(Fm_{\mathcal{L}}^{\delta}) = (G(Fm_{\mathcal{L}}^{\delta}), \circ, \varepsilon)$ to the unital groupoid freely generated by $Fm_{\mathcal{L}}^{\delta} \cup \{\varepsilon\}$, where ε is a new constant.

[1] Sometimes referred to as right-residuation law; for this reason left-residuated ℓ-groupoids are occasionally called right-residuated as well [10].

[2] We note that in the context of residuated ℓ-groupoids, $0/\varphi$ is usually denoted by $-\varphi$, while $\varphi\backslash 0$ is typically denoted by $\sim\varphi$.

Now, consider a new symbol $\_$. The set $G(Fm_{\mathcal{L}}^{\delta})^{\alpha}$ is defined inductively as follows:

- $\_ \in G(Fm_{\mathcal{L}}^{\delta})^{\alpha}$ and
- $a \in G(Fm_{\mathcal{L}}^{\delta})^{\alpha}$ and $x \in G(Fm_{\mathcal{L}}^{\delta})$ entail that $a \circ x, x \circ a \in G(Fm_{\mathcal{L}}^{\delta})^{\alpha}$.

Given $u \in G(Fm_{\mathcal{L}}^{\delta})^{\alpha}$ and $x \in G(Fm_{\mathcal{L}}^{\delta})^{\alpha} \cup G(Fm_{\mathcal{L}}^{\delta})$, we will denote by $u[x]$ the word obtained by replacing the (unique!) occurrence of $\_$ in u by x. For example, if $u = (a \circ (b \circ \_))$, $v = (a \circ \_)$, and $x = (a \circ b)$, then $u[v] = (a \circ (b \circ (a \circ \_)))$ and $u[x] = (a \circ (b \circ (a \circ b)))$. To make the operation more explicit we allow ourselves to denote $u[x]$ by $u \star x$. The elements of $G(Fm_{\mathcal{L}}^{\delta})^{\alpha}$ are called *G-contexts* or simply *contexts*.

A sequent is any pair in $G(Fm_{\mathcal{L}}^{\delta}) \times G(Fm_{\mathcal{L}}^{\delta})$. Following the customary conventions, we will denote any sequent (X, Y) by $X \Rightarrow Y$. The set of all such sequents is denoted by $\mathrm{Seq}(\mathcal{L})$.

The calculi that follow will have infinitely many rules of inference organized in meta-inference rules, or (*metarules*). In order to formally define metarules, we need three pairwise disjoint sets of *meta-variables*: A (of sort S_A), G (of sort S_G with $S_A \subseteq S_G$) and U (of sort S_U). In our system, we will have $A = Fm_{\mathcal{L}}(F)$, where F is a set of propositional variables distinct from $Var_{\mathcal{L}}$ above. A *meta-Sasaki word* will be any term of type S_G built up by means of the following operations: ${}^{\sim} : S_A \to S_G$, $\circ : S_G \times S_G \to S_G$, $\star : S_U \times S_G \to S_G$, and the constant ε of sort S_G. For any u, X of type S_U resp. S_G, we set $u \star X = u[X]$. A meta-sequent (X, Y) is any pair such that X and Y are meta-Sasaki words. Meta-rules are pairs (S, s) such that $S \cup \{s\}$ is a set of meta-sequents. An inference rule is said to be an *instance* of a metarule, if all metavariables from A, G and U are instantiated to elements of $Fm_{\mathcal{L}}$, $G(Fm_{\mathcal{L}}^{\delta})$ and $G(Fm_{\mathcal{L}}^{\delta})^{\alpha}$, respectively, metasequent operators ε, $\circ$ and $\star$ are replaced by the corresponding sequent operators, and for any instance φ of a metavariable $\alpha \in A$, $\alpha^{\sim}$ is mapped to $\varphi^{\sim} \in Fm_{\mathcal{L}}^{\sim}$.

For each formula $\varphi \in Fm_{\mathcal{L}}$, by φ' we abbreviate the formula $0/\varphi$. We define inductively the following maps $(\cdot)^{\delta} : G\left(Fm_{\mathcal{L}}^{\delta}\right) \to G\left(Fm_{\mathcal{L}}^{\delta}\right)$, $(\cdot)^{\Pi} : G\left(Fm_{\mathcal{L}}^{\delta}\right) \to Fm_{\mathcal{L}}$ and $(\cdot)^{\Sigma} : G\left(Fm_{\mathcal{L}}^{\delta}\right) \to Fm_{\mathcal{L}}$:

$$\varepsilon^{\delta} = \varepsilon \;\Big|\; \varphi^{\delta} = \varphi^{\sim} \;\Big|\; (\varphi^{\sim})^{\delta} = \varphi \;\Big|\; (X \circ Y)^{\delta} = X^{\delta} \circ Y^{\delta},$$

$$\varepsilon^{\Pi} = 1 \;\Big|\; \varphi^{\Pi} = \varphi \;\Big|\; (\varphi^{\sim})^{\Pi} = \varphi' \;\Big|\; (X \circ Y)^{\Pi} = X^{\Pi} \cdot Y^{\Pi},$$

$$\varepsilon^{\Sigma} = 0 \;\Big|\; \varphi^{\Sigma} = \varphi \;\Big|\; (\varphi^{\sim})^{\Sigma} = \varphi' \;\Big|\; (X \circ Y)^{\Sigma} = \left((X^{\Sigma})' \cdot (Y^{\Sigma})'\right)'.$$

3.2. The calculus and its heuristic explanation. The inference rules for the calculus OGC are given in Figure 1, where we use $X \Leftrightarrow Y$ as an abbreviation for the conjunction of sequents $X \Rightarrow Y^{\Pi}$ and $Y \Rightarrow X^{\Pi}$, respectively called the $\Rightarrow$- and $\Leftarrow$-*sequent*,[3] and the double horizontal lines in (δ) to indicate that it is a bidirectional rule. While the letters $\alpha, \beta, \ldots$ can be instantiated only by members of $Fm_{\mathcal{L}}$, the letters $a, b, \ldots$ can be instantiated by any members of $Fm_{\mathcal{L}}^{\delta}$.

[3] We caution the reader that the abbreviation $X \Leftrightarrow Y$ is *not equivalent* to the conjunction of $X \Rightarrow Y$ and $Y \Rightarrow X$.

Axioms:

$$\frac{}{\alpha \Rightarrow \alpha}\,(\mathrm{id}) \qquad \frac{}{0 \Rightarrow \varepsilon}\,(0\mathrm{l}) \qquad \frac{}{\varepsilon \Rightarrow 1}\,(1\mathrm{r})$$

Structural Rules:

$$\frac{X \Leftrightarrow a \quad u[a] \Rightarrow Y}{u[X] \Rightarrow Y}\,(\mathrm{cut}) \qquad \frac{u[X \circ X] \Rightarrow Y}{u[X] \Rightarrow Y}\,(\mathrm{con_l}) \qquad \frac{a \circ X \Rightarrow a \quad u[a \circ X] \Rightarrow Y}{u[X \circ a] \Rightarrow Y}\,(\mathrm{el_1})$$

$$\frac{Z \circ X \Rightarrow V \circ Y}{(U \circ Z) \circ X \Rightarrow (W \circ V) \circ Y}\,(\mathrm{w}) \qquad \frac{X \circ Z \Rightarrow Y}{X \Rightarrow Y \circ Z^{\delta}}\,(\delta) \qquad \frac{a \circ X \Rightarrow a \quad u[X \circ a] \Rightarrow Y}{u[a \circ X] \Rightarrow Y}\,(\mathrm{el_2})$$

Operational Rules:

$$\frac{\alpha \circ X \Rightarrow Y}{(\alpha \wedge \beta) \circ X \Rightarrow Y}\,(\wedge\mathrm{l_1}) \qquad \frac{\beta \circ X \Rightarrow Y}{(\alpha \wedge \beta) \circ X \Rightarrow Y}\,(\wedge\mathrm{l_2}) \qquad \frac{X \Rightarrow \alpha \circ Y \quad X \Rightarrow \beta \circ Y}{X \Rightarrow (\alpha \wedge \beta) \circ Y}\,(\wedge\mathrm{r})$$

$$\frac{\alpha \circ X \Rightarrow Y \quad \beta \circ X \Rightarrow Y}{(\alpha \vee \beta) \circ X \Rightarrow Y}\,(\vee\mathrm{l}) \qquad \frac{X \Rightarrow \alpha \circ Y}{X \Rightarrow (\alpha \vee \beta) \circ Y}\,(\vee\mathrm{r_1}) \qquad \frac{X \Rightarrow \beta \circ Y}{X \Rightarrow (\alpha \vee \beta) \circ Y}\,(\vee\mathrm{r_2})$$

$$\frac{u[\alpha \circ \beta] \Rightarrow Y}{u[\alpha \cdot \beta] \Rightarrow Y}\,(\cdot\mathrm{l}) \qquad \frac{X \Rightarrow \alpha \quad Y \Leftrightarrow \beta}{X \circ Y \Rightarrow \alpha \cdot \beta}\,(\cdot\mathrm{r}) \qquad \frac{u[\varepsilon] \Rightarrow Y}{u[1] \Rightarrow Y}\,(1\mathrm{l})$$

$$\frac{X \Leftrightarrow \alpha \quad \beta \Rightarrow Z}{\beta/\alpha \circ X \Rightarrow Z}\,(/\mathrm{l}) \qquad \frac{X \circ \alpha \Rightarrow \beta}{X \Rightarrow \beta/\alpha}\,(/\mathrm{r}) \qquad \frac{X \Rightarrow u[\varepsilon]}{X \Rightarrow u[0]}\,(0\mathrm{r})$$

Fig. 1. The calculus `OGC`.

In the interest of allowing the reader to intuitively digest the inference rules presented, a discussion about the intended (algebraic) interpretation is in order. As will be seen in Section 4, the intended interpretation of a sequent $X \Rightarrow Y$ will be the inequality $X^{\Pi} \leq Y^{\Sigma}$, i.e., the intended meaning of $\Rightarrow$ is $\leq$, structures X to the left of $\Rightarrow$ represent (Sasaki) products while structures Y on the right are (Sasaki) sums. With this interpretation in mind, we can now elucidate and motivate our choice of presentation.

On the one hand, this interpretation makes the genesis of the rule (δ) clear: in an orthomodular lattice $\mathbf{A}$ we have the following residuation law(s) relating the Sasaki operations with the lattice order:

$$x \cdot z \leq y \iff x \leq y/z \approx y + z'$$

for all $x, y, z \in A$. Similarly, the *conditional* exchange rules $(\mathrm{el_1})$ and $(\mathrm{el_2})$, where active structures in the right-hand premiss can be swapped if the left-hand premiss is also available, corresponds to the demand that the algebraic interpretations of a and X commute (see Lemma 1(5)); i.e., for all $a, x \in A$

$$a \cdot x \leq a \iff a \text{ commutes with } x \iff x \text{ commutes with } a \iff x \cdot a \leq x,$$

or in other words, a commutes with x if and only if $a \cdot x = a \wedge x = x \cdot a$.

On the other hand, the interpretation also illuminates the peculiarities of the remaining structural and operational rules based mainly around Remark 4; namely the failure of the Sasaki operations to preserve (or in the case of hook, reverse) order in the right-coordinate. This immediately explains, for example, the particular left-associated versions of familiar inferences rules like the *weakening* rule (w) as well as those rules for the lattice connectives $\wedge$ and $\vee$. Furthermore, this motivates the appearance of $\Leftrightarrow$, whose intended interpretation corresponds to $\approx$, in the rules $(\cdot\mathrm{r})$, $(/\mathrm{l})$, and (cut), which mirror the fact that in $\mathbb{OG}$ (or even $\mathbb{PLRG}$) only the following quasi-identities generally hold:

$$\begin{array}{lcl} x \leq a \;\&\; y \approx b & \Longrightarrow & x \cdot y \leq a \cdot b, \\ x \approx a \;\&\; b \leq z & \Longrightarrow & (b/a) \cdot x \leq z. \end{array} \tag{1}$$

Clearly, the 3-premiss rules $(\cdot\mathrm{r})$ and $(/\mathrm{l})$ violate the subformula property, since e.g., the formula Y^{Π} disappears in the conclusion of $(\cdot\mathrm{r})$. However, the search space of all possible premisses of applications of both rules is always finite.

It is worth noting that need for $\Leftrightarrow$ in the cut-rule, for example, can be alleviated with a related, but syntactically different, version of this calculus. Namely, let $\mathtt{OGC}'$ be the calculus obtained from: replacing the binary operation $/$ with a unary operation $'$ in $\mathcal{L}$; replacing the rule (cut) by (cut_1) [see Lemma 13]; and replacing $(/\mathrm{l})$ and $(/\mathrm{r})$ by $(\prime\mathrm{l})$ and $(\prime\mathrm{r})$ [see below Lemma 7]. Due to the generic nature of $(\prime\mathrm{l})$ and $(\prime\mathrm{r})$, it is not difficult to prove the following claims [see Section 4.2]:

- $\mathtt{OGC}'$ satisfies (cut).
- $\mathtt{OGC}'$ satisfies the rules $(/\mathrm{l})$ and $(/\mathrm{r})$, where $\varphi/\psi := (\varphi' \cdot \psi)'$.

In fact, following an almost identical argument as seen in Section 4, one can show that $\mathtt{OGC}'$ is algebraisable with a variety of algebras in the signature $\langle \wedge, \vee, \cdot, ', 0, 1\rangle$ that is term-equivalent to $\mathbb{OG}$.

However, ridding entirely the appearance of $\Leftrightarrow$ does not seem possible, as it is necessary for proving completeness [see Lemma 17], for instance, with the (Sasaki) product operation. It is for this reason, as well as our intention of providing a syntactic calculus for orthomodular groupoids, that we use this particular presentation.

3.3. Elementary properties of **OGC.** For the sake of readability, in what follows we will write x^{uv} as an abbreviation for $(x^u)^v$, where $u, v \in \{\prime, \sim, \delta, \Pi, \Sigma\}$ and x is a (meta-) variable of the proper sort. Also, given a G-context u, we inductively define the G-context u^{δ} as follows: $\_^{\delta} = \_$; $(v \circ w)^{\delta} = v^{\delta} \circ w^{\delta}$ for $v, w \in G(Fm_{\mathcal{L}}^{\delta})^{\alpha} \cup G(Fm_{\mathcal{L}}^{\delta})$. Hence, $u[X]^{\delta} = u^{\delta}[X^{\delta}]$ for any structure X.

Also, for the sake of readability, some $\mathtt{OGC}$-derivations contain informal uses of rules not proper to the calculus; e.g.,

$$\frac{s'}{s}\,[=] \qquad \& \qquad \frac{s_1 \quad \cdots \quad s_n}{s_0}\,[\star],$$

where $[=]$ means syntactically equal, while $[\star]$ is shorthand for the sub-derivation in $\mathtt{OGC}$ resulting from some lemma, derived rule, or assumption $\star$.

LEMMA 5. $X^{\delta\delta} = X$, *and consequently* $X \Rightarrow Y \circ Z \dashv\vdash_{\mathtt{OGC}} X \circ Z^{\delta} \Rightarrow Y$.

Proof. The first claim easily follows by induction on the complexity of X by the definition of δ. Using this and the fact that (δ) is a bidirectional rule, we obtain the second claim. □

In light of the above lemma, we will freely use the fact that $X^{\delta\delta} = X$ without reference, and we will abuse notation and write (δ) to represent $X \Rightarrow Y \circ Z \dashv\vdash_{\mathtt{OGC}} X \circ Z^{\delta} \Rightarrow Y$.

LEMMA 6. $X \Rightarrow Y \vdash_{\mathtt{OGC}} Y^{\delta} \Rightarrow X^{\delta}$. *Consequently,* $\vdash_{\mathtt{OGC}} a \Rightarrow a$.

Proof. Observe:

$$\dfrac{\dfrac{\dfrac{X \Rightarrow Y}{\varepsilon \Rightarrow Y \circ X^{\delta}}\,(\delta)}{Y^{\delta} \circ X \Rightarrow \varepsilon}\,(\delta)}{Y^{\delta} \Rightarrow X^{\delta}}\,(\delta)\;.$$

The second claim follows from (id) for $a \in Fm_{\mathcal{L}}$, and also the above for $a \in Fm_{\mathcal{L}}^{\sim}$. $\square$

LEMMA 7. *$\vdash_{\texttt{OGC}} \varphi^{\sim} \Leftrightarrow \varphi'$ and $\vdash_{\texttt{OGC}} \varphi \Leftrightarrow \varphi''$. Furthermore,*

$$u[\varphi'] \Rightarrow X \dashv\vdash_{\texttt{OGC}} u[\varphi^{\sim}] \Rightarrow X \quad \textit{and} \quad u[\varphi] \Rightarrow X \dashv\vdash_{\texttt{OGC}} u[\varphi''] \Rightarrow X,$$

$$X \Rightarrow u[\varphi'] \dashv\vdash_{\texttt{OGC}} X \Rightarrow u[\varphi^{\sim}] \quad \textit{and} \quad X \Rightarrow u[\varphi] \dashv\vdash_{\texttt{OGC}} X \Rightarrow u[\varphi''].$$

Proof. Recall $\varphi' := 0/\varphi$. First, we observe [$\star$]: $\vdash_{\texttt{OGC}} \varphi^{\sim} \Rightarrow \varphi'$ and [$\star'$]: $\vdash_{\texttt{OGC}} \varphi' \Rightarrow \varphi^{\sim}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi^{\sim} \Rightarrow \varphi^{\sim}}\,[\text{Lem.6}]}{\varphi^{\sim} \circ \varphi \Rightarrow \varepsilon}\,(\delta)}{\varphi^{\sim} \circ \varphi \Rightarrow 0}\,(0\mathrm{r})}{\varphi^{\sim} \Rightarrow 0/\varphi}\,(/\mathrm{r}) \qquad \dfrac{\dfrac{\dfrac{}{\varphi \Leftrightarrow \varphi}\,(\mathrm{id}) \quad \dfrac{}{0 \Rightarrow \varepsilon}\,(0\mathrm{l})}{0/\varphi \circ \varphi \Rightarrow \varepsilon}\,(/\mathrm{l})}{0/\varphi \Rightarrow \varphi^{\sim}}\,(\delta)\;.$$

Hence [$\star_1$]: $\vdash_{\texttt{OGC}} \varphi^{\sim} \Leftrightarrow \varphi'$ by [$\star$] and that fact that $\vdash_{\texttt{OGC}} \varphi' \Rightarrow \varphi^{\sim\Pi}$ is an instance of (id) since $\varphi^{\sim\Pi} = \varphi'$ by definition of the operator Π. For [$\star_2$]: $\vdash_{\texttt{OGC}} \varphi \Leftrightarrow \varphi''$, observe:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi' \Rightarrow \varphi^{\sim}}\,[\star']}{\varphi \Rightarrow \varphi'^{\sim}}\,[\text{Lem.6}]}{\varphi \circ \varphi' \Rightarrow \varepsilon}\,(\delta)}{\varphi \circ \varphi' \Rightarrow 0}\,(0\mathrm{r})}{\varphi \Rightarrow \varphi''}\,(/\mathrm{r}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi^{\sim} \Leftrightarrow \varphi'}\,[\star_1] \quad \dfrac{}{0 \Rightarrow \varepsilon}\,(0\mathrm{l})}{0/\varphi' \circ \varphi^{\sim} \Rightarrow \varepsilon}\,(/\mathrm{l})}{\varphi'' \circ \varphi^{\sim} \Rightarrow \varepsilon}\,[=]}{\varphi'' \Rightarrow \varphi}\,(\delta)\;.$$

Thus, for either $\{a, b\} = \{\varphi', \varphi^{\sim}\}$ or $\{a, b\} = \{\varphi, \varphi''\}$, we have [$\star_3$] :$\vdash_{\texttt{OGC}} a \Leftrightarrow b$ and hence:

$$\dfrac{\dfrac{}{a \Leftrightarrow b}\,[\star_3] \qquad u[b] \Rightarrow X}{u[a] \Rightarrow X}\,(\mathrm{cut})\;.$$

Whence $u[\varphi'] \Rightarrow X \dashv\vdash_{\texttt{OGC}} u[\varphi^{\sim}] \Rightarrow X$ and $u[\varphi] \Rightarrow X \dashv\vdash_{\texttt{OGC}} u[\varphi''] \Rightarrow X$. Lastly, we observe:

$$\begin{array}{rl} X \Rightarrow u[\varphi'] & \dashv\vdash_{\texttt{OGC}} X \circ u[\varphi']^{\delta} \Rightarrow \varepsilon \\ & = X \circ u^{\delta}[\varphi'^{\sim}] \Rightarrow \varepsilon \\ & \dashv\vdash_{\texttt{OGC}} X \circ u^{\delta}[\varphi''] \Rightarrow \varepsilon \\ & \dashv\vdash_{\texttt{OGC}} X \circ u^{\delta}[\varphi] \Rightarrow \varepsilon \\ & \dashv\vdash_{\texttt{OGC}} X \Rightarrow u[\varphi^{\sim}] \end{array} \quad \text{and} \quad \begin{array}{rl} X \Rightarrow u[\varphi] & \dashv\vdash_{\texttt{OGC}} X \circ u^{\delta}[\varphi^{\sim}] \Rightarrow \varepsilon \\ & \dashv\vdash_{\texttt{OGC}} X \circ u^{\delta}[\varphi'] \Rightarrow \varepsilon \\ & \dashv\vdash_{\texttt{OGC}} X \circ u^{\delta}[\varphi'''] \Rightarrow \varepsilon \\ & \dashv\vdash_{\texttt{OGC}} X \circ u^{\delta}[\varphi''^{\sim}] \Rightarrow \varepsilon \\ & \dashv\vdash_{\texttt{OGC}} X \Rightarrow u[\varphi''] \end{array}\;.$$

$\square$

As a consequence of the lemma above, the following rules are derivable in `OGC`, for any $\varphi \in Fm_{\mathcal{L}}$, $X, Y \in G(Fm_{\mathcal{L}}^{\delta})$, and $u \in G(Fm_{\mathcal{L}}^{\delta})^{\alpha}$:

$$\dfrac{u[\varphi^{\sim}] \Rightarrow Y}{u[\varphi'] \Rightarrow Y}\,({}'\mathrm{l}) \qquad \dfrac{X \Rightarrow u[\varphi^{\sim}]}{X \Rightarrow u[\varphi']}\,({}'\mathrm{r})\;.$$

LEMMA 8. $u[X] \Rightarrow Y \dashv\vdash_{\mathtt{OGC}} u\left[X^{\Pi}\right] \Rightarrow Y$. *Consequently,* $\vdash_{\mathtt{OGC}} X \Rightarrow X^{\Pi}$.

Proof. We proceed by induction on the construction of X. For $X = \varepsilon$, then the $\vdash$-direction is the rule (11), while the $\dashv$-direction is derived below:

$$\dfrac{\dfrac{}{\varepsilon \Rightarrow 1}\,(1\mathrm{r}) \qquad \dfrac{\dfrac{}{1 \Rightarrow 1}\,(\mathrm{id})}{1 \Rightarrow \varepsilon^{\Pi}}\,[=] \qquad u[1] \Rightarrow Y}{u[\varepsilon] \Rightarrow Y}\,(\mathrm{cut}).$$

For $X = \varphi$, then $X = X^{\Pi}$ and there is nothing to prove. For $X = \varphi^{\sim}$, then $X^{\Pi} = \varphi'$, and the result follows from Lemma 7.

For the inductive step, assume the claim (IH) holds for words of complexity strictly less than that of X, and suppose $X = Z \circ V$. For the $\vdash$-direction, observe:

$$\dfrac{\dfrac{\dfrac{u[Z \circ V] \Rightarrow Y}{u[Z^{\Pi} \circ V^{\Pi}] \Rightarrow Y}\,[\mathrm{IH}]}{u[Z^{\Pi} \cdot V^{\Pi}] \Rightarrow Y}\,(\cdot\mathrm{l})}{u[(Z \circ V)^{\Pi}] \Rightarrow Y}\,[=].$$

And for the $\dashv$-direction:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Z^{\Pi} \Rightarrow Z^{\Pi}}\,(\mathrm{id})}{Z \Rightarrow Z^{\Pi}}\,[\mathrm{IH}] \qquad \dfrac{\dfrac{}{V^{\Pi} \Leftrightarrow V^{\Pi}}\,(\mathrm{id})}{V \Leftrightarrow V^{\Pi}}\,[\mathrm{IH}]}{Z \circ V \Rightarrow Z^{\Pi} \cdot V^{\Pi}}\,(\cdot\mathrm{r}) \qquad \dfrac{\dfrac{}{Z^{\Pi} \cdot V^{\Pi} \Rightarrow Z^{\Pi} \cdot V^{\Pi}}\,(\mathrm{id})}{Z^{\Pi} \cdot V^{\Pi} \Rightarrow (Z \circ V)^{\Pi}}\,[=] \qquad \dfrac{u[(Z \circ V)^{\Pi}] \Rightarrow Y}{u[Z^{\Pi} \cdot V^{\Pi}] \Rightarrow Y}\,[=]}{u[Z \circ V] \Rightarrow Y}\,(\mathrm{cut}).$$

□

LEMMA 9. $\vdash_{\mathtt{OGC}} X^{\delta\Pi\sim} \Leftrightarrow X^{\Sigma}$ *and* $\vdash_{\mathtt{OGC}} X^{\delta\Pi} \Leftrightarrow X^{\Sigma\sim}$.

Proof. For fixed $X \in G(Fm^{\delta}_{\mathcal{L}})$, call the second claim $(\star_X)$. Clearly, if the first claim holds for X then $(\star_X)$ follows by the derivations below:

$$\dfrac{\dfrac{}{X^{\delta\Pi\sim} \Rightarrow X^{\Sigma}}\,[\text{Claim 1}]}{X^{\Sigma\sim} \Rightarrow X^{\delta\Pi}}\,[\text{Lem.}6] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{X^{\Sigma} \Rightarrow X^{\delta\Pi\prime}}\,[\text{Claim 1}]}{X^{\Sigma} \Rightarrow X^{\delta\Pi\sim}}\,[\text{Lem.}7]}{X^{\delta\Pi} \Rightarrow X^{\Sigma\sim}}\,[\text{Lem.}6]}{X^{\delta\Pi} \Rightarrow X^{\Sigma\prime}}\,({\prime}\mathrm{r}).$$

We now prove the first claim by induction on the complexity of X. Let $X = \varepsilon$. By the definitions of the operators, clearly $\varepsilon^{\delta\Pi\sim} = \varepsilon^{\Pi\sim} = 1^{\sim}$, while $\varepsilon^{\Sigma} = 0$. Below we observe the derivation for the $\Rightarrow$-sequent on the left and the $\Leftarrow$-sequent on the right:

$$\dfrac{\dfrac{\dfrac{}{\varepsilon \Rightarrow 1}\,(1\mathrm{r})}{1^{\sim} \Rightarrow \varepsilon}\,(\delta)}{1^{\sim} \Rightarrow 0}\,(0\mathrm{r}) \qquad \dfrac{\dfrac{}{0 \Rightarrow \varepsilon}\,(0\mathrm{l})}{0 \Rightarrow 1'}\,(\mathrm{w}).$$

For $X = \varphi$, observe $\varphi^{\delta\Pi\sim} = (\varphi^{\sim})^{\Pi\sim} = (\varphi')^{\sim}$ and $\varphi^{\Sigma} = \varphi$. Hence the claim follows by Lemma 7. Similarly for $X = \varphi^{\sim}$, we have $(\varphi^{\sim})^{\delta\Pi\sim} = \varphi^{\Pi\sim} = \varphi^{\sim}$ while $(\varphi^{\sim})^{\Sigma} = \varphi'$, and again the claim follows from Lemma 7.

For the inductive step, assume the claim holds for structures of complexity strictly less than that of X, and suppose $X = Z \circ V$. In particular, the inductive hypothesis implies that both $(\star_Z)$ and $(\star_V)$ hold. By definition, we have

$$(Z \circ V)^{\delta\Pi\sim} = (Z^{\delta} \circ V^{\delta})^{\Pi\sim} = (Z^{\delta\Pi} \cdot V^{\delta\Pi})^{\sim} \quad \text{and} \quad (Z \circ V)^{\Sigma} = (Z^{\Sigma'} \cdot V^{\Sigma'})'.$$

Below we derive the $\Rightarrow$-sequent on the left and the $\Leftarrow$-sequent on the right:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Z^{\Sigma\sim} \Rightarrow Z^{\delta\Pi}}\,[\star_Z] \qquad \dfrac{}{V^{\Sigma\sim} \Leftrightarrow V^{\delta\Pi}}\,[\star_V]}{Z^{\Sigma\sim} \circ V^{\Sigma\sim} \Rightarrow Z^{\delta\Pi} \cdot V^{\delta\Pi}}\,(\cdot\mathrm{r})}{Z^{\Sigma'} \cdot V^{\Sigma'} \Rightarrow Z^{\delta\Pi} \cdot V^{\delta\Pi}}\,({\prime}\mathrm{l}),(\cdot\mathrm{l})}{(Z^{\delta\Pi} \cdot V^{\delta\Pi})^{\sim} \Rightarrow (Z^{\Sigma'} \cdot V^{\Sigma'})^{\sim}}\,[\text{Lem.}6]}{(Z^{\delta\Pi} \cdot V^{\delta\Pi})^{\sim} \Rightarrow (Z^{\Sigma'} \cdot V^{\Sigma'})'}\,({\prime}\mathrm{r}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Z^{\delta\Pi} \Rightarrow Z^{\Sigma\prime}}\,[\star_Z] \qquad \dfrac{}{V^{\delta\Pi} \Leftrightarrow V^{\Sigma\sim}}\,[\star_V]}{Z^{\delta\Pi} \circ V^{\delta\Pi} \Rightarrow Z^{\Sigma'} \cdot V^{\Sigma'}}\,(\cdot\mathrm{r})}{Z^{\delta\Pi} \cdot V^{\delta\Pi} \Rightarrow Z^{\Sigma'} \cdot V^{\Sigma'}}\,(\cdot\mathrm{l})}{(Z^{\Sigma'} \cdot V^{\Sigma'})^{\sim} \Rightarrow (Z^{\delta\Pi} \cdot V^{\delta\Pi})^{\sim}}\,[\text{Lem.}6]}{(Z^{\Sigma'} \cdot V^{\Sigma'})' \Rightarrow (Z^{\delta\Pi} \cdot V^{\delta\Pi})'}\,({\prime}\mathrm{l}),({\prime}\mathrm{r}) \; .$$

□

LEMMA 10. $X \Rightarrow u[Y] \dashv\vdash_{\mathtt{OGC}} X \Rightarrow u[Y^{\Sigma}]$. *Consequently*, $X^{\Sigma} \vdash_{\mathtt{OGC}} X$.

Proof. Below we derive the $\vdash$-direction on the top and the $\dashv$-direction on the bottom:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Y^{\Sigma\sim} \Leftrightarrow Y^{\delta\Pi}}\,[\text{Lem.}9] \qquad \dfrac{\dfrac{\dfrac{X \Rightarrow u[Y]}{X \circ u[Y]^{\delta} \Rightarrow \varepsilon}\,(\delta)}{X \circ u^{\delta}[Y^{\delta}] \Rightarrow \varepsilon}\,[=]}{X \circ u^{\delta}[Y^{\delta\Pi}] \Rightarrow \varepsilon}\,[\text{Lem.}8]}{X \circ u^{\delta}[Y^{\Sigma\sim}] \Rightarrow \varepsilon}\,(\mathrm{cut})}{X \circ u[Y^{\Sigma}]^{\delta} \Rightarrow \varepsilon}\,[=]}{X \Rightarrow u[Y^{\Sigma}]}\,(\delta)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Y^{\delta\Pi} \Leftrightarrow Y^{\Sigma\sim}}\,[\text{Lem.}9] \qquad \dfrac{\dfrac{X \Rightarrow u[Y^{\Sigma}]}{X \circ u[Y^{\Sigma}]^{\delta} \Rightarrow \varepsilon}\,(\delta)}{X \circ u^{\delta}[Y^{\Sigma\sim}] \Rightarrow \varepsilon}\,[=]}{X \circ u^{\delta}[Y^{\delta\Pi}] \Rightarrow \varepsilon}\,(\mathrm{cut})}{X \circ u^{\delta}[Y^{\delta}] \Rightarrow \varepsilon}\,[\text{Lem.}8]}{X \circ u[Y]^{\delta} \Rightarrow \varepsilon}\,[=]}{X \Rightarrow u[Y]}\,(\delta) \; .$$

□

By Lemmas 8 and 10, we obtain the following.

COROLLARY 11. $X \Rightarrow Y \dashv\vdash_{\mathtt{OGC}} X^{\Pi} \Rightarrow Y^{\Sigma}$

The derivable structural rules in the next lemma express, against the backdrop of the informal reading suggested for sequents of the form $a \circ b \Rightarrow a$ (Section 3.2), some of the algebraic properties of the commuting relations pinned down in Lemma 1(1) and (2).

LEMMA 12. *The following inference rules are derivable in* OGC :

$$\dfrac{a\circ Y\Rightarrow a}{a^\delta\circ Y\Rightarrow a^\delta}\,(\mathrm{co}_1)\qquad\qquad \dfrac{a\circ b\Rightarrow a}{b\circ a\Rightarrow b}\,(\mathrm{co}_2)\qquad\qquad \dfrac{a\circ b\Rightarrow a}{a\circ b^\delta\Rightarrow a}\,(\mathrm{co}_3)$$

$$\dfrac{X\Rightarrow u[Y\circ Y]}{X\Rightarrow u[Y]}\,(\mathrm{con}_\mathrm{r})\quad \dfrac{a\Rightarrow a\circ Y\quad X\Rightarrow u[a\circ Y]}{X\Rightarrow u[Y\circ a]}\,(\mathrm{er}_1)\quad \dfrac{a\Rightarrow a\circ Y\quad X\Rightarrow u[Y\circ a]}{X\Rightarrow u[a\circ Y]}\,(\mathrm{er}_2).$$

Proof. Below, we establish (co_1) on the left and (er_1) on the right:

$$\dfrac{\dfrac{\dfrac{a\circ Y\Rightarrow a}{a^\delta\Rightarrow(a\circ Y)^\delta}\,[\text{Lem.6}]}{a^\delta\Rightarrow a^\delta\circ Y^\delta}\,[=]}{a^\delta\circ Y\Rightarrow a^\delta}\,(\delta)\qquad\qquad \dfrac{\dfrac{\dfrac{a\Rightarrow a\circ Y}{a^\delta\circ Y^\delta\Rightarrow a^\delta}\,[\text{Lem.6}]\quad \dfrac{X\Rightarrow u[a\circ Y]}{X\circ u^\delta[a^\delta\circ Y^\delta]\Rightarrow\varepsilon}\,(\delta)}{X\circ u^\delta[Y^\delta\circ a^\delta]\Rightarrow\varepsilon}\,(\mathrm{el}_1)}{X\Rightarrow u[Y\circ a]}\,(\delta).$$

Now, the rules (er_2) and $(\mathrm{con}_\mathrm{r})$ are proven similarly to (er_1). Lastly, below we derive (co_2) on the left and using it with (co_1) we derive (co_3) on the right:

$$\dfrac{a\circ b\Rightarrow a\quad \dfrac{\dfrac{}{b\Rightarrow b}\,[\text{Lem.6}]}{a\circ b\Rightarrow b}\,(\mathrm{w})}{b\circ a\Rightarrow b}\,(\mathrm{el}_1)\qquad\qquad \dfrac{\dfrac{\dfrac{a\circ b\Rightarrow a}{b\circ a\Rightarrow b}\,(\mathrm{co}_2)}{b^\delta\circ a\Rightarrow b^\delta}\,(\mathrm{co}_1)}{a\circ b^\delta\Rightarrow a}\,(\mathrm{co}_2).$$

□

LEMMA 13. *The following inference rules are derivable in* OGC :

$$\dfrac{X\Rightarrow a\quad a\circ Y\Rightarrow Z}{X\circ Y\Rightarrow Z}\,(\mathrm{cut}_1)\qquad\qquad \dfrac{X\circ Y\Rightarrow a\quad Y\circ a\Rightarrow Z}{X\circ Y\Rightarrow Z}\,(\mathrm{cut}_2).$$

Proof. For (cut_1), let $\alpha=a^\Pi$. Observe:

$$\dfrac{\dfrac{\dfrac{X\Rightarrow a}{X\Rightarrow\alpha}\,(R)\quad \dfrac{}{X\Rightarrow X^\Pi}\,[\text{Lem.8}]}{X\Rightarrow\alpha\wedge X^\Pi}\,(\wedge\mathrm{r})\quad \dfrac{\dfrac{}{X^\Pi\Rightarrow X^\Pi}\,(\mathrm{id})}{\alpha\wedge X^\Pi\Rightarrow X^\Pi}\,(\wedge\mathrm{l}_2)\quad \dfrac{\dfrac{a\circ Y\Rightarrow Z}{\alpha\circ Y\Rightarrow Z}\,(R')}{(\alpha\wedge X^\Pi)\circ Y\Rightarrow Z}\,(\wedge\mathrm{l}_1)}{X\circ Y\Rightarrow Z}\,(\mathrm{cut}).$$

where $(R)=(R')=[=]$ if $a\in Fm_\mathcal{L}$, otherwise $(R)=(/\mathrm{r})$ and $(R')=(/\mathrm{l})$ when $a\in Fm^\delta_\mathcal{L}$. As for (cut_2), observe:

$$\dfrac{\dfrac{X\circ Y\Rightarrow a\quad \dfrac{\dfrac{X\circ Y\Rightarrow a}{a\circ(X\circ Y)\Rightarrow a}\,(\mathrm{w})\quad \dfrac{Y\circ a\Rightarrow Z}{(X\circ Y)\circ a\Rightarrow Z}\,(\mathrm{w})}{a\circ(X\circ Y)\Rightarrow Z}\,(\mathrm{el}_2)}{(X\circ Y)\circ(X\circ Y)\Rightarrow Z}\,(\mathrm{cut}_1)}{X\circ Y\Rightarrow Z}\,(\mathrm{con}_\mathrm{l}).$$

□

LEMMA 14. *The following rules are derivable in* OGC :

$$\dfrac{X\Rightarrow\varphi\quad X\Rightarrow\psi}{X\Rightarrow\varphi\cdot\psi}\,(\cdot\mathrm{r}_1^*)\qquad\qquad \dfrac{X\circ Y\Rightarrow\varphi\quad Y\Rightarrow\psi}{X\circ Y\Rightarrow\varphi\cdot\psi}\,(\cdot\mathrm{r}_2^*).$$

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Proof. For $(\cdot \mathrm{r}_1^*)$, we observe:

$$\dfrac{\dfrac{X \Rightarrow \psi}{\varepsilon \circ X \Rightarrow \psi}[=] \quad \dfrac{X \Rightarrow \varphi \quad \overline{\psi \Leftrightarrow \psi}\ (\mathrm{id})}{X \circ \psi \Rightarrow \varphi \cdot \psi}(\cdot \mathrm{r})}{\dfrac{\varepsilon \circ X \Rightarrow \varphi \cdot \psi}{X \Rightarrow \varphi \cdot \psi}[=]}(\mathrm{cut}_2) \quad .$$

Now, $(\cdot \mathrm{r}_2^*)$ is derived by applying (w) on its right premise and then applying $(\cdot \mathrm{r}_1^*)$. ☐

§4. Algebraisability of `OGC` and orthomodular groupoids.

4.1. `OGC` ***and*** $\mathbb{OG}$. Recall, to show that `OGC` is algebraisable with equivalent variety semantics given by $\mathbb{OG}$, it is enough to find transformers $\tau : \mathrm{Seq}(\mathcal{L}) \to \wp(Fm^2_{\mathcal{L}})$ and $\rho : Fm^2_{\mathcal{L}} \to \wp(\mathrm{Seq}(\mathcal{L}))$ both commuting with substitutions, such that, for all $S \cup \{s\} \subseteq \mathrm{Seq}(\mathcal{L})$ and for all $\mathcal{L}$-formulas φ, ψ:

- $S \vdash_{\mathtt{OGC}} s$ iff $\tau[S] \models_{\mathbb{OG}} \tau(s)$ and
- $\varphi \approx \psi \dashv\models_{\mathbb{OG}} \tau[\rho(\varphi \approx \psi)]$.

We will see that the appropriate transformers are defined as follows, for all $X, Y \in G(Fm^{\delta}_{\mathcal{L}})$ and $\varphi, \psi \in Fm_{\mathcal{L}}$:

$$\tau(X \Rightarrow Y) := \{X^{\Pi} \leq Y^{\Sigma}\} \qquad \& \qquad \rho(\varphi \approx \psi) := \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}.$$

Clearly, τ and ρ commute with substitutions.

LEMMA 15. *For all* $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$, $\varphi \approx \psi \dashv\models_{\mathbb{OG}} \tau[\rho(\varphi \approx \psi)]$.

Proof. We observe,

$$\begin{aligned} \varphi \approx \psi \dashv\models_{\mathbb{OG}} \tau[\rho(\varphi \approx \psi)] \quad &\text{iff} \quad \varphi \approx \psi \dashv\models_{\mathbb{OG}} \tau[\varphi \Rightarrow \psi, \psi \Rightarrow \varphi], \\ &\text{iff} \quad \varphi \approx \psi \dashv\models_{\mathbb{OG}} \{\varphi \leq \psi, \psi \leq \varphi\}. \end{aligned}$$

However, this is obvious given the reflexivity and antisymmetry of the induced order of the lattice reduct in $\mathbb{OG}$. ☐

LEMMA 16 (Soundness). *For all* $S \cup \{s\} \subseteq \mathrm{Seq}(\mathcal{L})$, *if* $S \vdash_{\mathtt{OGC}} s$ *then* $\tau[S] \models_{\mathbb{OG}} \tau(s)$.

Proof. Let s denote the sequent $X \Rightarrow Y$. That is, we must prove

$$S \vdash_{\mathtt{OGC}} s \quad \text{implies} \quad \tau[S] \models_{\mathbb{OG}} X^{\Pi} \leq Y^{\Sigma}.$$

A customary induction on the length of the `OGC`-proof of $X \Rightarrow Y$ from S will suffice. If $X \Rightarrow Y$ is an axiom, then it is either (id), (1r), or (0l); and it is immediate to observe that the translations of these axioms make instances of the identity $x \leq x$, valid in all $\mathbf{A} \in \mathbb{OG}$. If $s \in S$, our claim is trivial. The other rules can be checked with some occasional help from Lemma 1 and Theorem 2. For the sake of illustration, we exemplify the inductive step through the rule $(\cdot \mathrm{r})$. Suppose $X \circ Y \Rightarrow \alpha \cdot \beta$ has been derived from S in `OGC` via a final application of $(\cdot \mathrm{r})$ to the premises $X \Rightarrow \alpha$, $Y \Rightarrow \beta$, and $\beta \Rightarrow Y^{\Pi}$. By the inductive hypothesis, we have that

$$\begin{aligned} \tau[S] &\models_{\mathbb{OG}} \quad X^{\Pi} \leq \alpha, \\ \tau[S] &\models_{\mathbb{OG}} \quad Y^{\Pi} \leq \beta, \\ \tau[S] &\models_{\mathbb{OG}} \quad \beta \leq Y^{\Pi}. \end{aligned}$$

Note that, by the last two consequences, we obtain $\tau[S] \models_{\mathbb{OG}} Y^\Pi \approx \beta$. Now, let $\mathbf{A} \in \mathbb{OG}$ and h be a valuation to $\mathbf{A}$, and suppose that $\tau[S] \subseteq \ker(h)$. Hence $h(Y^\Pi) = h(\beta)$, and so by Equation (1),

$$h\left((X \circ Y)^\Pi\right) = h(X^\Pi) \cdot h(Y^\Pi) = h(X^\Pi) \cdot h(\beta) \leq h(\alpha) \cdot h(\beta) = h\left((\alpha \cdot \beta)^\Sigma\right),$$

whence our conclusion follows. □

LEMMA 17 (Completeness). *For all* $S \cup \{s\} \subseteq \mathrm{Seq}(\mathcal{L})$, *if* $\tau[S] \models_{\mathbb{OG}} \tau(s)$ *then* $S \vdash_{\mathtt{OGC}} s$.

Proof. Suppose contrapositively that $S \nvdash_{\mathtt{OGC}} s$, where s is the sequent $X \Rightarrow Y$. We require the Lindenbaum algebra construction, so let T be the ξ-theory generated by S, where ξ is the closure operator associated with $\vdash_{\mathtt{OGC}}$. Define, for all $\varphi, \psi \in Fm_{\mathcal{L}}$:

$$\langle \varphi, \psi \rangle \in \Theta_T \iff \rho(\varphi \approx \psi) \subseteq T.$$

We need an algebra $\mathbf{A} \in \mathbb{OG}$ and a valuation h to $\mathbf{A}$ such that $\tau[S] \subseteq \ker(h)$, while $X^\Pi \leq Y^\Sigma \notin \ker(h)$. Unsurprisingly enough, we choose $\mathbf{A} = \mathbf{Fm}_{\mathcal{L}}/\Theta_T$, and h as the valuation uniquely determined by the condition $h(x) = x/\Theta_T$. For our choice to be viable, we must verify the following two assertions:

(1) Θ_T is a congruence on $\mathbf{Fm}_{\mathcal{L}}$ and
(2) The quotient $\mathbf{Fm}_{\mathcal{L}}/\Theta_T$ belongs to $\mathbb{OG}$.

For (1), Θ_T is clearly reflexive, symmetric, and transitive by (id) and (cut_1). Thus we need only show compatibility with product, implication, meet, and join, i.e., $\varphi\, \Theta_T\, \psi$ and $\chi\, \Theta_T\, \upsilon$ implies $(\varphi \star \chi)\, \Theta_T (\psi \star \upsilon)$ for each $\star \in \{\cdot, /, \wedge, \vee\}$. By definition of Θ_T and ρ, we must therefore show $\varphi \Leftrightarrow \psi, \chi \Leftrightarrow \upsilon \in T$ implies $(\varphi \star \chi) \Leftrightarrow (\psi \star \upsilon) \in T$, where we abuse the notation $x \Leftrightarrow y \in T$ to mean $\rho(x \approx y) \subseteq T$. It suffices to show $(\varphi \star \chi) \Rightarrow (\psi \star \upsilon) \in T$, since the $\Leftarrow$-sequent is obtained by a symmetric derivation. Below we catalogue the derivations for each $\star \in \{\cdot, /, \wedge, \vee\}$:

$$\dfrac{\dfrac{\varphi \Rightarrow \psi \quad \chi \Leftrightarrow \upsilon}{\varphi \circ \chi \Rightarrow \psi \cdot \upsilon}\,(\cdot\mathrm{r})}{\varphi \cdot \chi \Rightarrow \psi \cdot \upsilon}\,(\cdot\mathrm{l}) \qquad \dfrac{\dfrac{\upsilon \Leftrightarrow \chi \quad \varphi \Rightarrow \psi}{\varphi/\chi \circ \upsilon \Rightarrow \psi}\,(/\mathrm{l})}{\varphi/\chi \Rightarrow \psi/\upsilon}\,(/\mathrm{r}) \qquad \dfrac{\dfrac{\varphi \Rightarrow \psi}{\varphi \wedge \chi \Rightarrow \psi}\,(\wedge\mathrm{l}_1) \quad \dfrac{\chi \Rightarrow \upsilon}{\varphi \wedge \chi \Rightarrow \upsilon}\,(\wedge\mathrm{l}_2)}{\varphi \wedge \chi \Rightarrow \psi \wedge \upsilon}\,(\wedge\mathrm{r}).$$

The derivation for $\vee$ is similar to that for $\wedge$. Hence Θ_T is a congruence.

For (2), observe that $\varphi/\Theta_T \leq^{\mathbf{A}} \psi/\Theta_T$ iff $\varphi \Leftrightarrow \varphi \wedge \psi \in T$ iff $\varphi \Rightarrow \psi \in T$. We base on this observation the verification that $\mathbf{A}$ belongs to $\mathbb{OG}$.

First we show that $\mathbf{A} \in \mathbb{PLRG}$. As the derivations for verifying $\mathbf{A}$ has a lattice reduct as well as a multiplicative (left-)unit are standard, we only show the defining inequations for residuation (see [30]):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{id}) \quad \dfrac{}{\psi \Leftrightarrow \psi}\,(\mathrm{id})}{\varphi \circ \psi \Rightarrow \varphi \cdot \psi}\,(\cdot\mathrm{r})}{\varphi \circ \psi \Rightarrow \varphi \cdot \psi \vee \chi}\,(\vee\mathrm{r}_1)}{\varphi \Rightarrow (\varphi \cdot \psi \vee \chi)/\psi}\,(/\mathrm{r}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\psi \Leftrightarrow \psi}\,(\mathrm{id}) \quad \dfrac{}{\chi \Rightarrow \chi}\,(\mathrm{id})}{\chi/\psi \circ \psi \Rightarrow \chi}\,(/\mathrm{l})}{(\varphi \wedge \chi/\psi) \circ \psi \Rightarrow \chi}\,(\wedge\mathrm{l}_2)}{(\varphi \wedge \chi/\psi) \cdot \psi \Rightarrow \chi}\,(\cdot\mathrm{l}).$$

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So $\mathbf{A} \in \mathbb{PLRG}$. Hence, by Proposition 3, it is enough to show that $\mathbf{A}$ is strongly idempotent and strongly involutive. For strong idempotency, observe:

$$\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{id}) \qquad \dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{id})}{\varphi \Rightarrow \varphi \vee \psi}\,(\vee\mathrm{r}_1)}{\varphi \Rightarrow \varphi \cdot (\varphi \vee \psi)}\,(\cdot\mathrm{r}_1^*) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{id})}{\varphi \Rightarrow \varphi \vee \psi}\,(\vee\mathrm{r}_1)}{(\varphi \vee \psi) \circ \varphi \Rightarrow \varphi \vee \psi}\,(\mathrm{w})}{\varphi \circ (\varphi \vee \psi) \Rightarrow \varphi}\,(\mathrm{co}_2)}{\varphi \cdot (\varphi \vee \psi) \Rightarrow \varphi}\,(\cdot\mathrm{l}) \quad .$$

For strong involutivity, observe:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\psi^\sim \Rightarrow \psi'}\,[\text{Lem.7}] \qquad \dfrac{}{\varphi \Leftrightarrow \varphi}\,(\mathrm{id})}{\psi^\sim \circ \varphi \Rightarrow \psi' \cdot \varphi}\,(\cdot\mathrm{r})}{\psi^\sim \Rightarrow (\psi' \cdot \varphi) \circ \varphi^\sim}\,(\delta)}{(\psi' \cdot \varphi)^\sim \circ \varphi \Rightarrow \psi}\,[\text{Lem.6}]}{(\psi' \cdot \varphi)' \circ \varphi \Rightarrow \psi}\,({}'\mathrm{l})}{(\psi' \cdot \varphi)' \Rightarrow \psi / \varphi}\,(/\mathrm{r}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Leftrightarrow \varphi}\,(\mathrm{id}) \qquad \dfrac{}{\psi \Rightarrow \psi}\,(\mathrm{id})}{\psi/\varphi \circ \varphi \Rightarrow \psi}\,(/\mathrm{l})}{\psi/\varphi \Rightarrow \psi \circ \varphi^\sim}\,(\delta)}{\psi/\varphi \circ (\psi^\sim \circ \varphi) \Rightarrow \varepsilon}\,(\delta)}{\psi/\varphi \circ (\psi' \cdot \varphi) \Rightarrow \varepsilon}\,({}'\mathrm{l}),(\cdot\mathrm{l})}{\psi/\varphi \Rightarrow (\psi' \cdot \varphi)^\sim}\,(\delta)}{\psi/\varphi \Rightarrow (\psi' \cdot \varphi)'}\,({}'\mathrm{r}) \quad .$$

Hence Θ_T and $\mathbf{A}$ are as claimed, and so we need only verify $\tau[S] \subseteq \ker(h)$ but $\tau[s] \notin \ker(h)$, written in full:

$$\{Z^\Pi \Rightarrow V^\Sigma : Z \Rightarrow V \in S\} \subseteq \ker(h) \quad \& \quad X^\Pi \Rightarrow Y^\Sigma \notin \ker(h).$$

Now, by Corollary 11, $W \Rightarrow U \in \xi(S)$ iff $W^\Pi \Rightarrow V^\Sigma \in \xi(S)$. Since $T = \xi(S)$, it follows that $\tau[S] \subseteq \ker(h)$, while $\tau(s) \notin \ker(h)$ since $S \nvdash_{\mathtt{OGC}} s$. Therefore $\tau[S] \nvDash_{\mathbb{OG}} \tau(s)$. $\square$

Therefore, by Lemmas 15–17 we conclude:

THEOREM 18. *The calculus* OGC *is algebraisable with* $\mathbb{OG}$ *as equivalent variety semantics.*

***4.2. The alternative calculus* $\mathtt{OGC}'$.** As it has been already mentioned above, in $\mathbb{OG}$ the identity $x/y \approx (x' \cdot y)'$ holds. Therefore, orthomodular groupoids can be equivalently regarded as algebras in the signature $\{\wedge, \vee, \cdot, ', 0, 1\}$, which are unital lattice-ordered groupoids with an antitone involution satisfying further conditions. Indeed, it can be shown that once OGC is formulated into the alternative calculus $\mathtt{OGC}'$ [see Section 3.2] by assuming as a primitive connective $'$, and replacing $(/\mathrm{l})$ and $(/\mathrm{r})$ by

$$\dfrac{u[\alpha^\sim] \Rightarrow X}{u[\alpha'] \Rightarrow X}\,({}'\mathrm{l}) \quad \text{and} \quad \dfrac{X \Rightarrow u[\alpha^\sim]}{X \Rightarrow u[\alpha']}\,({}'\mathrm{r}),$$

then the resulting calculus can be still shown to be algebraisable with respect to orthomodular groupoids in the new signature (by means of the same τ and ρ). Moreover, $\mathtt{OGC}'$ seems to behave better proof-theoretically. First of all, it can be shown that $(/\mathrm{l})$ and $(/\mathrm{r})$ can be derived without making use of (cut). In fact we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X\circ\alpha\Rightarrow\beta}{X\Rightarrow\beta\circ\alpha^{\sim}}\,(\delta)}{X\circ(\beta^{\sim}\circ\alpha)\Rightarrow\varepsilon}\,(\delta)}{X\circ(\beta'\circ\alpha)\Rightarrow\varepsilon}\,({\prime}\mathrm{l})}{X\circ(\beta'\cdot\alpha)\Rightarrow\varepsilon}\,(\cdot\mathrm{l})}{X\Rightarrow(\beta'\cdot\alpha)'}\,(\delta)+({\prime}\mathrm{r})$$

as well as

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta\Rightarrow Z}{Z^{\delta}\Rightarrow\beta^{\sim}}\,[\text{Lem.6}]}{Z^{\delta}\Rightarrow\beta'}\,({\prime}\mathrm{r})\qquad X\Leftrightarrow\alpha}{Z^{\delta}\circ X\Rightarrow\beta'\cdot\alpha}}{Z^{\delta}\Rightarrow(\beta'\cdot\alpha)\circ X^{\delta}}\,(\delta)}{(\beta'\cdot\alpha)^{\sim}\circ X\Rightarrow Z}\,[\text{Lem.6}]}{(\beta'\cdot\alpha)'\circ X\Rightarrow Z}\,({\prime}\mathrm{l})$$

Secondly, it can be shown that (cut) can be replaced by (cut$_1$). To see this, we first prove the following lemmas.

LEMMA 19. *For any $X\in G(Fm^{\sim}_{\mathcal{L}'})$ there exists a cut-free proof of* $\vdash_{\mathtt{OGC}} X\Rightarrow X^{\Pi}$.

Proof. We proceed by induction on X. If $X=\varepsilon$, then $\vdash_{\mathtt{OGC}} 1\Rightarrow 1$ follows by means of a simple application of (id). If $X=\varphi^{\sim}$, one has:

$$\dfrac{\dfrac{}{\varphi\Rightarrow\varphi}\,(\mathrm{id})}{\varphi^{\sim}\Rightarrow\varphi^{\sim}}\,[\text{Lem.6}].$$

Finally, if $X=Y\circ Z$, then one has by induction hypothesis that $\vdash_{\mathtt{OGC}} Y\Rightarrow Y^{\Pi}$ as well as $\vdash_{\mathtt{OGC}} Z\Rightarrow Z^{\Pi}$. Therefore one has:

$$\dfrac{\dfrac{Y\Rightarrow Y^{\Pi}\qquad Z\Leftrightarrow Z^{\Pi}}{Y\circ Z\Rightarrow Y^{\Pi}\cdot Z^{\Pi}}\,(\cdot\mathrm{r}).}{X\Rightarrow X^{\Pi}}\,[=]$$

□

LEMMA 20. $X\Leftrightarrow a\vdash_{\mathtt{OGC}} u[X]\Leftrightarrow u[a]$.

Proof. We prove the statement by induction on the complexity of $u[-]$ ($||u||$). If u is _ then there is nothing to prove. Suppose that $||u||>0$. Two cases may occur:

(i) $u=Y\circ u_1[\_]$. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{}{Y\Rightarrow Y^{\Pi}}\,[\text{Lem.19}]\qquad\dfrac{}{u_1[X]\Rightarrow u_1[a]^{\Pi}}\,[\text{IH}]\qquad\dfrac{\dfrac{}{u_1[a]\Rightarrow u_1[X]^{\Pi}}\,[\text{IH}]}{u_1[a]^{\Pi}\Rightarrow u_1[X]^{\Pi}}\,({\prime}\mathrm{l})+(\cdot\mathrm{l}).}{Y\circ u_1[X]\Rightarrow Y^{\Pi}\cdot u_1[a]^{\Pi}}\,(\cdot\mathrm{r})}{u[X]\Rightarrow u[a]^{\Pi}}\,[=]$$

Similarly, one proves $X\Leftrightarrow a\vdash_{\mathtt{OGC}} u[a]\Rightarrow u[X]^{\Pi}$.

(ii) $u = u_1[\_] \circ Y$.

$$\dfrac{\dfrac{}{u_1[X] \Rightarrow u_1[a]^\Pi}\,[\mathrm{IH}] \qquad \dfrac{}{Y \Rightarrow Y^\Pi}\,[\mathrm{Lem.19}] \qquad \dfrac{}{Y^\Pi \Rightarrow Y^\Pi}\,(\mathrm{id})}{\dfrac{u_1[X] \circ Y \Rightarrow u_1[a]^\Pi \cdot Y^\Pi}{u[X] \Rightarrow u[a]^\Pi}\,[=]}.$$

Similarly, we have $X \Leftrightarrow a \vdash_{\mathtt{OGC}} u[a] \Rightarrow u[X]^\Pi$.

□

THEOREM 21. *In* $\mathtt{OGC}'$(cut) *can be equivalently replaced by* (cut_1).

Proof. Assuming (cut) we derive (cut_1) by Lemma 13. Conversely, by Lemma 20, we have $X \Leftrightarrow a \vdash_{\mathtt{OGC}} u[X] \Rightarrow u[a]^\Pi$. Therefore, we have:

$$\dfrac{u[X] \Rightarrow u[a]^\Pi \qquad \dfrac{u[a] \Rightarrow Z}{u[a]^\Pi \Rightarrow Z}\,(\cdot\mathrm{l})+(/\mathrm{l})}{u[X] \Rightarrow Z}\,(\mathrm{cut}_1).$$

□

Indeed, the above results show that, in presence of ($/$r) and ($/$l), we do not need to assume (cut_1) in its full generality. In fact the same results follows by replacing (cut_1) by the simple cut:

$$\dfrac{X \Rightarrow \alpha \qquad \alpha \Rightarrow Y}{X \Rightarrow Y}.$$

§5. A calculus for $\mathbb{PLRG}$. We now discuss the calculus $\mathtt{PLRGC}$, whose equivalent algebraic semantics are shown to be $\mathbb{PLRG}$ in Theorem 23 by specialising the arguments from Sections 3 and 4. The calculus $\mathtt{OGC}$ can be viewed as an extension of $\mathtt{PLRGC}$, but we present the latter as a restriction of the former.

A sequent in $\mathtt{PLRGC}$ is of the form $X \Rightarrow \Pi$, where X is a (possibly empty) groupoid word built from formulas and Π (called a *stoup*) is either a formula or the empty word ε; i.e., it is pair in $G(Fm_{\mathcal{L}}) \times (Fm_{\mathcal{L}} \cup \{\varepsilon\})$.[4] We note that sequents in $\mathtt{PLRGC}$ are restrictions of those in $\mathtt{OGC}$ in two ways: firstly that the groupoid words are generated from $Fm_{\mathcal{L}}$ (instead of $Fm^\delta_{\mathcal{L}}$), and secondly that the right-hand side contains at most a single formula. The concept of metavariables remains essentially the same.

The rule schemata for $\mathtt{PLRGC}$ are essentially the same as in $\mathtt{OGC}$, in particular it has the same axioms as $\mathtt{OGC}$, but the following exceptions/modifications are required:

- The only structural rule in $\mathtt{PLRGC}$ is (cut), but replacing structures Y with stoups Π in the sequents and a restricted to $Fm_{\mathcal{L}}$.
- The same operational rules are present as in $\mathtt{OGC}$, but modified accordingly so that all the right-hand sides of sequents contain either a formula-symbol or a stoup, e.g.,

$$\dfrac{\alpha \circ X \Rightarrow \Pi \qquad \beta \circ X \Rightarrow \Pi}{(\alpha \vee \beta) \circ X \Rightarrow \Pi}\,(\vee\mathrm{l}) \qquad \dfrac{X \Rightarrow \alpha}{X \Rightarrow \alpha \vee \beta}\,(\vee\mathrm{r}_1),$$

[4] Sequents of this type are often called *normal*.

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and furthermore the introduction rules for the constants 1 and 0 are replaced by the following[5]:

$$\frac{X \Rightarrow \Pi}{1 \circ X \Rightarrow \Pi}\ (1\mathrm{l}) \qquad \frac{X \Rightarrow \varepsilon}{X \Rightarrow 0}\ (0\mathrm{r}).$$

Remark 22. *Observe the use of* $\Leftrightarrow$ *in* PLRGC (*and* OGC) *for the rules* (cut), ($\cdot$r), *and* (/l) *in contrast to their respective variants in* GL:

$$\frac{X \Rightarrow \alpha \quad u[\alpha] \Rightarrow \Pi}{u[X] \Rightarrow \Pi} \qquad \frac{X \Rightarrow \alpha \quad Y \Rightarrow \beta}{X \circ Y \Rightarrow \alpha \cdot \beta} \qquad \frac{X \Rightarrow \alpha \quad \beta \Rightarrow \Pi}{(\beta/\alpha) \circ X \Rightarrow \Pi}\ .$$

This is necessitated by the observations made in Remark 4 and Theorem 18.

It is easy to see that the analogue to the rule (cut_1) is derivable in PLRGC from Lemma 13, where structures Z are replaced with stoups Π and a restricted to $Fm_{\mathcal{L}}$. Similarly, it is also easy to see that an analogue to Corollary 11 holds for PLRGC, namely that $X \Rightarrow \Pi \dashv\vdash_{\mathtt{PLRGC}} X^{\Pi} \Rightarrow \Pi^{\Sigma}$ for all $X \in G(Fm_{\mathcal{L}})$ and $\Pi \in Fm_{\mathcal{L}} \cup \{\varepsilon\}$.[6] Hence, using basically the same transformers $\tau(X \Rightarrow \Pi) := X^{\Pi} \leq \Pi^{\Sigma}$ and $\rho(\varphi \approx \psi) := \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}$, and the same relevant arguments contained in Lemmas 15–17, we therefore obtain the following theorem:

Theorem 23. *The calculus* PLRGC *is algebraisable with* $\mathbb{PLRG}$ *as equivalent variety semantics.*

§6. Conclusion and open problems. At the present stage, we do not know whether the rule (cut) is admissible either in OGC or in PLRGC. As already observed, this is not the unique rule that flouts the subformula property in either calculus, but no other rule forces the complete bottom-up proof search tree to contain infinite antichains. Also, notice that rules like (con_1) block the immediate conversion of any possible cut-elimination procedure into a decision procedure for the calculus.

Based on the results we have obtained so far, therefore, the calculi we presented appear less useful for applications than they are informative about the substructural character of the logics of orthomodular lattices and pointed left-residuated ℓ-groupoids. However, it is not out of the question that some helpful insights may come from their study. As hinted above, the problem as to whether the equational theory of orthomodular lattices is decidable has been open for some decades now [7]. Borrowing hitherto untried ideas from neighbouring fields is vital to make some headway.

Added in Proof. After submitting this article, we became acquainted (courtesy of the author) with a paper by Kornell [25] which seems to present similarities with our approach. In future work, we hope to undertake a comparison of these works.

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5 Note that the rule (1l) is much weaker than the rule of the same name in OGC: this is because, in $\mathbb{PLRG}$ the constant 1 is only a left-identity for product, while in $\mathbb{OG}$ the constant 1 is a two-sided identity for product.

6 Using the same basic argument in Lemma 8, it follows that $X \Rightarrow \alpha \dashv\vdash_{\mathtt{PLRGC}} X^{\Pi} \Rightarrow \alpha$, for α a formula. On the other hand, $X \Rightarrow \varepsilon \dashv\vdash_{\mathtt{PLRGC}} X^{\Pi} \Rightarrow 0$ easily follows by (0r) for the $\vdash$-direction, and (cut_1) with (0l) for the $\dashv$-direction.

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