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TIGHT UNIVERSAL SUMS OF m-GONAL NUMBERS

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Abstract

For a positive integer n, let $\mathcal{T}(n)$ denote the set of all integers greater than or equal to n. A sum of generalised m-gonal numbers g is called tight $\mathcal{T}(n)$-universal if the set of all nonzero integers represented by g is equal to $\mathcal{T}(n)$. We prove the existence of a minimal tight $\mathcal{T}(n)$-universality criterion set for a sum of generalised m-gonal numbers for any pair (m, n). To achieve this, we introduce an algorithm giving all candidates for tight $\mathcal{T}(n)$-universal sums of generalised m-gonal numbers. Furthermore, we provide some experimental results on the classification of tight $\mathcal{T}(n)$-universal sums of generalised m-gonal numbers.

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1. Introduction

A positive definite integral quadratic form

$$f = f(x_1, x_2, \dots, x_k) = \sum_{1\le i,j\le k} a_{ij}x_ix_j \quad (a_{ij} = a_{ji} \in \mathbb{Z})$$

is called *universal* if it represents all positive integers. Lagrange's four-square theorem states that the quaternary quadratic form $x^2 + y^2 + z^2 + w^2$ is universal. Ramanujan [15] found all diagonal quaternary universal quadratic forms. In 1993, Conway and Schneeberger announced the '15-Theorem' which says that a (positive definite integral) quadratic form representing all positive integers up to 15 actually represents every positive integer. Bhargava [1] introduced an algorithm, called the escalation method, which yields the classification of universal quadratic forms (see also [4]). The escalation method shows that if an integral quadratic form f represents nine integers $1, 2, 3, 5, 6, 7, 10, 14$ and 15, then f is universal. Kim *et al.* [10] generalised this result and proved that for any infinite set S of quadratic forms of bounded rank, there is a finite subset S_0 of S such that any (positive definite integral) quadratic form

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representing every form in S_0 represents all of S. Following [11], we call such a set S_0 *an S-universality criterion set*. An S-universality criterion set S_0 is called *minimal* if no proper subset S_0' of S_0 is an S-universality criterion set.

For an integer $m \geq 3$, we define a polynomial $P_m(x)$ by

$$P_m(x) = \frac{(m-2)x^2 - (m-4)x}{2}.$$

An integer of the form $P_m(u)$ for some integer u is called a generalised m-gonal number. A polynomial of the form

$$a_1 P_m(x_1) + a_2 P_m(x_2) + \cdots + a_k P_m(x_k)$$

with positive integers $a_1, a_2, \ldots, a_k$ is called *a sum of generalised m-gonal numbers* or *an m-gonal form*. In [9], Kane and Liu proved that there is a constant γ_m such that if a sum of generalised m-gonal numbers represents all positive integers up to γ_m, then it represents all positive integers. By applying the escalation method to sums of generalised m-gonal numbers, they showed the existence of such a γ_m and found an asymptotic upper bound of γ_m in terms of m.

For each positive integer n, we define $\mathcal{T}(n)$ to be the set of all integers greater than or equal to n. An m-gonal form g is called *tight* $\mathcal{T}(n)$*-universal* if the set of all nonzero integers represented by g is equal to $\mathcal{T}(n)$. We introduce an algorithm giving all tight $\mathcal{T}(n)$-universal m-gonal forms and provide some experimental results from the algorithm. In Section 2, some basic notation and terminology will be given. In Section 3, we introduce an algorithm which gives the classification of tight $\mathcal{T}(n)$-universal m-gonal forms for each given pair (m, n). This algorithm is analogous to the escalation algorithm described by Bhargava and, when $n = 1$, it coincides with the algorithm for universal m-gonal forms in [9]. In Section 4, we provide some experimental results from the algorithm described in Section 3, including candidates for tight $\mathcal{T}(n)$-universal m-gonal forms for $m = 7, 9, 10$ and 11.

2. Preliminaries

For $k = 1, 2, 3, \ldots$, we define a set $\mathcal{N}(k)$ to be the set of all vectors of positive integers with length k and coefficients in ascending order, that is,

$$\mathcal{N}(k) = \{\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k : a_1 \leq a_2 \leq \cdots \leq a_k\}.$$

Put $\mathcal{N} = \bigcup_{k=1}^{\infty} \mathcal{N}(k)$. For two vectors $\mathbf{a} \in \mathcal{N}(k)$ and $\mathbf{b} \in \mathcal{N}(s)$ with $k \leq s$, we write

$$\mathbf{a} \preceq \mathbf{b} \quad (\mathbf{a} \prec \mathbf{b})$$

if the sequence $(a_i)_{1 \leq i \leq k}$ is a (proper) subsequence of $(b_j)_{1 \leq j \leq s}$, where

$$\mathbf{a} = (a_1, a_2, \ldots, a_k) \quad \text{and} \quad \mathbf{b} = (b_1, b_2, \ldots, b_s).$$

Given a vector $\mathbf{a} \in \mathcal{N}(k)$ and a positive integer a, we define a vector $\mathbf{a} * a$ by

$$\mathbf{a} * a = (a_1, a_2, \ldots, a_i, a, a_{i+1}, a_{i+2}, \ldots, a_k) \in \mathcal{N}(k+1),$$

where i is the maximum index satisfying $a_i \le a$, that is, $\mathbf{a} * a$ is the vector in $\mathcal{N}(k+1)$ with coefficients $a_1, a_2, \ldots, a_k$ and a. For $\mathbf{a} \in \mathcal{N}(k)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_s) \in \mathcal{N}(s)$, we define $\mathbf{a} * \mathbf{b}$ to be the vector

$$\mathbf{a} * b_1 * b_2 * \cdots * b_s \in \mathcal{N}(k+s).$$

We identify $\mathcal{N}(1)$ with $\mathbb{N}$, so that, for example, $3 * 7 * 2 * 5$ denotes the vector $(2, 3, 5, 7) \in \mathcal{N}(4)$. Let S be a set of nonnegative integers containing 0 and 1 and let n be a positive integer. For a vector $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathcal{N}(k)$, we define

$$R_S(\mathbf{a}) = \{a_1 s_1 + a_2 s_2 + \cdots + a_k s_k : s_i \in S\} \quad \text{and} \quad R'_S(\mathbf{a}) = R_S(\mathbf{a}) - \{0\}.$$

Let $\mathcal{GP}_m$ be the set of generalised m-gonal numbers, that is,

$$\mathcal{GP}_m = \{P_m(u) : u \in \mathbb{Z}\}.$$

Then an m-gonal form

$$a_1 P_m(x_1) + a_2 P_m(x_2) + \cdots + a_k P_m(x_k) \quad (a_1 \le a_2 \le \cdots \le a_k)$$

corresponds to the pair $(\mathcal{GP}_m, \mathbf{a})$, where $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathcal{N}(k)$. A pair $(\mathcal{GP}_m, \mathbf{a})$ ($\mathbf{a} \in \mathcal{N}(k)$) will also be called a k-ary m-gonal form. Let n be a positive integer. An m-gonal form $(\mathcal{GP}_m, \mathbf{a})$ is called *$\mathcal{T}(n)$-universal* if $R'_{\mathcal{GP}_m}(\mathbf{a}) \supseteq \mathcal{T}(n)$ and *tight $\mathcal{T}(n)$-universal* if $R'_{\mathcal{GP}_m}(\mathbf{a}) = \mathcal{T}(n)$. A tight $\mathcal{T}(n)$-universal m-gonal form $(\mathcal{GP}_m, \mathbf{a})$ is called *new* if $R'_{\mathcal{GP}_m}(\mathbf{b}) \subsetneq \mathcal{T}(n)$ for every vector $\mathbf{b} \in \mathcal{N}$ satisfying $\mathbf{b} \prec \mathbf{a}$. When $n = 1$, we use the expression 'universal' along with 'tight $\mathcal{T}(1)$-universal' to follow the convention.

LEMMA 2.1. *Let m be an integer greater than or equal to 3 and n be a positive integer. Then there exists a vector $\mathbf{a}$ such that $R'_{\mathcal{GP}_m}(\mathbf{a}) = \mathcal{T}(n)$.*

PROOF. Let $\mathbf{b} = (n, n, \ldots, n) \in \mathcal{N}(m)$ be the vector of length m with every coefficient equal to n. By Fermat's polygonal number theorem,

$$R_{\mathcal{GP}_m}(\mathbf{b}) = \{nu : u \in \mathbb{Z}_{\ge 0}\}.$$

From this, one may easily deduce that

$$R'_{\mathcal{GP}_m}(\mathbf{b} * (n+1) * (n+2) * \cdots * (2n-1)) = \mathcal{T}(n).$$

This completes the proof. □

3. An algorithm for tight $\mathcal{T}(n)$-universal sums of m-gonal numbers

We introduce an algorithm which gives all new tight $\mathcal{T}(n)$-universal m-gonal forms. Let m be an integer ≥ 3 and n be a positive integer. For $\mathbf{a} \in \mathcal{N}$, we denote by $\Psi(\mathbf{a})$

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the set of integers in $\mathcal{T}(n)$ which are not represented by the m-gonal form $(\mathcal{GP}_m, \mathbf{a})$, that is,

$$\Psi(\mathbf{a}) = \Psi_{m,n}(\mathbf{a}) = \mathcal{T}(n) - R'_{\mathcal{GP}_m}(\mathbf{a}).$$

We define a function $\psi = \psi_{m,n} : \mathcal{N} \to \mathcal{T}(n) \cup \{\infty\}$ by

$$\psi(\mathbf{a}) = \begin{cases} \min(\Psi(\mathbf{a})) & \text{if } \Psi(\mathbf{a}) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

For a vector $\mathbf{a}$ with $\psi(\mathbf{a}) < \infty$, we define the set $\mathcal{E}(\mathbf{a})$ by

$$\mathcal{E}(\mathbf{a}) = \{g \in \mathbb{Z} : n \leq g \leq \psi(\mathbf{a}) - n\} \cup \{\psi(\mathbf{a})\}.$$

Note that if $\psi(\mathbf{a}) < 2n$, then $\mathcal{E}(\mathbf{a}) = \{\psi(\mathbf{a})\}$. For $k = 1, 2, 3, \ldots$, we define subsets $E(k), U(k), NU(k)$ and $A(k)$ of $\mathcal{N}(k)$ recursively as follows. Put $E(1) = \{(n)\}$. Define

$$U(k) = \{\mathbf{a} \in E(k) : \psi(\mathbf{a}) = \infty\}.$$

Let $NU(k)$ be the set of all vectors $\mathbf{a}$ in $U(k)$ such that $\mathbf{b} \notin \bigcup_{i=1}^{k-1} U(i)$ for every $\mathbf{b} \in \mathcal{N}$ satisfying $\mathbf{b} \prec \mathbf{a}$. Let $A(k) = E(k) - U(k)$ and

$$E(k+1) = \bigcup_{\mathbf{a} \in A(k)} \{\mathbf{a} * g : g \in \mathcal{E}(\mathbf{a})\}.$$

The algorithm terminates once $A(k) = \emptyset$.

THEOREM 3.1. *With the notation given above, for a vector* $\mathbf{a} \in \mathcal{N}(k)$*, a k-ary m-gonal form* $(\mathcal{GP}_m, \mathbf{a})$ *is new tight* $\mathcal{T}(n)$*-universal if and only if* $\mathbf{a} \in NU(k)$*.*

PROOF. The 'if' part is clear by construction. To prove the 'only if' part, let $\mathbf{a} \in \mathcal{N}(k)$ be a vector such that $(\mathcal{GP}_m, \mathbf{a})$ is tight $\mathcal{T}(n)$-universal. Since $R'_{\mathcal{GP}_m}(\mathbf{a}) = \mathcal{T}(n)$, it clearly follows that $a_{i_1} = n$, where we put $i_1 = 1$. Note that the set $R_{\mathcal{GP}_m}(a_{i_1})$ does not contain any positive integer less than n and it does contain 0 and all integers from n to $\psi(a_{i_1}) - 1$. From this and $\psi(a_{i_1}) \in \mathcal{T}(n) = R'_{\mathcal{GP}_m}(\mathbf{a})$, one may easily deduce that there must be an index i_2 different from i_1 such that

$$a_{i_2} \in \mathcal{E}(a_{i_1}) = \{n, n+1, n+2, \ldots, \psi(a_{i_1}) - n\} \cup \{\psi(a_{i_1})\}.$$

Thus $a_{i_1} * a_{i_2} \preceq \mathbf{a}$, where $a_{i_1} * a_{i_2} \in E(2)$. Note that $\psi(a_{i_1}) \in R'_{\mathcal{GP}_m}(a_{i_1} * a_{i_2})$. Assume $R'_{\mathcal{GP}_m}(a_{i_1} * a_{i_2}) \subsetneq \mathcal{T}(n)$ so that $\psi(a_{i_1} * a_{i_2}) < \infty$. One may easily show that there should be an index i_3 different from both i_1 and i_2 such that

$$a_{i_3} \in \mathcal{E}(a_{i_1} * a_{i_2}) = \{n, n+1, n+2, \ldots, \psi(a_{i_1} * a_{i_2}) - n\} \cup \{\psi(a_{i_1} * a_{i_2})\}$$

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in a similar manner. We have $a_{i_1} * a_{i_2} * a_{i_3} \in E(3)$ by construction. Note that

$$\psi(a_{i_1} * a_{i_2} * \cdots * a_{i_j}) < \infty$$

for every $j = 1, 2, \ldots, k-1$ since otherwise, $(\mathcal{GP}_m, \mathbf{a})$ cannot be new. Repeating this, we arrive at

$$\mathbf{a} = a_{i_1} * a_{i_2} * \cdots * a_{i_k} \in E(k).$$

Since $(\mathcal{GP}_m, \mathbf{a})$ is new tight $\mathcal{T}(n)$-universal, one may easily see that $\mathbf{a} \in NU(k)$. This completes the proof. □

Although the proof of the following lemma appeared in the proof of [9, Lemma 2.1], we provide it for completeness. For two positive integers d and r, we define a set

$$\mathcal{AP}_{d,r} = \{dg + r : g \in \mathbb{N} \cup \{0\}\} \ (\subseteq \mathbb{N}).$$

Lemma 3.2. *With the notation given above, there is a positive integer $l = l(m, n)$ depending on m and n such that $A(l) = \emptyset$.*

Proof. Let t be a positive integer greater than 4 and let $\mathbf{a} = (a_1, a_2, \ldots, a_t)$ be a vector in $A(t) = E(t) - U(t)$ so that $\psi(\mathbf{a}) < \infty$. Note that, for any $\mathbb{Z}$-lattice L of rank ≥ 4 with $Q(\text{gen}(L)) \subsetneq \mathbb{N}$,

$$\mathbb{N} - Q(\text{gen}(L)) = \bigcup_{i=1}^{\nu_1'} \mathcal{AP}_{d_i', r_i'}$$

for some positive integers ν_1', d_i' and r_i' with $r_i' < d_i'$ by the results in [14]. From this and [3, Theorem 4.9] (see also [5]), one may easily deduce that

$$\mathcal{T}(n) - R'_{\mathcal{GP}_m}(\mathbf{a}) = \bigcup_{i=1}^{\nu_1} \mathcal{AP}_{d_i, r_i} \cup \{e_1, e_2, \ldots, e_{\nu_2}\}$$

for some nonnegative integers ν_1, ν_2 not both 0 and some positive integers d_i, r_i, e_j with $e_j \notin \bigcup_{i=1}^{\nu_1} \mathcal{AP}_{d_i, r_i}$ for all $j = 1, 2, \ldots, \nu_2$. Suppose that g_1 is a positive integer with $n \leq g_1 \leq \psi(\mathbf{a}) - n$ or $g_1 = \psi(\mathbf{a})$ so that $\mathbf{a} * g_1 \in E(t+1)$. If

$$Q(\text{gen}(\langle a_1, a_2, \ldots, a_t \rangle)) \subsetneq Q(\text{gen}(\langle a_1, a_2, \ldots, a_t, g_1 \rangle)),$$

then

$$\mathcal{T}(n) - R'_{\mathcal{GP}_m}(\mathbf{a} * g_1) = \bigcup_{w=1}^{\nu_3} \mathcal{AP}_{d_{i_w}, r_{i_w}} \cup \{e_1', e_2', \ldots, e_{\nu_4}'\},$$

where ν_3 is an integer with

$$0 \leq \nu_3 < \nu_1, \quad (i_1, i_2, \ldots, i_{\nu_3}) \prec (1, 2, \ldots, \nu_1),$$

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and v_4 is a nonnegative integer. When

$$Q(\text{gen}(\langle a_1, a_2, \ldots, a_t \rangle)) = Q(\text{gen}(\langle a_1, a_2, \ldots, a_t, g_1 \rangle)),$$

it follows that

$$\mathcal{T}(n) - R'_{\mathcal{GP}_m}(\mathbf{a} * g_1) = \bigcup_{i=1}^{v_1} \mathcal{AP}_{d_i, r_i} \cup \{e_{j_1}, e_{j_2}, \ldots, e_{j_{v_5}}\},$$

where v_5 is a nonnegative integer less than v_2 and $(j_1, j_2, \ldots, j_{v_5}) \prec (1, 2, \ldots, v_2)$.

Let $\mathbf{b}$ be a vector in $A(5) = E(5) - U(5)$. From what we observed above, we may define a positive integer $w = w(\mathbf{b})$ to be the maximal positive integer w satisfying

$$b * g_1 * g_2 * \cdots * g_i \in A(5+i) - U(5+i), \quad g_i \in \mathcal{E}(b * g_1 * g_2 * \cdots * g_{i-1}),$$

for every $i = 1, 2, \ldots, w-1$. Since the set $E(5)$ is finite by construction, we may take l as

$$l = 5 + \max\{w(\mathbf{b}) : \mathbf{b} \in E(5) - U(5)\}.$$

This completes the proof. □

We now introduce our main result which gives a natural generalisation of the Conway–Schneeberger 15-Theorem to the case of tight $\mathcal{T}(n)$-universal m-gonal forms.

Theorem 3.3. *With the notation given above, there is a finite set $CS(m, n)$ such that $R'_{\mathcal{GP}_m}(\mathbf{a}) = \mathcal{T}(n)$ if and only if $R'_{\mathcal{GP}_m}(\mathbf{a}) \cap \{1, 2, \ldots, n-1\} = \emptyset$ and $CS(m, n) \subset R'_{\mathcal{GP}_m}(\mathbf{a})$ for any vector $\mathbf{a} \in \mathcal{N}$.*

Proof. Using Lemma 3.2, we take the smallest positive integer l satisfying $A(l) = \emptyset$. Define a finite set

$$CS(m, n) = \{n\} \cup \bigcup_{k=1}^{l-1} \{\psi(\mathbf{a}) : \mathbf{a} \in A(k)\}.$$

Let $\mathbf{a} \in \mathcal{N}$ be a vector with $R'_{\mathcal{GP}_m}(\mathbf{a}) \cap \{1, 2, \ldots, n-1\} = \emptyset$ such that $R'_{\mathcal{GP}_m}(\mathbf{a}) \supset CS(m, n)$. From the condition that $R'_{\mathcal{GP}_m}(\mathbf{a}) \supset CS(m, n)$, one may easily see that there is a vector $\mathbf{b} \in \mathcal{N}$ with $\mathbf{b} \preceq \mathbf{a}$ such that $\mathbf{b} \in U(k)$ for some k less than or equal to l. It follows that

$$\mathcal{T}(n) = R'_{\mathcal{GP}_m}(\mathbf{b}) \subseteq R'_{\mathcal{GP}_m}(\mathbf{a}).$$

This completes the proof. □

Remark 3.4. In Theorem 3.3, the set $CS(m, n)$ is minimal in the sense that for any $g \in CS(m, n)$, there is a vector $\mathbf{b} \in \mathcal{N}$ such that $R'_{\mathcal{GP}_m}(\mathbf{b}) = \mathcal{T}(n) - \{g\}$. To see this, we take $\mathbf{b} = \mathbf{c} * \mathbf{d}$, where $\psi(\mathbf{c}) = g$ and $R'_{\mathcal{GP}_m}(\mathbf{d}) = \mathcal{T}(g+1)$. The existence of such vectors $\mathbf{c}$ and $\mathbf{d}$ follows from the definition of the set $CS(m, n)$ and Lemma 2.1, respectively.

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In the spirit of Remark 3.4 and [11], we may call the set $CS(m,n)$ a *minimal tight $\mathcal{T}(n)$-universality criterion set for m-gonal forms*.

PROPOSITION 3.5. *Let m be an integer greater than or equal to* 3 *and different from* 5 *and let n be an integer greater than* 1. *With the notation given above:*

(i) $\{n, n+1, n+2, \dots, 2n\} \subseteq CS(m,n)$;
(ii) $E(k) = \{(n, n+1, n+2, \dots, n+k-1)\}$ *for* $k = 1, 2, \dots, n$;
(iii) $U(k) = \emptyset$ (*or equivalently*, $A(k) = E(k)$) *for* $k = 1, 2, \dots, n$;
(iv) $E(n+1) = \{(n, n, n+1, n+2, \dots, 2n-1), (n, n+1, n+2, \dots, 2n-1, 2n)\}$.

PROOF. Note that $2 \notin \mathcal{GP}_m$ since $m \neq 5$. For $i = 1, 2, \dots, n-1$, one may easily show that $\psi(n) = n+1$ and

$$\psi(n, n+1, n+2, \dots, n+i) = n+i+1.$$

The proposition follows directly from this. □

REMARK 3.6. Proposition 3.5(i), (ii) and (iii) also hold for the case of pentagonal forms, that is, when $m = 5$. However, Proposition 3.5(iv) is no longer true when $m = 5$. In fact, since $2 = P_5(-1) \in \mathcal{GP}_5$, we have

$$2n \in R'_{\mathcal{GP}_5}(n) \subset R'_{\mathcal{GP}_5}(n, n+1, n+2, \dots, 2n-1),$$

and thus we would have $\psi(n, n+1, n+2, \dots, 2n-1) > 2n$.

4. Some experimental results

We provide some experimental results based on the escalation algorithm for tight $\mathcal{T}(n)$-universal m-gonal forms introduced in Section 3. We first note that, in practice, we use the set

$$\Psi(\mathbf{a}) = \Psi_{m,n}(\mathbf{a}) = \{u \in \mathcal{T}(n) : u \leq 10^6\} - R'_{\mathcal{GP}_m}(\mathbf{a})$$

instead of the original definition $\Psi(\mathbf{a}) = \mathcal{T}(n) - R'_{\mathcal{GP}_m}(\mathbf{a})$ in the algorithm so that

$$\{u \in \mathbb{N} : n \leq u \leq 10^6\} \subset R'_{\mathcal{GP}_m}(\mathbf{a}) \quad \text{for all } \mathbf{a} \in \bigcup_{k=1}^{\infty} U(k).$$

In Table 1, we give the sets $CS(m,n)$ for some pairs (m,n). In the table, the pair (m,n) is marked with † when the tight $\mathcal{T}(n)$-universal m-gonal forms are already completely classified so that the set $CS(m,n)$ in the table has been proved to be equal to the set $CS(m,n)$ in the algorithm in Section 3.

For the classification of tight $\mathcal{T}(n)$-universal m-gonal forms, we refer the reader to [1] for $(m,n) = (4,1)$, [2] for $(m,n) = (3,1)$, [8] for $(m,n) = (8,1)$, [6] for $(m,n) = (5,1)$, [13] for $m = 4$ and $n \geq 2$, and [12] for the others. The tight universal m-gonal forms are classified for $m = 4, 3$, and tight $\mathcal{T}(n)$-universal octagonal forms for all $n \geq 2$

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TABLE 1. $CS(m,n)$ for some pairs (m,n).

m	n	$CS(m,n)$
3	$1^\dagger$	$\{1,2,4,5,8\}$
	$2^\dagger$	$\{2,3,4,8,10,16,19\}$
	$3^\dagger$	$\{3,4,5,6,16\}$
	$\geq 4^\dagger$	$\{n,n+1,n+2,\ldots,2n\}$
4	$1^\dagger$	$\{1,2,3,5,6,7,10,14,15\}$
	$2^\dagger$	$\{2,3,4,6,9,10,13,15,17,23\}$
	$3^\dagger$	$\{3,4,5,6,13,14,18,25,35,46\}$
	$\geq 4^\dagger$	$\{n,n+1,n+2,\ldots,2n\}$
5	$1^\dagger$	$\{1,3,8,9,11,18,19,25,27,43,98,109\}$
	2	$\{2,3,9,53,77,141\}$
	3	$\{3,4,5,22,47,52,62\}$
	$4 \leq n \leq 6$	$\{n,n+1,n+2,\ldots,2n-1\}$
	$\geq 7^\dagger$	$\{n,n+1,n+2,\ldots,2n-1\}$
7	1	$\{1,2,3,5,6,9,10,15,16,19,23,31,131\}$
	2	$\{2,3,4,6,9,10,13,15,18,27,30,32,50\}$
	3	$\{3,4,5,6,13,14,18\}$
	$4 \leq n \leq 10$	$\{n,n+1,n+2,\ldots,2n\}$
	$\geq 11^\dagger$	$\{n,n+1,n+2,\ldots,2n\}$
8	$1^\dagger$	$\{1,2,3,4,6,7,9,12,13,14,18,60\}$
	2	$\{2,3,4,6,8,9,11,12,14,18\}$
	3	$\{3,4,5,6,13,14,16,17,21,22,27,36\}$
	4	$\{4,5,6,7,8,23,28\}$
	$5 \leq n \leq 10$	$\{n,n+1,n+2,\ldots,2n\}$
	$\geq 11^\dagger$	$\{n,n+1,n+2,\ldots,2n\}$
9	1	$\{1,2,3,4,5,7,8,10,11,14,16,17,20,22,23,29,32,34,69\}$
	2	$\{2,3,4,6,8,9,10,11,13,14,16,17,19,23,25,28,34,37,58\}$
	3	$\{3,4,5,6,13,14,16,17,19,20,21,25,26,28,38,46,53\}$
	4	$\{4,5,6,7,8,23,25,27,28,32,33\}$
	$5 \leq n \leq 12$	$\{n,n+1,n+2,\ldots,2n\}$
	$\geq 13^\dagger$	$\{n,n+1,n+2,\ldots,2n\}$
≥ 10	$\geq 2m-5^\dagger$	$\{n,n+1,n+2,\ldots,2n\}$

are treated in [7]. In this spirit, we provide the candidates for tight $\mathcal{T}(n)$-universal pentagonal forms in the cases of $n=2,3$ in Tables 2 and 3, respectively. Note that there is exactly one candidate for tight $\mathcal{T}(n)$-universal pentagonal forms for each $n=4,5,6$, which is $(\mathcal{GP}_5,(n,n+1,n+2,\ldots,2n-1))$.

For any integer $m \geq 3$ and a positive integer n, we define $\gamma_{m,n}$ to be the maximum element in the set $CS(m,n)$, as in the proof of Theorem 3.3. By Theorem 3.3,

TABLE 2. Candidates for new tight $\mathcal{T}(2)$-universal pentagonal forms $(\mathcal{GP}_5, (a_1, a_2, \ldots, a_k))$.

a_1	a_2	a_3	a_4	Conditions on a_k $(3 \le k \le 4)$
2	2	3		
2	3	a_3		$6 \le a_3 \le 9,\ a_3 \ne 8$
2	3	3	a_4	$3 \le a_4 \le 77,\ a_4 \ne 6, 7, 9, 76$
2	3	4	a_4	$4 \le a_4 \le 141,\ a_4 \ne 6, 7, 9, 140$
2	3	5	a_4	$5 \le a_4 \le 53,\ a_4 \ne 6, 7, 9, 52$

TABLE 3. Candidates for new tight $\mathcal{T}(3)$-universal pentagonal forms $(\mathcal{GP}_5, (a_1, a_2, \ldots, a_k))$.

a_1	a_2	a_3	a_4	a_5	Conditions on a_k $(4 \le k \le 5)$
3	3	4	5		
3	4	4	5		
3	4	5	a_4		$6 \le a_4 \le 22,\ a_4 \ne 10, 15, 20, 21$
3	4	5	5	a_5	$a_5 = 5, 10, 15, 20, 21, 62$ or $23 \le a_5 \le 59$
3	4	5	10	a_5	$a_5 = 10, 15, 20, 21, 47$ or $23 \le a_5 \le 44$
3	4	5	15	a_5	$a_5 = 15, 20, 21, 52$ or $23 \le a_5 \le 49$

TABLE 4. γ_m for $3 \le m \le 11$.

m	$3^\dagger$	$4^\dagger$	$5^\dagger$	7	$8^\dagger$	9	10	11
γ_m	8	15	109	131	60	69	46	45

TABLE 5. Candidates for new universal heptagonal forms $(\mathcal{GP}_7, (a_1, a_2, \ldots, a_k))$.

a_1	a_2	a_3	a_4	a_5	Conditions on a_k $(4 \le k \le 5)$
1	1	1	a_4		$1 \le a_4 \le 10,\ a_4 \ne 6$
1	1	2	a_4		$2 \le a_4 \le 23$
1	1	3	a_4		$4 \le a_4 \le 5$
1	2	2	a_4		$2 \le a_4 \le 19$
1	2	3	a_4		$3 \le a_4 \le 31$
1	2	4	a_4		$4 \le a_4 \le 131$
1	2	5	a_4		$5 \le a_4 \le 10,\ a_4 \ne 6$
1	1	1	6	a_5	$a_5 = 6$ or $11 \le a_5 \le 16$
1	1	3	3	a_5	$a_5 = 3$ or $6 \le a_5 \le 9$
1	1	3	6	a_5	$6 \le a_5 \le 15$
1	2	5	6	a_5	$a_5 = 6$ or $11 \le a_5 \le 16$

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TABLE 6. Candidates for new universal nonagonal forms $(\mathcal{GP}_9, (a_1, a_2, \ldots, a_k))$.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	Conditions on a_k $(4 \le k \le 7)$
1	1	1	a_4				$a_4 = 2, 4$
1	1	2	a_4				$2 \le a_4 \le 5$
1	1	3	a_4				$a_4 = 4, 7$
1	2	2	a_4				$a_4 = 3, 4, 7$
1	2	3	a_4				$a_4 = 4, 5$
1	2	4	a_4				$4 \le a_4 \le 12$, $a_4 \ne 6, 9$
1	1	1	1	a_5			$a_5 = 1, 3, 5$
1	1	1	3	a_5			$3 \le a_5 \le 17$, $a_5 \ne 4, 7$
1	1	3	3	a_5			$5 \le a_5 \le 11$, $a_5 \ne 6, 7$
1	1	3	5	a_5			$5 \le a_5 \le 16$, $a_5 \ne 7$
1	1	3	6	a_5			$6 \le a_5 \le 14$, $a_5 \ne 7$
1	1	3	8	a_5			$8 \le a_5 \le 16$
1	2	2	2	a_5			$2 \le a_5 \le 34$, $a_5 \ne 3, 4, 7$
1	2	2	5	a_5			$5 \le a_5 \le 22$, $a_5 \ne 7$
1	2	2	6	a_5			$6 \le a_5 \le 22$, $a_5 \ne 7$
1	2	3	3	a_5			$a_5 = 3$ or $6 \le a_5 \le 10$
1	2	3	6	a_5			$6 \le a_5 \le 23$
1	2	3	7	a_5			$7 \le a_5 \le 17$, $a_5 \ne 15$
1	2	4	6	a_5			$a_5 = 6, 9$ or $13 \le a_5 \le 20$
1	2	4	9	a_5			$a_5 = 9$ or $13 \le a_5 \le 29$
1	2	4	13	a_5			$13 \le a_5 \le 69$
1	2	4	14	a_5			$14 \le a_5 \le 34$
1	1	3	3	3	a_6		$a_6 = 6$ or $12 \le a_6 \le 14$
1	1	3	3	6	a_6		$15 \le a_6 \le 17$
1	2	3	7	15	a_6		$a_6 = 15$ or $18 \le a_6 \le 32$
1	1	3	3	3	3	a_7	$a_7 = 3, 15, 16, 17$

if an m-gonal form g does not represent any integer less than n and does represent all integers from n to $\gamma_{m,n}$, then g is tight $\mathcal{T}(n)$-universal. For $m = 3, 4, \ldots,$ we define

$$\gamma_m = \gamma_{m,1} = \max(C(m, 1)).$$

Now we consider universal m-gonal forms. In Table 4, γ_m is given for $3 \le m \le 11$ and the proved cases are marked †. We provide all candidates of new universal m-gonal forms, for $m = 7, 9, 10, 11$, in Tables 5–8, since the universal m-gonal forms are of particular interest.

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TABLE 7. Candidates for new universal decagonal forms $(\mathcal{GP}_{10}, (a_1, a_2, \ldots, a_k))$.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	Conditions on a_k $(4 \le k \le 8)$
1	1	1	4					
1	1	2	a_4					$2 \le a_4 \le 5$
1	2	2	a_4					$3 \le a_4 \le 4$
1	2	3	a_4					$a_4 = 4, 6$
1	2	4	a_4					$a_4 = 4, 5, 8$
1	1	1	1	a_5				$a_5 = 2, 3, 5$
1	1	1	2	6				
1	1	1	3	a_5				$5 \le a_5 \le 16$
1	1	3	3	a_5				$a_5 = 5, 8$
1	1	3	4	a_5				$4 \le a_5 \le 16$
1	1	3	5	a_5				$5 \le a_5 \le 24$
1	1	3	6	a_5				$7 \le a_5 \le 11, a_5 \ne 9$
1	2	2	2	a_5				$a_5 = 2$ or $5 \le a_5 \le 8$
1	2	2	5	a_5				$6 \le a_5 \le 13$
1	2	2	6	a_5				$7 \le a_5 \le 19, a_5 \ne 14$
1	2	3	3	a_5				$3 \le a_5 \le 11, a_5 \ne 4, 6, 8$
1	2	3	5	a_5				$5 \le a_5 \le 16, a_5 \ne 6$
1	2	3	7	a_5				$7 \le a_5 \le 26$
1	2	3	8	a_5				$8 \le a_5 \le 16, a_5 \ne 12, 15$
1	2	4	6	a_5				$6 \le a_5 \le 23, a_5 \ne 8$
1	2	4	7	a_5				$7 \le a_5 \le 39, a_5 \ne 8$
1	1	1	1	1	a_6			$a_6 = 1, 6$
1	1	1	3	3	a_6			$a_6 = 3, 17, 18, 19$
1	1	3	3	3	a_6			$4 \le a_6 \le 12, a_6 \ne 5, 6, 8$
1	1	3	3	4	a_6			$17 \le a_6 \le 19$
1	1	3	3	6	a_6			$a_6 = 6, 9$ or $12 \le a_6 \le 15$
1	1	3	3	7	a_6			$7 \le a_6 \le 19, a_6 \ne 8$
1	1	3	3	9	a_6			$9 \le a_6 \le 18$
1	1	3	6	6	a_6			$a_6 = 9$ or $12 \le a_6 \le 18$
1	1	3	6	9	a_6			$a_6 = 9$ or $12 \le a_6 \le 24$
1	1	3	6	12	a_6			$12 \le a_6 \le 24$
1	2	2	5	5	a_6			$a_6 = 5$ or $14 \le a_6 \le 18$
1	2	2	6	6	a_6			$a_6 = 6, 14$ or $20 \le a_6 \le 25$
1	2	2	6	14	a_6			$a_6 = 14$ or $20 \le a_6 \le 39$
1	2	3	3	8	a_6			$12 \le a_6 \le 19, a_6 \ne 13, 14, 16$
1	2	3	8	12	a_6			$12 \le a_6 \le 46, a_6 \ne 13, 14, 16$
1	2	3	8	15	a_6			$15 \le a_6 \le 34, a_6 \ne 16$
1	1	3	3	3	3	a_7		$a_7 = 6, 13, 14, 15$
1	1	3	3	3	6	a_7		$16 \le a_7 \le 18$
1	1	3	6	6	6	a_7		$a_7 = 6$ or $19 \le a_7 \le 24$
1	1	3	3	3	3	3	a_8	$a_8 = 3, 16, 17, 18$

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TABLE 8. Candidates for new universal hendecagonal forms $(\mathcal{GP}_{11}, (a_1, a_2, \ldots, a_k))$.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	Conditions on a_k $(4 \le k \le 8)$
1	1	2	a_4					$a_4 = 3, 4$
1	2	2	4					
1	2	3	4					
1	2	4	a_4					$4 \le a_4 \le 8$
1	1	1	1	a_5				$a_5 = 3, 4, 5$
1	1	1	2	a_5				$a_5 = 2, 5, 6$
1	1	1	3	a_5				$4 \le a_5 \le 7$
1	1	1	4	a_5				$4 \le a_5 \le 18$
1	1	2	2	a_5				$a_5 = 2, 5, 6, 7$
1	1	2	5	a_5				$5 \le a_5 \le 20$
1	1	3	3	a_5				$a_5 = 4, 5, 6, 9$
1	1	3	4	a_5				$a_5 = 5, 8, 9$
1	1	3	5	a_5				$6 \le a_5 \le 18$
1	1	3	6	a_5				$6 \le a_5 \le 13$, $a_5 \ne 10$
1	2	2	2	a_5				$2 \le a_5 \le 9$, $a_5 \ne 4$
1	2	2	3	a_5				$3 \le a_5 \le 9$, $a_5 \ne 4$
1	2	2	5	a_5				$5 \le a_5 \le 14$
1	2	2	6	a_5				$6 \le a_5 \le 20$, $a_5 \ne 17$
1	2	3	3	a_5				$5 \le a_5 \le 12$, $a_5 \ne 6, 9$
1	2	3	5	a_5				$5 \le a_5 \le 12$
1	2	3	6	a_5				$7 \le a_5 \le 15$
1	2	3	7	a_5				$8 \le a_5 \le 38$
1	2	4	9	a_5				$9 \le a_5 \le 18$
1	1	1	1	1	a_6			$a_6 = 2, 6$
1	1	1	1	2	7			
1	1	1	3	3	a_6			$a_6 = 3, 8$ or $10 \le a_6 \le 21$
1	1	3	3	3	a_6			$a_6 = 3, 7, 8, 11, 12, 13$
1	1	3	3	7	a_6			$7 \le a_6 \le 20$, $a_6 \ne 9$
1	1	3	3	8	a_6			$8 \le a_6 \le 21$, $a_6 \ne 9$
1	1	3	3	10	a_6			$10 \le a_6 \le 20$
1	1	3	4	4	a_6			$4 \le a_6 \le 21$, $a_6 \ne 5, 8, 9$
1	1	3	4	6	a_6			$a_6 = 10$ or $14 \le a_6 \le 27$
1	1	3	4	7	a_6			$a_6 = 7$ or $10 \le a_6 \le 17$
1	1	3	4	10	a_6			$10 \le a_6 \le 27$
1	1	3	5	5	a_6			$a_6 = 5$ or $19 \le a_6 \le 23$
1	1	3	6	10	a_6			$a_6 = 10$ or $14 \le a_6 \le 23$
1	2	2	6	17	a_6			$a_6 = 17$ or $21 \le a_6 \le 37$
1	2	3	3	3	a_6			$a_6 = 9, 13, 14, 15$
1	2	3	3	6	a_6			$a_6 = 6, 16, 17, 18$
1	2	3	3	9	a_6			$a_6 = 9$ or $13 \le a_6 \le 21$
1	2	3	6	6	a_6			$a_6 = 6$ or $16 \le a_6 \le 21$
1	2	3	7	7	a_6			$a_6 = 7$ or $39 \le a_6 \le 45$
1	1	1	1	1	1	a_7		$a_7 = 1, 7$
1	1	3	3	3	10	a_7		$21 \le a_7 \le 23$
1	2	3	3	3	3	a_7		$a_7 = 6, 16, 17, 18$
1	2	3	3	3	6	a_7		$19 \le a_7 \le 21$
1	2	3	3	3	3	3	a_8	$a_8 = 3, 19, 20, 21$

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